Specialized Ricatti Equation Manipulations/Equivalences Arising in Kalman Filters and LQ Optimal Regulators

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<u>We seek to</u>:

1. DEMONSTRATE that the so designated *Joseph's form* of the discrete-time covariance update equation (that arises within a proper Kalman filter implementation):

$$P(k|k) = [I - K_k H_k]\, P(k|k-1)\, [I - K_k H_k]^T + K_k R_k K_k^T \tag{1}$$

   is mathematically equivalent to the historically older (yet still prevalent but less numerically well-behaved) covariance update equation:

$$P(k|k) = [I - K_k H_k]\, P(k|k-1) \ , \tag{2}$$

   where the expresion for the discrete-time Kalman gain K_k appearing throughout the above two expressions is:

$$K_k = P(k|k-1) H_k^T \left(H_k P(k|k-1) H_k^T + R_k \right)^{-1} \ . \tag{3}$$

2. DEMONSTRATE that the original continuous-time $(n \times n)$ matrix differential equation (known as a Riccati equation) of the form:

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) - P(t)H(t)R^{-1}(t)H(t)P(t) \ , \tag{4}$$

   (nonlinear because of the presence of the last term, where $P(t)$ occurs twice) with initial condition:

$$P(t_o) = P_o \tag{5}$$

   is actually satisfied by $(n \times n)$ components within the solution of an associated $(2n \times n)$ linear matrix differential equation, when the candidate Ricatti equation solution is defined as follows:

$$P(t) = \Gamma_1(t)\Gamma_2^{-1}(t) \ , \tag{6}$$

   as constructed in terms of the constituent components $\Gamma_1(t)$ and $\Gamma_2(t)$, being partitions of the following associated $(2n \times n)$ linear matrix differential equation of the form:

$$\begin{bmatrix} \dot{\Gamma}_1 \\ \cdots \\ \dot{\Gamma}_2 \end{bmatrix} = \begin{bmatrix} F(t) & \vdots & G(t)Q(t)G^T(t) \\ \cdots & \cdots & \cdots \\ H(t)R^{-1}(t)H^T(t) & \vdots & -F^T(t) \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \cdots \\ \Gamma_2 \end{bmatrix} , \tag{7}$$

   with initial condition:

$$\begin{bmatrix} \Gamma_1(t_o) \\ \cdots \\ \Gamma_2(t_o) \end{bmatrix} = \begin{bmatrix} P_o \\ \cdots \\ I_{n\times n} \end{bmatrix} . \tag{8}$$

Since Eq. 7 is linear and of the form:

$$\dot{x}(t) = F_2(t)\, x(t) \ , \tag{9}$$

it, in turn, is known to have a solution of the form:

$$x(t) = \Phi_2(t, t_o) x_o \ , \tag{10}$$

where $\Phi_2(t, \tau)$ satisfies:

$$\frac{\partial \Phi_2(t,\tau)}{\partial t} = F_2(t)\, \Phi_2(t,\tau) \text{ with } \Phi_2(s,s) = I_{n \times n} \text{ for all } s \tag{11}$$

where here

$$F_2(t) = \begin{bmatrix} F(t) & \vdots & G(t)Q(t)G^T(t) \\ \cdots & \cdots & \cdots \\ H(t)R^{-1}(t)H^T(t) & \vdots & -F^T(t) \end{bmatrix} \tag{12}$$

where in the above the state $x(t)$ is:

$$x(t) = \begin{bmatrix} \Gamma_1(t) \\ \cdots \\ \Gamma_2(t) \end{bmatrix} , \tag{13}$$

and the initial condition is:

$$x(t_o) = \begin{bmatrix} \Gamma_1(t_o) \\ \cdots \\ \Gamma_2(t_o) \end{bmatrix} = \begin{bmatrix} P_o \\ \cdots \\ I_{n \times n} \end{bmatrix} . \tag{14}$$

Hint: Recall as lemma 1 here the technique for and result of obtaining the derivative of the inverse of a matrix as, say, $\frac{d}{dt}A^{-1}(t)$ proceeds after first forming the identity:

$$A(t)A^{-1}(t) = I_{n \times n} \ , \tag{15}$$

then differentiating the above product throughout using the chain-rule to yield

$$\dot{A}(t)A^{-1}(t) + A(t)\frac{d}{dt}A^{-1}(t) = 0 \ , \tag{16}$$

which can then be rearranged to yield the final result:

$$\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t) \ . \tag{17}$$

ANSWERS: 1. Substituting the result of Eq. 3 for the Kalman gain K_k in Eq. 1, then expanding out terms on the right hand side yields:

$$
\begin{aligned}
P(k|k) &= \left[I - K_k H_k\right] P(k|k-1) \left[I - K_k H_k\right]^T + K_k R_k K_k^T \\
&= \left[I - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k\right] P(k|k-1) \left[I - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k\right]^T \\
&\quad + P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} R_k \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&= \left[P(k|k-1) - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1)\right] \left[I - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k\right]^T H_k P(k|k-1) \\
&\quad + P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} R_k \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&= P(k|k-1) - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&\quad - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&\quad + P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&\quad + P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} R_k \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&= P(k|k-1) - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&\quad - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&\quad + P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} \underbrace{\left[H_k P(k|k-1)H_k^T + R_k\right] \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1}}_{I} H_k P(k|k-1) \\
&= P(k|k-1) - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&\quad \underbrace{- P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) + P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1)}_{0} \\
&= P(k|k-1) - P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1} H_k P(k|k-1) \\
&= \left[I - \underbrace{P(k|k-1)H_k^T \left(H_k P(k|k-1)H_k^T + R_k\right)^{-1}}_{K_k} H_k\right] P(k|k-1) \\
&= \left[I - K_k H_k\right] P(k|k-1)
\end{aligned}
\tag{18}
$$

2.Expanding out the individual partitions of Eq. 7 yields:

$$\dot{\Gamma}_1(t) = F(t)\,\Gamma_1(t) + G(t)Q(t)G^T(t)\,\Gamma_2(t) \tag{19}$$

and

$$\dot{\Gamma}_2(t) = H^T(t)R^{-1}(t)H(t)\,\Gamma_1(t) - F^T(t)\,\Gamma_2(t)\ . \tag{20}$$

Now for a candidate solution $P(t)$ constructed to be of the form:

$$P(t) = \Gamma_1(t)\Gamma_2^{-1}(t)\ , \tag{21}$$

it can now be differentiated with respect to time via the chain rule (and lemma 1 for handling the derivative of an inverse) to yield:

$$
\begin{aligned}
\dot{P}(t) &= \dot{\Gamma}_1(t)\Gamma_2^{-1}(t) + \Gamma_1(t)\tfrac{d}{dt}\Gamma_2^{-1}(t) \\
&= \dot{\Gamma}_1(t)\Gamma_2^{-1}(t) - \Gamma_1(t)\Gamma_2^{-1}(t)\dot{\Gamma}_2(t)\Gamma_2^{-1}(t)\ ,
\end{aligned}
\tag{22}
$$

then substituting Eqs. 19 and 20 for $\dot{\Gamma}_1(t)$ and $\dot{\Gamma}_2(t)$, respectively, in the above yields

$$
\begin{aligned}
\dot{P}(t) &= \left[F(t)\Gamma_1(t) + G(t)Q(t)G^T(t)\Gamma_2(t)\right]\Gamma_2^{-1}(t) \\
&\quad -\Gamma_1(t)\Gamma_2^{-1}(t)\dot{\Gamma}_2(t)\Gamma_2^{-1}(t) \\
&= F(t)\Gamma_1(t)\Gamma_2^{-1}(t) + G(t)Q(t)G^T(t)\underbrace{\Gamma_2(t)\Gamma_2^{-1}(t)}_{I} \\
&\quad -\Gamma_1(t)\Gamma_2^{-1}(t)\left[H^T(t)R^{-1}(t)H(t)\Gamma_1(t) - F^T(t)\Gamma_2(t)\right]\Gamma_2^{-1}(t) \\
&= F(t)\underbrace{\Gamma_1(t)\Gamma_2^{-1}(t)}_{P(t)} + G(t)Q(t)G^T(t) \\
&\quad -\underbrace{\Gamma_1(t)\Gamma_2^{-1}(t)}_{P(t)}\left[H^T(t)R^{-1}(t)H(t)\underbrace{\Gamma_1(t)\Gamma_2^{-1}(t)}_{P(t)} - F^T(t)\underbrace{\Gamma_2(t)\Gamma_2^{-1}(t)}_{I}\right] \\
&= F(t)P(t) + G(t)Q(t)G^T(t) \\
&\quad -P(t)\left[H^T(t)R^{-1}(t)H(t)P(t) - F^T(t)\right] \\
&= F(t)P(t) + G(t)Q(t)G^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) - P(t)F^T(t) \\
&= F(t)P(t) - P(t)F^T(t) + G(t)Q(t)G^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t)\ .
\end{aligned}
\tag{23}
$$

The above final expression is recognized to be identically Eq. 4 thus demonstrating that the construction of Eq. 6 does suffice to create a solution as long as its constituent components satisfy Eq. 7 and initial condition of Eq. 8 so that

$$
P(t_o) = \Gamma_1(t_o)\Gamma_2^{-1}(t_o) = P_o I^{-1} = P_o\ . \tag{24}
$$

In passing, it is mentioned that for time-varying parameters in Eq. 12, Eq. 9 (and Eq. 7) can, in general, only be solved by numerical integration to solve the associated Eqs. 10 and 11. However, if the parameters in Eq. 12 (and Eq. 7) are time-invariant constant matrices, then the transition matrix of Eqs. 10 and 11 may be obtained directly from the matrix exponential as a considerable simplification.