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THE CONTROVERSY OVER USE OF SPRT AND GLR TECHNIQUES AND OTHER LOOSE-ENDS IN FAILURE DETECTION

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1. Introduction

While reformulations of Sequential Probability Ratio Tests (SPRT) and Generalized Likelihood Ratio (GLR) tests have been used (at least for initial investigations) within the last ten years for failure detection and other event or incident detection applications, there appear to be several theoretical and implementation problems underlying SPRT and GLR for these applications that have yet to be resolved. Somewhat suprisingly, questions and caveats pertaining to some of these very problems were first raised in the preceeding decade by even earlier investigators as they grappled with the fundamental assumptions. For this reason, results and criticisms are depicted here within a more expansive historical context than that offered by considering just the last decade. A perspective on the current status of SPRT and GLR for failure and event detection is also provided herein.

Section 2 provides an overview of unresolved issues in SPRT and Section 3 provides an examination of unresolved issues in GLR development for failure detection. Highlights of several problems are touched upon in this paper. Unresolved issues relating to other failure detection techniques are critically examined in Section 4 while Section 5 constructively provides new results for one failure detection approach.

2. Loose-Ends in Adapting SPRT To Failure Detection

One of the earliest approaches to detecting soft failures in navigation systems [3] used Wald's Sequential Probability Likelihood Ratio Test (SPRT) [4]. While there was some initial controversy [5], [6] (surprisingly not over the assumption of independent measurements, where such difficulties are elaborated upon on p. 96 of [13]), it was generally agreed [7] that there was no better approach available at that time to this particularly difficult problem. Later approaches to detecting the same type of navigation system gyro and accelerometer failures [8], [9] also sought to use modification of the SPRT.

Fairly obvious objections to the use of SPRT in failure detection are that:

- The SPRT is not really appropriate for detecting transitions from one underlying system mode (such as non-failure) to a second mode (such as a failure occuring) during a trial (or measurement sampling interval) since the rigorous foundation of SPRT ([13], [14]) is strictly as a binary hypothesis test using an upper and lower decision threshold to only describe which of the two situations has persisted since test inception, as indicated by the test statistic going above the upper threshold or dipping below the lower threshold. There are so many aspects to the type, magnitude, and time of failure that can occur in applications involving dynamic systems that the underlying hypotheses are clearly not just simple binary, but mixed.
- The modifications of SPRT to accomodate detection of failures as mode changes are widely admitted to be mere attempts to come up with useful contrivances that may work acceptably (e.g., in [8] when a no-failure decision is declared, the SPRT test statistic is immediately reset back to the neutral midpoint between the two thresholds, where, in effect, no decision has been arrived at but merely more data is to be collected to facilitate a later decision). This mechanization obviously alters the random decision times of the SPRT as an otherwise optimal decision test as it was originally conceived of and justified by Wald [4] (and Selin [13]). Theoretical justification for the recent modifications are yet to be supplied.
- The test offered in [8] must have a priori knowledge provided/specified of whether magnitude of failure being investigated is "large" or "small" in order to provide unambiguous test resolution. That SPRT approaches require such assumptions (as prior knowledge of failure magnitude) during attempts to apply SPRT in situations possessing unknown parameters is agreed upon by a rigorous statistical route, as reported on p. 102 of [13].
- The use of initiating "triggers" in [9] prior to commencing full SPRT calculations (analogous to use of a trigger to set-off a bistable multivibrator) introduces obvious complications of dual decision threshold determination/specification for both the trigger and the subsequent trailing-window SPRT decisions that are consequently cross-correlated to it. No such issue is addressed in [9] of how to properly specify dual cross-coupled decision thresholds for both the SPRT and its leading trigger test.

3. Loose-Ends in GLR Development

While Generalized Likelihood Ratios (GLR) (where maximum likelihood estimates of parameters are utilized within the likelihood ratios in lieu of the parameters being unknown) are presented and developed by Davenport and Root [10], Root went further

[11] to investigate applicability of GLR techniques in the radar detection problem of resolving closely spaced targets in either a background of known arbitrary correlated Gaussian noise or Gaussian white noise. However, Root [11] obtained explicit criteria that could be applied to indicate conditions under which one could reasonably expect to not be able to resolve two known signals (of unknown amplitudes and parameters) and additionally pointed out a difficulty of using GLR for this purpose. Selin [12] found that some of the unknown parameters (such as unknown relative carrier phase) must also be estimated in order to maximize the a posteriori probability in the estimation of two similar signals in white Gaussian noise. Selin further identified four standard caveats (p. 106 of [13]) associated with the use of a maximum likelihood estimate of the unknown parameters in a likelihood ratio (as utilized in GLR). McAulay and Denlinger [14] advocated use of GLR in conjunction with a Kalman Filter in decision-directed adaptive radar trackers for air traffic control applications. Finally, Stuller [15] defined an M-ary GLR test that ostensibly overcame Root's original objections [11] to GLR for this type of application. ([15] also provides a limited history of GLR developments for radar, excepting no mention of [14], which possibly eluded him.)

The use of GLR for failure detection was pioneered by Willsky and Jones [16], [17] using an identical GLR formulation as presented by McAulay and Denlinger [14]. While Willsky claims [p. 607, first column, last paragraph, 18] that GLR is the "optimum decision rule for failure detection" and McAulay and Denlinger claim [p. 229, 14] that they present "an optimum maneuver detector" by utilizing GLR, an explicit indication of what the criterion of optimality is, or a demonstration of optimality, or references to where it is established have eluded this author to date. Similarly, in [p. 109, paragraph following Eq. 21, 16] it is stated that some simple reasoning yields the result that the estimate (provided in the paper) is precisely the optimal estimate given the measurements, but given no a priori information on the covariance (i.e., initial covariance being infinite). Only an anonymous reviewer was cited in [p. 109, 16] for this important conclusion without demonstration, explanation, or indication of what the above indicated "simple reasoning" required for justification actually is. The real pity is that other emerging approaches to event detection as well as previous approaches are unfairly stiffled somewhat by being measured against GLR with its published "claims" of optimality and "claims" of completed threshold specification/evaluation. Because of an assortment of perceived inconsistences and other apparently unsubstantiated GLR claims as GLR relates to the failure and event detection problem (e.g., claimed but apparently unsubstantiated decision threshold determination), an itemized scrutiny of pertinent issues has been prepared as [1], some of which are previewed below.

INDICATED REQUIREMENT FOR A REAL-TIME PSEUDO-INVERSE CALCULATION

In [18, p. 608, first sentence], [41, p. 852, last paragraph], [16, p. 112, last paragraph], the reason given for having the upper limit of

$$\theta \leq k-N \tag{1}$$

in the constraint of Appendix A, Eq. A-1 was to avoid observability problems. Any observability problems were alleged (but never previously or evidently subsequently demonstrated or proven*) to be realized in the non-invertibilty of $C(k;\theta)$. However, the single numerical example being common to [16], [17], [18] has parameters in the model being

$$\phi = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}; \; H = [1 \;\; 0] \; ; \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 0.003 \end{bmatrix}$$

$$\Delta = \text{time step} \tag{2}$$

which can be demonstrated to be completely observable and completely stochastically controllable by the Kalman rank test since the observability and controllability grammians, respectively, are of rank=2. Yet the N in the upper limit of Eq. 1 for this two-dimensional example is taken in [16], [17], [41] to be

$$N=6 \tag{3}$$

For an observable n^{th}-order stationary discrete-time system, observability as gauged in terms of the nonsingularity of the observability matrix is theoretically guaranteed to occur in at least n steps. For the second order example of [16], [17], [41] as summarized in Eq. 2, the nonsingularity of $C(k;\theta)$ should be achieved within an allotted two steps if $C(k;\theta)$ is actually directly related to the observability of the system†. It is apparently somewhat contradicatory to require use of N=6 for the totally observable system as done in the example of [16], [17], [41] when it should only require at most two steps before the necessary $C(k;\theta)$ is invertible.

The telescoping property of the definition of $C(k;\theta)$ (as Eq. 12 in [16], Eq. 29 in [41], Eq. 46 in [17]) is used in establishing a means of recursively generating $C(k;\theta)$ as:

$$C(k;\theta)=G^T(k;\theta)V^{-1}(k)G(k;\theta)+C(k-1;\theta) \tag{4}$$

In [17, eq. 56], the matrix inversion lemma is applied to Eq. 4 to result in

* However [pp. 44-48, 12] does provide some first steps in the direction of resolving the observability issue, but it is done only asymptotically for the more restrictive time-invariant steady-state case (using notation for $G(k;\theta)$ [as defined in Eqs. 16, 19, 23 of [41], Eqs. 32, 37, 39 of [17], Eqs. 10, A3 of [16]) that suppresses the crucial dependence of the failure time θ, by representing it as $G(k)$ only].

† Otherwise, the unproven theoretical link to system observability (as asserted in [16], [18], [41]) is suspect.

$$C^{-1}(k;\theta) = [G^TV^{-1}G+C(k-1;\theta)]^{-1}$$
$$= C^{-1}(k-1;\theta)-C^{-1}(k-1;\theta)G^T[GC^{-1}(k-1;\theta)G^T+V]^{-1} GC^{-1}(k-1;\theta) \quad (5)$$

However in [17, p. 17, last sentence of paragraph two], it is asserted that the matrix inversion lemma can be used to propagate the psuedo-inverse recursively (in case all the G(.;.) are not of full rank and the strict inverse of C(k-1;θ) is not guaranteed to exist). However, this assertion was never proved nor referenced in [17], nor properly qualified as being merely a conjecture. Indeed, a counterexample to the property asserted in [17] that the pseudo-inverse may be recursively propagated via Eq. 5 is available as

$$C(k-1;\theta) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}; \; C^{\dagger}(k-1;\theta)= \frac{1}{25}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad (6)$$

$$G(k;\theta) = I_2; \; V(k) = V^{-1}(k) = I_2 \quad (7)$$

which when used in Eq. 4 yields

$$C(k;\theta) = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \quad (8)$$

having a valid pseudo-inverse (which in this case is the same as the inverse) being

$$C^{\dagger}(k;\theta)=C^{-1}(k;\theta) = \frac{1}{6}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \quad (9)$$

However, the following erroneous result

$$C^{\dagger}(k;\theta) = \frac{1}{30}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad (10)$$

is obtained when the matrix inversion lemma is utilized in an attempt to recursively generate the pseudo-inverse*. More recent results [61] describe how to reduce (or possibly avoid) matrix inversions in a discrete time GLR implementation by more circuitous recursive computations.

Another recent application of the generalized likelihood ration [71] to a system of the same general form as considered in [14], [16], [17], [41] is in detecting "failures in the Italian power network" (modeled in [71] with only a 4 state linear system). [71] distinguishes between this "different" GLR formulation and that of [16], [17], [41] as not using a Kalman filter, rather relying instead on solving a Fredholm integral equation for the system weighting function (to be used as optimal linear weightings of the measurements that are utilized in the likelihood ratio). However, the extremely important contribution encountered in the landmark derivation of Schweppe's likelihood ratio [72], (as clearly identified on pp. 659-660 of [73]) is that the solution of this difficult (if not intractable) Fredholm equation can be avoided entirely through recourse to an equivalent tractably calculated Kalman filter. Thus, it is revealed here that the "new" formulation of the likelihood ratio is identical to those previously presented in [14], [16], [41], but less tractably obtained.

On p. 92 of [84], attention is called to the fact that the GLR is not a Uniformly Most Powerful (UMP) test, while p. 96 of [84] offers recognition that cases exist where use of GLR can give bad results. That a maximum likelihood estimate (MLE) is not necessarily statistically consistent in general is explicitly demonstrated in a counterexample on p. 146 of [74]. Moreover, the difficulty usually encountered in attempting to demonstrate consistency of MLE's (as specifically used in identifying unknown parameters of linear dynamical systems) is conveyed in [76], [77], [78]. Within the theoretical formulations of the ideal or exact GLR for failure detection (discussed in [16], [17], [18], [41]), both the failure magnitude (including possible a priori specified likely "failure directions" or finite enumeration of possible "failure modes" that could physically occur for a particular system) and the time-of-failure are obtained from MLE estimators. These MLE's are subsequently substituted back into the likelihood ratio to serve in the role of the unknown parameters. All alternative approximate implementations of GLR (offered to data as [41], [16], [17], [18], [37], [41]) apparently avoid use of the explicit exact MLE of failure time in order to circumvent the impractical requirement that would otherwise be present of having to implement a bank-of-Kalman-filters whose number grows linearly with time ([14, following Fig. 34], [41, following Eq. 32], [17, p. 12], [18, following Eq. 45]). A new but equivalent reformulation of the calculation for exact GLR implementation [79] is more efficient by only requiring a logarithmic increase with time in the computational burden, but obviously any increase with time in the computational burden is unacceptable for real-time applications involving long run times or long monitoring intervals.

Use of the exact MLE of failure time is avoided completely in all of the above GLR approaches for approximating the exact GLR test statistic in order to avoid the requirement of neeeding a growing bank-of-filters. Unfortunately, the sufficient conditions (Item 6, p. 145 of [74]) that guarantee asymptotic efficiency of the estimates therefore cannot be verified for these approximate implementations since this sufficient condition requires use of the exact MLE of failure time [as needed to obtain first, second, and third partials of the underlying pdf. and to then seek to

* Cautionary counterexamples are offered in [2] to prevent widespread reliance on an improper definition of pseudoinverse as occurs in [70], on p. 19, and to further reveal current problems in correct numerical approximation of the associated "condition number" in a popular SVD computational algorithm for pseudo-inverse calculation.

demonstrate that these partials are bounded by integrable functions over (-oo, oo)]. The consistency of estimates supplied by the practical approximate implementaitons of GLR is therefore still an open question.

Bewildering or perhaps merely somewhat anomalous aspects appear in four recent applications of GLR [30], [31], [32], [59] with regard to how the decision threshold is used or, rather, not used in the GLR implementation. However, [1] offers constructive suggestions for GLR improvement in both test statistic calculation and decision threshold approximation. A brief survey of rigorously substantiated results pertaining to both the ordinary likelihood ratio and the GLR tests is presented in the introduction to [69].

4. An Assortment of Perceived Loose-Ends

An extremely rigorous alternate approach to detecting unexpected changes from a prescribed time-series model is provided in [19]; however, the theoretical properties that are exploited in [19] are equivalent to testing for whiteness of the residuals of a Kalman filter. Even in the unfailed case, the residuals are nominally corrupted or non-white due to the practical computational constraint in most engineering applications of only allowing use of a reduced-order filter. Alternate approaches to the failure detection problem that don't critically rely on filter residuals being white are available using the theory of confidence regions as discussed in [20]-[24]*, where the theory is tailored for a specific inertial navigation system application [25]-[29]. A complete statistical analysis is offered in [23] [24] to rigorously evaluate P_D and P_{FA} for the CR2 failure detection approach. [87] exemplifies recent additional experience of the author in evaluating detection probabilities. Alternate approaches to failure detection that are not confidence region-based are also surveyed in [21]-[24]. Promising approaches for detecting particular types of failures are offered in [67], [68], [60].

Probability-of-false-alarm (P_{FA}) and probability-of-correct-detection (P_D) are two well-known statistics that are at the heart of all detection decisions and correspond to performance evaluations of the probability of making errors of the "first and second kind (α and β)" as encountered in the terminology of statistical hypothesis testing. However, evaluations of these two descriptive fundamental statistics are nonexistent for most of the failure detection approaches previously proposed (as surveyed in [18], [23], [24]) and a noticable lack of rigor can be observed to afflict many of the remainder of those detection approaches that do get around to evaluating P_{FA} and P_D. Five examples are cited below to point out specific technical problems** in these earlier evaluations of P_{FA} and P_D.

Example 1: On p. 745 of [58], the sums of three triangularly distributed random variables of zero mean without unit variance are taken to be Chi-square with 3 degrees-of-freedom, dispite the well-known definition of Chi-square as only resulting from the sums of squares of uncorrelated, zero mean, unit variance, Gaussian.

Example 2: On p. 110 of [16], the full GLR is asserted to be Chi-square distributed under no failure; however, each of the individual likelihood ratios $\ell(k,\theta)$ is known to be a weighted Chi-square, and the nonlinear operation of maximization to yield the full GLR test statistic as

$$\ell(k)=\max_{\theta} \ell(k,\theta) \qquad (11)$$

does not preserve the property of being Chi-square (where the maximizing θ is the estimate of the time-of-failure and the computational burden corresponds to an additional Kalman filter for each additional candidate failure time considered). Please see pp. 193-194 of [57] for analytic verification that the maximization operation alters the final pdf even if each individual pdf were identical.

Example 3: Unfortunate notational ambiguity in the definition and notational usage for probability-of-false-alarm and for probability-of-correct-detection in the 1978 failure detection application of [49], as now described. In Eq. 18 of [49], the probability-of-false-alarm is defined to be

$$P_{FA} = \text{Probability } \{|p_N| > T_D | b=0\} \quad (12)$$

where

$|p_N| \triangleq$ magnitude of the test statistic used and is shown in Eq. 11 of [49] to be Rayleigh distributed, based on prior assumptions of measurement structure (Eq. 1 of [49]) and constraints inherent in an underlying parity matrix V (as defined following Eq. 2 and in Eq. 3 of [49]).

$T_D \triangleq$ a constant decision threshold that should be specified a prior (based on standard statistical considerations characterizing the well-known trade-off between making it small enough to enforce an acceptably large P_D; yet, not so small as to cause an intollerably large P_{FA})

* As discussed in [23], the author's two confidence region (CR2) approach can be used to detect more general time-varying failures, while [33] considers only detectability conditions for time-invariant systems. Recall that, in general, the linearization of a nonlinear system is time-varying (pp. 53, 54 of [43]) so failure detection algorithms that are compatible with time-varying linear systems apparently should be considered to be more practical, by being more widely applicable.

** The need to sometimes resort to engineering approximations, where warranted, for more expediently tractable evaluation is in fact greatly appreciated. However, the instances cited herein seem to exceed being mere approximations but appear instead to be gross "hacking attempts", unaccompanied by the slightest hint or acknowledgement in the cited reference of its first or subsequent appearance that what was being presented was not exact.

$b \overset{\Delta}{=}$ magnitude of apparent sensor bias shift (as defined in Eq. 5 of [49] as the failure to be detected).

Hence Eq. 12 above defines the probability of the test statistic $|p_N|$ exceeding the decision threshold, when it is known that the magnitude of the failure is zero (i.e., no failure has occurred). Furthermore, in Eq. 13 of [49], the probability-of-correct-detection is defined to be

$$P_D = \text{Probability } \{|p_F| > \sqrt{T_D}\} \tag{13}$$

A minor objection is that both $|p_F|$ and $|p_N|$ as used above (following the lead of [49]) is unfortunate and confusing* notation for the same entity being used as the common underlying test statistic as:

p_N being used under conditions of no failure (as stated in Eq. 4 of [49])

p_F being used under conditions of failure (as stated in Eq. 5 of [49])

and use of this notation involving p is to represent parity residuals (as defined following Eq. 2 of [49]), rather than probabilities. A blatant inconsistency between Eq. 12 and 13 above (as repeated from [49]) is that in Eq. 13, $\sqrt{T_D}$ must be exceeded by the test statistic; while in Eq. 12, $(\sqrt{T_D})^2 = T_D$ must be exceeded. Unfortunately, this otherwise minor inconsistency between use of T_D or $\sqrt{T_D}$ also carries over into the further evaluations presented in Eqs. 19, 39, 40 of [49] for explicit evaluation.

Another instance of an apparent oversight (rendering most of the subsequent evaluation results of [49] arguably useless) is that in the extension from a consideration of the five sensor case in Section III of [49] to a consideration of the six sensor case treated in Section IV of [49], an evaluation of probability-of-correct-detection for five (5) sensors as

$$P_D = \frac{1}{\pi} \int_0^\infty (\pi - \delta)\, f_{|p_N|}(\alpha)\, d\alpha \tag{14}$$

is altered to correspond to the following expression for six (6) sensors:

$$P_D = \frac{1}{\sqrt{2\pi}} \int_0^\infty (1 + \cos\delta)\, f_{|p_N|}(\alpha)\, d\alpha \tag{15}$$

corresponding exactly to Eqs. 17 and 38 of [49], respectively) where (from Eqs. 15 and 16 of [49]) in the above

$$\delta \overset{\Delta}{=} \cos^{-1}\left\{ g\left(\frac{\left(\frac{n-3}{n}\right) b^2 + |p_N|^2 - T_D}{2\sqrt{\left(\frac{n-3}{n}\right)}\, b\, |p_N|} \right) \right\} \tag{16}$$

and

$$g \overset{\Delta}{=} \begin{cases} -1 & \text{for } x < -1 \\ x & \text{for } -1 \le x \le 1 \\ 1 & \text{for } 1 < x \end{cases} \tag{17}$$

$n \overset{\Delta}{=}$ number of sensors being considered or monitored for failure occurrence in each case (i.e., n=5 or 6);

$f_{|p_N|} \overset{\Delta}{=}$ probability distribution function of the magnitude of the test statistic in the unfailed case (b=0).

The discrepancy being explicitly called out here is between the integrands of Eqs. 14 and 15 above (repeated from [49]) which appear to exhibit unexplainable differences. Especially disconcerting is the presence of the cos δ term in Eq. 15 (Eq. 38 of [49]) since, by the definition of Eq. 16 (repeated from Eq. 15 of [49])

$$\cos\delta = \cos(\cos^{-1}\{g(.)\}) = g(.) \tag{18}$$

as an obvious simplification, but which apparently cannot be physically justified as requiring an arccos in the integrand of Eq. 14 for 5 sensors and none in the integrand of Eq. 15 for 6 sensors. Further discrepancies between the mix-up between use of $\sqrt{T_D}$ and T_D, similar to occurrances already commented upon above, are also evident in Section IV in Eqs. 35 and 39 of [49].

Example 4: A significantly improved computational reformulation of the GLR for event detection is provided in [61]. However, while acknowledging (in the last sentence in the paragraph following Eq. 2.11a of [61]) the need to specify the decision threshold (corresponding to the role of T_D in Eq. 13, above), there is no indication in [61] of how it is to be set explicitly nor is a reference offered where such setting is provided as having already been worked out.

Example 5: Instead of offering rigorously substantiated evaluations of the fundamental statistical parameters P_{FA} and P_D underlying any detection test, some researchers [18], [37] have diverted attention away to other quantities such as (p. 25 of [37]):

- Probability of cross-detection (sic),
- Probability of wrong time-of-failure indicated,
- Probability of time-to-detection (sic),

which they assert are even more important. (Notable is that while definiing the parameters that these quantities depend upon, the decision threshold was left off even though it appears explicitly in the definition).

While a measure of "delay time for failure detection" first appears explicitly on p. 25 of [37] as a definition of "probability of time to detect" (P_{TD}), unfortunately no procedure is provided in [38] for evaluating or approximating this probability. Similarly, a definition of "probability of wrong time" (P_{WT}) is also provided on the same page of [37], without benefit of any evaluation procedure. On p. 608 of [18] it is indicated that for the Simplified Generalized Likelihood Ratio (SGLR), the special case of GLR failure

* Adding to the confusion is the fact that the distinction between occurrences of upper and lower case p's in P_{FA} and P_F, respectively, is blurred in [49] due to the type face used throughout.

detection (where the magnitude and type of failure is perhaps unrealistically known a priori as discussed in Eq. 47 of [18]), then the expected time delay in detection can be calculated. However, explicit mention of where that calculation has been performed is wanting.

It is perhaps worth mentioning, that a measure of the delay time for failure detection is inherently available from a rigorous evaluation of P_D. As can be seen from the expression for the CR2 instantaneous probability of correct detection (Eq. 49 of [23]), that depends on the signal-to-noise response of the system to a specific failure and can also be seen from Eq. 9 of [23], as the signal-to-noise ratio* (justified as Eq. 35 of [23]):

$$SNR(k) \triangleq \sqrt{\underline{d}^T(k)\ P(k)^{-1}\ \underline{d}(k)} \qquad (19)$$

becomes larger (with increasing k corresponding to the elapsing of time) then the probability-of-detection, which is a monotonely increasing function of SNR, increases accordingly. In an application [23] of detecting uncompensated ramp gyro-drift-rate as a failure, the ramp is initally small, but grows with time until it is sufficiently large to be well-above the background noise as indicated in an evaluation of the corresponding signal-to-noise ratio. A plot of the signal-to-noise ratio (SNR) versus time provides sufficient information to determine when P_D as a function of SNR will reach a prescribed level. Scaling can also be used to determine how the detecting time will be altered by those same failure types of greater or lesser magnitude.

On p. 676 of [39], it is stated that the formulas of Rice for level-crossing problems (as provided in Eq. 10.3.1 on p. 194 of [62], [57]) can be used as an analytic technique to compute estimates of the false alarm rate. Use of the above mentioned formulas of Rice (pre-1950) for standard level-crossing problems involving Markov processes usually indicate that an infinite number of level-crossings are expected to occur in any time interval (no matter how short) [55]. (See Section A, p. 304, of [63] for additional elaboration.) A lead toward an appropriate modification to allow a proper handling of this problem from a practical point of view is offered in [56]. An exception is also taken here to the comment (on p. 677 of [39]) that "when digital logic is used in detection schemes, analytic methods [for level-crossings analysis] are rarely available in their routine evaluation". While it is true that such analytic methods are rarely available for most failure detection schemes that have been proposed, a particular digitally implemented failure detection technique has been developed for the Navy [27], [28] that does in fact possess an analytically tractable underlying level-crossing analysis [24].

The so-called model following approach, pursued by Beard and Jones in [48] and [42], respectively, requires that the failure detector have the same mathematical structure as a Kalman filter (i.e., incorporating a system model). However, in [43] the filter gains are chosen not to minimize the mean square error of estimation, as done in an optimal Kalman filter; but are chosen instead to emphasize or enhance the estimates of the failure mode states and to not necessarily satisfy any other objectives such as acceptably tracking the other important system states that necessitated the use of a Kalman filter in the first place. The approach of these two authors also makes use of a novel decomposition* of the state space into the controllable (observable) and uncontrollable (unobservable) subspaces. This decomposition is especially amenable to purely deterministic systems subject to failures, but some questions relating to extent of applicability are raised when these same concepts are extended to apply to the failure detection of systems having plant and measurement noises (as are frequently encountered in most navigation applications). The random contribution of the effects of noises can defy confinement of the failure response to the controllable (observable) subspace as is otherwise exploited to an advantage in the case of failures in a purely deterministic system.

Other researchers first indicated that the standard reliability framework and techniques were not general enough [34]-[36] to accomodate rigorous analyses of the effects of failure detection on the systems management of the overall problem of control in dynamic stochastic linear systems subject to failures of the type characterized in [18]. However, more recent 1980 results [38] use exactly the same discrete-time Markov reliability techniques [40] as were used by the author [27], [22] in 1976 for the failure detection portion of the same type of problem.

The previous critisms about having trigger decisions cross-correlated with the later more sophisticated maximum likelihood estimates subsequently utilized in failure detection applies to [47] also. Additionally, please notice that the only failures illustrated by simulation in [47] are those that would correspond to a constant bias occuring directly in the measurements, modeled as (continuous time analogue of Eq. 43 of [18]):

$$\underline{\text{Type 1}}: \qquad y = Hx+v+\nu\delta(t-\sigma) \qquad (20)$$

while practical avionics/marine navigation systems are typically also vulnerable to gyro and accelerometer bias-shifts corresponding to a dynamic system model of the form (being a continuous time analogue of Eq. 40 of [18]) as

$$\underline{\text{Type 2}}: \qquad \dot{x} = Fx+Gu + w+\nu\delta(t-\sigma) \qquad (21)$$

which represents more of a challenge in trying to extricate reasonable estimates of the bias-shift. Caution is being extended here to place in proper context the illustrative examples offered in [47] as being representative of only relatively benign Type

* In Eq. 19, $\underline{d}$ is the vector signal to be detected and P is the covariance of the effective noise background.

* The analytic methodology provided in [42], [48] for implementing the decomposition is valid only for time-invariant linear system models.

1 systems (as designated in Eqs. 3, 4, 6 of [47]) rather than the more likely Type 2 (designated in Eq. 40 of [18]) or a mix of Type 1 and 2. The above notation and designation are explained in [18].

The failure detection approach developed in [45]-[47] is based on having to solve a Two Point Boundary Value Problem (TPBVP), as formulated by Friedland in 1966 [51]. Since most practical applications for navigation systems seek real time indications of failures when they occur, the need to solve a TPBVP in real time appears to be quite a computational barrier. However, the approach of [45]-[47] proceeds by attempting an approximate solution of this TPBVP via use of a Kalman filter. Only scalar and two dimensional examples are used to illustrate the performance of the approach of [45]-[47], rather than use of the higher dimensional examples routinely encountered in real navigation applications.

Certain TPBVP's can be converted to initial value problems (more ameanable to real time solution) via the technique of Invariant Imbedding. If the original problem is purely linear, then Invariant Imbedding provides a solution that is exact (pp. xi-xiii of [50]) rather than merely approximate, as occurs in the nonlinear case. TPBVP's involving Lyapunov and Riccati Equations (such as in [52], [53], and [54]) can be converted, if necessary, to the associated linear equations (of twice the dimension) [80] where Invariant Imbedding could offer exact solutions as initial value problems.

While it is in fact fairly unusual for a Kalman filter to be used to solve a TPBVP as done in [45]-[47], it is perhaps interesting to note that an approach for deriving the Kalman filter is via a TPBVP [81] [82, pp. 291-295] as has also been generalized for deriving some decentralized filters (as illustrated in Section 2.3 of [83] with a detailed view being offered in Appendix A of [86]). An important example of a TPBVP (involving both a Riccati equation to be solved forward in time and a Lyapunov equation to be solved backwards in time) is provided in [53] for a navigation system application having realistic dimensions.

Occassionally, the so-called sampling approximation [64] is used as an approach to obtain a close approximation to the probability of false alarm over a time interval consisting of N consecutive check instants as (e.g., [65, p. D-9]):

$$P_{FA}[N_1,N_2] \approx 1-(1-P_{FA})^N \qquad (22)$$

where $P_{FA} \stackrel{\Delta}{=}$ instantaneous probability of false alarm. However, it is demonstrated in [64] using an analytically tractable level-crossing formulation for a simplified representation of a square-law detection device [66], that the approximation of Eq. 22 provides an indicated false alarm rate that is better (i.e., smaller) than is actually present ([64, p. 24]). A recent simulation approach for evaluating P_{FA} is reported in [39].

5. New Results

The following discussion is offered to aid those interested in implementing the Two Confidence Region (CR2) approach of [21]-[24]. The following result is a response to the charge of an anonymous reviewer of [24] that the filter convariances would grow too quickly with time in most applications to be capable of utilization in a failure detection algorithm in the manner claimed for CR2 and that use of a Kalman filter or a CR2 failure detection algorithm would be useless for detecting failures in a system whose linear truth model is unstable. The reply follows below.

The covariance values that are used in the CR2 hypothesis test of [21]-[24] are provided in Eq. 11 of [24]. While Eq. 11 may indeed result in covariance values that may grow with time (if the failure modes being monitored are modeled as random walks or integrated Markovs or integrated random walks), the covariances only grow to appropriately reflect how the system was modeled. Even growing covariances cause no numerical problems in the calculation of the CR2 test statistic, as can be conveniently seen for the scalar case in Eq. 14 of [24], where the covariances only appear in the denominator. That growing covariances cause no particular numerical problem in the calculation of the decision threshold can be conveniently seen for the scalar case through a rearrangement of Eq. 16 of [24] as

$$K_1(k)=b^2 \cdot \frac{1-\sqrt{P_1(k)/P_2(k)}}{1+\sqrt{P_1(k)/P_2(k)}} \qquad (23)$$

(where b, P_1, and P_2 are consistently defined in [23], [24]) which remains bounded even if $P_2(k)$ goes unbounded (and $P_1(k)$ always being $<P_2(k)$ since the covariance without using measurements is greater than or equal to the covariance making use of measurements).

The following two cases are more routine and more likely to be encountered in an actual failure detection application:

- Covariances that do not grow, but with initial uncertainty that is not exceptionally large,
- Covariances that quickly die down even if the initial covariances comprising P_0 are large.

The benign latter case is not a completely hypothetical supposition since it is frequently encountered in navigation error models as Markov gyro drift-rates.

While Kalman filtering is applicable to an unstable system to track well as long as the system is observable and controllable (or detectable and stabilizable as weaker more generally met necessary and sufficient conditions [85, pp. 82-83]), failure detection in a unstable system is not a usual goal. The big problem is the unstable system that is going unbounded, not the soft failure of minor consequence. There are no problems of tracking or failure detection in a marginally stable (oscillatory) system.

The fundamental consideration in determining whether only large failures will result in detections is the underlying Signal-to-Noise-Ratio (SNR). For the failure detection application, the failure signal consists of the response of the linear system to the failure (scaled by the magnitude of failure as discussed following Eq. 9 and in Section III of [23]) while the corrupting background noise consists of the effects of measurements and process noises and to a very small degree on the initial uncertainty. Only if the SNR is small will detection be difficult (due only to the application environment). If the magnitude of measurement and process noises are large, then a failure of comparable magnitude will be required for detection.

An expression is provided in Eq. 35 of [23] for the multidimensional signal-to-noise ratio (SNR) and Appendix A of [23] offers a proof (i.e., Theorem 1) that the two dimensional SNR is greater than or equal to the maximum of the two associated scalar SNRs that don't account for any existing cross-correlation between them. While the above mentioned SNR relationship is demonstrably true, a proper appreciation for the real benefit of using a two dimensional CR2 test over two corresponding scalar CR2 tests can only be instilled after comparing explicit quantifications. To illustrate just how much larger the two dimensional SNR (associated with use of the two dimensional CR2 test) is over what is offered by the underlying SNRs of the corresponding two scalar tests, consider the following example:

$$\underline{d} = \begin{bmatrix} 9 \\ 16 \end{bmatrix} \; ; \; P = \begin{bmatrix} 9 & -9.6 \\ -9.6 & 16 \end{bmatrix} \qquad (24)$$

Using this example, the two scalar SNRs are 3 and 4 units, respectively, while the two dimensional SNR (repeated herein as Eq. 19) is 14.86 units (being greater even than the sums of the two scalar SNRs). For navigation applications, use of a two dimensional CR2 is most appropriate for monitoring for gyro drift-rate or accelerometer degredations beyond what is acceptable for the mission (incurred as soft failures) in two-degree-of-freedom gyros having roters with two input axes.

APPENDIX A: DIFFICULTIES OF SUGGESTED PARTIAL GLR IMPLEMENTATION WITHOUT USE OF MLE FAILURE TIME

While the rigorous implementation of GLR over a sliding data window of fixed size (N+1) corresponds to candidate failure times being restricted to the time interval of $k-N \le \theta \le k$, a further narrowing of the candidate time interval (as explained in last paragraph of [41]) to

$$k-M \le \theta \le k-N \qquad (A-1)$$

(0<N<M) is made in [17, pp. 16, 17], [41, p. 852, last paragraph], [16, between Eqs. 15 and 16], [18, p. 607, bottom of page]. The stated purpose of the additional restriction, using N on the right side of Eq. A-1 is imposed to avoid problems with "failure observability and false alarming". It is further mentioned in [17, p. 31] that an implementation alternative is to utilize a partial GLR algorithm* in which the optimization to arrive at an estimate of the time of failure is replaced by the explicit assumption that

$$\theta_p(k) = k-M+1 \qquad (A-2)$$

without requiring any maximizations.

Use of Eq. A-2 or any estimate of θ other than the maximum likelihood estimate yields only a pseudo-GLR test. While the version of a pseudo-GLR test obtained by utilizing the suggested Eq. A-2 is desirable from the point of view of eliminating all the optimization operations and confining the calculations to just one Kalman filter, the fundamental question of whether the resulting test statistic is adequate or close enough to the objective of an exact GLR test statistic sufficient for practical engineering purposes is not always answered in the affirmative (and perhaps seldom answered in the affirmative) as shown next.

Since the proposed partial GLR algorithm uses an estimate of $\theta(k)$ other than the maximum likelihood failure time estimate, problems can potentially arise. A numerical example is now provided to illustrate one problem. Consider the following maximization problem:

$$\max_{\theta,v} \{f(\theta,v)\} \qquad (A-3)$$

where

$$f(\theta,v) = (1+\theta^2) - (v-2\theta)^2 \qquad (A-4a)$$

$$= 1 - 3\theta^2 - v^2 + 4\theta v \qquad (A-4b)$$

and

$$k-M \le \theta \le k-N \qquad (A-5)$$

while v is arbitrary.

The specific maximization indicated in Eq. 24 of [41] (following [14]) can be abstracted as:

$$\max\{a(\theta) - \underline{b}^T(\theta,\underline{v})\underline{b}(\theta,\underline{v})\} \qquad (A-6)$$

where

- o the scalar $a(\theta)>0$ for all θ under consideration (A-7)
- o the scalar $\underline{b}^T(\theta,\underline{v})\underline{b}(\theta,\underline{v})\ge 0$ for all $\theta,\underline{v}$ (A-8)

Maximization of the abstracted representation of the problem above is achieved for $\underline{v}=\underline{v}$ chosen such that

$$\underline{b}(\theta,\underline{v}) \equiv \underline{0} \qquad (A-9)$$

Applying this general result to the specific cost function of Eq. A-4a yields

$$v-2\theta = 0 \text{ or, equivalently, } v = 2\theta \qquad (A-10)$$

and

$$\max_{\theta,\, v} \{f(\theta,v)\} = \max_{\substack{k-M\le\theta\le k-N \\ v=2\theta}} \{1+\theta^2\} \qquad (A-11)$$

The conclusion as to which value of θ at time k should be the true maximum liklihood failure time estimate by maximizing the cost function of Eq. A-4 is

* Also referred to by the term "fixed-lag GLR".

$$\hat{\theta}(k) = k-N \quad \text{(A-12)}$$

by being the largest allowable value of the constraint of Eq. A-5, yields a GLR of

$$\ell(k) = (1/2) \max_{\theta,v} \{f(\theta,v)\} = (1/2)\{1+(k-N)^2\} \quad \text{(A-13)}$$

Using the choice advocated in Eq. A-2 as the failure time estimate to be used in an implementation of the partial GLR algorithm, yields a pseudo-GLR of

$$\ell_p(k)=(1/2)\max_v\{f(\theta,v)\}\Big|_{\theta=k-M+1} =(1/2)\{1+(k-M+1)^2\} \quad \text{(A-14)}$$

To illustrate* more explicitly how different the resulting GLR test statistics of Eq. A-13 at time k is from the resulting pseudo-GLR test statistic of Eq. A-14 at time k, use the values of

$$M = 12, N = 6 \quad \text{(A-15)}$$

as advocated for the single simple two dimensional numerical example repeated in [16], [17], [41]. For the quantification of Eq. A-11 at say time=13, the GLR test statistic of Eq. A-13 is

$$\ell(13) = (1/2)\{1 + (13-6)^2\} = 25 \quad \text{(A-16)}$$

while the pseudo-GLR test statistic of Eq. A-14 is

$$\ell_p(13)=(1/2)\{1 + (13-12+1)^2\} = 2.5 \quad \text{(A-17)}$$

a difference for this example and this check time† of a factor of ten. A recent approach for analyzing the performance of a moving window detector is offered in [75].

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* A deterministic example was selected here to illustrate what is actually occurring during the GLR maximization. In an actual application the details of the evaluation would be less transparent than the example offered here since the maximization is performed over a window of filter residuals which differ markedly between and within Monte Carlo runs, depending on where the sliding window is currently located.

† With this same example and parameters (for time k=14), $\ell_p(14)=5$ vice $\ell(14)=32.5$.

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