Critical Survey of Approaches to EWR Target Tracking: *endorsing use of an EKF's structure*

Thomas H. Kerr III, Ph.D., *Life Senior, IEEE, Assoc. Fellow, AIAA, Member ION & Life NDIA*

***Abstract*—New methodologies for target tracking and for evaluating its efficacy have recently emerged, all potentially being a "magic bullet". The questionable accuracy benefits, missing rigor (in some cases), and definitely large CPU-time computer loading drawbacks of the new estimation approaches are discussed as compared to conventional Extended Kalman Filters (EKF's) and the novel Batch Maximum Likelihood Least Squares algorithm, as the well-known previous candidates for use in land-based Early Warning Radar (EWR) target tracking. A reminder is that the existing 40 year old Cramer-Rao lower bound evaluation methodology is already rigorous and adequate for evaluating target tracking efficacy in $P_d < 1$ situations when confined to exo-atmospheric target tracking, as arises in EWR for NMD/GMD. We also discuss the more challenging and numerically sensitive angle-only filter methodologies, needed to handle target tracking when enemy escort jamming denies radar range measurements and impedes target tracking unless two or more radars cooperatively triangulate synchronously to thereby enable joint tracking of enemy jammers within the tight target threat complex. (Sometimes merely one sensor suffices for AOT, as discussed.) Finally, supporting technologies (some old, some new) are discussed for enhancing the performance of these EKF's with only a modest increase in computational burden. Motivation for these pursuits is the quest to gain more veracity in on-line filter covariance calculation to mitigate any tendency to be overly optimistic or pessimistic (which could otherwise adversely affect multi-target track associations which typically utilize such EKF covariances within its initial gating stage) occurring further downstream in the standard sequence of processing steps.**

***Index Terms*—EWR Target Tracking, Filter Models, Approximate Nonlinear Filter, EKF, IEKF, Batch Least Squares, Unscented Filter, IMM & MMM Filters, Particle Filter, Covariance Fidelity, Covariance Intersection, Probability 1, Multi-target Tracking Alrernatives.**

I. Introduction and Overview

STRATEGIC target tracking typically employs a (public domain) system dynamics target model that is nonlinear (with inverse square gravity along with the earth's second zonal harmonic [including J_2] appearing in its associated ODE description, similar to system and measurement models described in detail in [26], [107], [218], [284], [298] with parameter values as usually provided, but updated in [114]). The EWR target model is also nonlinear in the algebraic sensor observation equation, where range-Doppler ambiguity is compensated for within the plane of the antenna face [11], [295], using a transformation algorithm [133] (that should be updated [134], [356] to reduce the maximum error that can be incurred). Among the first cogent modern treatments of Reentry Vehicle (RV) target modeling for radar target tracking were[1] [290]-[292], [59]-[61] (and, afterwards, quickly followed by several others [62], [293]-[295]).

Fig. 1. Gating after each incremental EKF output enables MTT associations

EKF's are typically used for tracking the target state in a ballistic trajectory; but over the last 30+ years, some form of nonlinear batch least squares (BLS) algorithm has also been used for this purpose ([135], [14], [15]) on fast parallel processing machines by dynamically allocating and deallocating memory, as needed. However, EKF's are still relied upon for the measurement intensive routine data associations arising in first forming the initial hypotheses within multi-target tracking (MTT) before switching to BLS for later track enhancement of *mature targets* as part of the overall MTT process prior to Fire Control dispatching a (non-nuclear) kinetic-kill vehicle [2] to eliminate the hostile RV threat entirely during its exo-atmospheric midcourse transit [3]. Typical behavior of target-associated Confidence Regions (CR) during tracking are conceptually depicted in Fig. 1, as they go from initially being "a pancake" (at horizon break, upon first entering the radar fence (consisting of proprietary pre-programmed patterns of radar up-down pulse-pair chirps within multiple pencil beams sweeping and scanning for initial detections and subsequent confirmation) to later being "a football" as even more confirming radar sensor return "hits" are accumulated for this designated target. Target ID's are assigned by the radar controller/manager that subsequently "schedules" radar resources to systematically return to continue "viewing" these initially identified moving targets to enable further improved following of these objects of interest that are suspected to be

Manuscript received February 14, 2016. This research funded by TeK Associates' IR&D Contract No. 15-102.

T. H. Kerr III is CEO/Principal Investigator at **TeK Associates**, 21 Cummings Park, Suite 230, Woburn, Massachusetts 01801; E-mail: thomas_h_kerr@msn.com.

[1] Paralleling similar precedents, circa 1967, by earlier initial trailblazers: James R. Huddle (Litton), Stanley Schmidt & George Schmidt (Draper Lab.), and Arthur Gelb (TASC) in first recognizing the Kalman Filter's utility for Inertial Navigation Systems (INS).

[2] Possibly with a terrestrial-based *command guidance* or more likely with some form of on-board self-contained *proportional navigation* guidance.

[3] Successful *Terminal Phase* intercept during atmospheric reentry requires more than merely 6 filter states to account for the effect of lift and drag on the targets and sometimes considerations of micro-motions, all of which are beyond the scope of this paper. This paper considers only aspects historically related to UEWR, which has a goal of mid-course exo-atmospheric intercept.

potential threats (after having already passed a comparison test weeding them out from known background satellites and space debris listed in an up-to-date Space Object Catalog of 13,000+ entries [4]).

Fig. 2. Assumed Gaussian CR goes from being a pancake (at horizon break) to a football afterwards

Fig. 3. Functional overview of the MTT Data Association Aspect of Track Maintenance [53]

Among the new approaches to target tracking and for evaluating its efficacy are unscented filters [1]-[4]; Covariance Intersection (CI) filters [5], [6] (discussed in depth in Sec. II); Particle Filters (PF) [388], [8], [177]; and Farina et al's new formulation of Cramer-Rao Lower Bound evaluation for non-unity probability of detection (for $P_d < 1$) [9], all offering the lure of potentially being a "magic bullet". One of the clearest discussions of both historical and recent estimation algorithms (including EKF, UKF, MMM, INN, and Particle Filters [PF's]), their underlying assumptions/mechanization equations and that also provides extremely useful insight into important aspects and distinctions in their implementations is offered in [177, Chapts. 1-3]. We offer additional new practical observations herein in further considering how these algorithms relate to the EWR application. The questionable accuracy benefits, the missing rigor (in some cases), and definitely large computer loading drawbacks of the new estimation approaches all need to be considered (as discussed further in Secs. II to V) before deciding to push forward to implementation for new EWR applications. We also review the older $\alpha-\beta$ filters as they have again been recently reconsidered for EWR tracking [151].

Two of the new approaches were evaluated and cross-compared as being among the four reported in [10], but all with accuracy results obtained under a somewhat artificial scenario of use (viz., invoking only 4 tracking filter states and assuming only planar motion despite not being under the influence of central forces exclusively but projectile treated as if it were and the observing radar is only at the launch point and within the same plane as the target trajectory) thus leaving accuracy quantifications in [10] somewhat questionable as they relate to actual missile defense since this overly benign scenario apparently lacks realism, as explained in Sec. III. These approaches are discussed as compared to the standard load of two conventional Extended Kalman Filters such as Range Velocity Cartesian Coordinates (RVCC), and R-U-V [11]-[13], and to novel Batch Maximum Likelihood Least Squares (BLS) algorithms [14], [15], [189, App. 2], as three well-known historical candidates for use in radar target tracking within land-based EWR. (Recall that Ref. [113] has already demonstrated that partitioned filters can be unsatisfactory in some situations and are therefore undesirable even for crossing targets.)

In Sec. V, the existing 40+ year old Cramer-Rao lower bound evaluation methodology [16]-[24] is shown to be rigorous and flexible enough to adequately evaluate target tracking efficacy in $P_d < 1$ situations for specified detection threshold settings when confined to exo-atmospheric midcourse interception of a target and its prior tracking, as arises in EWR, where truth model process noise is theoretically zero. The *filter model* can have nonzero process noise tuning [5] and still abide by this $Q = 0$ constraint dictated by the *truth model's structure*. However, Farina et al's formulation [9] (based on [25]) is useful for evaluating Indo-atmospheric tracking, which is a more challenging situation, where system truth model process noise is definitely nonzero (reflecting atmospheric buffeting associated with reentry drag, or maneuvering, or with a projectile undergoing late stage thrusting).

As discussed in Sec. VI, motivation for these pursuits is the quest to gain more veracity in on-line filter covariance calculation to mitigate their tendency to be overly optimistic (or, much more rarely, pessimistic) since any hand-over or multi-target (MTT) associations rely on their veracity (Figs. 2, 6) and errors in the values of these covariances (being the only ones actually available in one-shot real world trials) are sensitive. In Sec. VII, we recommend pursuit of good tracking accuracy with probability one success (as pioneered by the late Frank Kozin in the 1960's and 1970's) over current Monte-Carlo-based mean square averaging techniques, which captures only aggregate behavior. In Sec. VIII, we mention the relevance of solutions for the Lambert Problem to EWR and summarize its status.

In Sec. IX, we discuss the more challenging and more initial-condition-sensitive angle-only tracking (AOT) filter methodologies [28]-[33], needed to handle target tracking when enemy escort jamming denies the radar its target range measurements and thus impedes tracking.

After reviewing the limitations of earlier trackers, supporting technologies (some old, some new) are discussed in Secs. IX and X for enhancing the performance of EKF's, which incur only a modest increase in the computational burden (all applicable as evolutionary low risk enhancements to the EKF's already present in EWR) as variations that more easily satisfy hard real-time constraints.

We offer our views here based on past experience in the

[4]In 2016, NASA and FAA seek to takeover compilation of Space Object Catalog from U.S. DoD, as now headed up by Dr. Moriba Jah, Director, Space Object Behavioral Sciences. Amount of "space junk" is now estimated at 22,000 objects to be tracked and cataloged.

[5]A quantitative rationale is offered in [285] for the appropriate magnitude of compensating fictitious process noise utilized in *tuning* an EKF filter model that seeks to track exo-atmospheric ballistic targets.

TABLE I
RELATIVE CPU BURDEN OF 4 FILTERS [59, P. 44]

EKF	$\alpha-\beta-\gamma$ Filter	EKF (adaptive)	2^{nd} order Filter	Iterated (IEKF)
1	0.1	1.3	1.5	2.0

theory [6] and applications of estimation, tracking, and Kalman filtering [43]-[58], [68], [75], [76] and by using the strong tradition of quantifying the relative computer burdens of various filter alternatives beforehand [59]-[63] for sequential implementations [7] using an awareness of the characteristics of what constitute acceptable exact and approximate solutions to nonlinear estimation problems [64] (cf., [65]), [73], [283], [286], [287]). We reference older surveys here [260], [259] [8] as respected precedents because of their correctness (see Table 1). A newer published survey [69] by Lincoln Lab in 1984 (specifically for RV target tracking) found no significant changes from the same approaches and 1970 principles of [59] other than looking at evolutionary changes and that the 1970 ranking of candidate tracking filter approaches had not changed much (or expanded, despite hundreds of subsequent researchers tackling the problem). Evidently, others also fear that many of the very recent new alternative approaches to the use of EKF's for target tracking are over-hyped (see [86, Sec. VI], [102], [112]). Further evidence supporting this view is offered here in Secs. III, IV, and X but we are receptive enough to also report their benefits. Constructive evolutionary EKF developments that have occurred in the last decade to improve performance are pointed out (especially in Sec. X) and we also point out other improvements, some as mundane as merely identifying new best parameter values [107], best practices for radar target tracking [114] (and sometimes as best models for INS/GPS navigation [9] [272] that also plays a useful role in locating moving antennas or in serving as a source of *true* position and velocity for targets equipped with GPS translators during validation tests), and in making the derivation of results simpler and more straight forward [181], [271]. Novelty and creativity are always encouraged but we also strongly desire that the test conditions for algorithm evaluations be realistic and actually representative of the application scenario.

Fig. 4. Our newer approach reduces the computational burden of Iterated Extended Kalman Filtering (from [35])

In 1989, we were able to make a slight improvement in the ranking of an Iterated EFK [35] [10] by simplifying its computer burden (Fig. 3). Instead of an IEKF being twice the computed load of a comparable EKF, as [59] reported (see Table 1), we offered an evolutionary modification and improvement that made our IEKF just $1.33\bar{3}$ the computer load of a comparable EKF yet yielded identical results of the same accuracy as the earlier more computationally intensive version. Also see [122]. (Carlson's 1973 version of Squareroot filtering now beats Gerald Bierman's later 1975 U-D-U^T formulation merely by the way computer processor hardware and its associated firmware algorithms are now implemented [58, App.]; prior to the mid 1990's, the maximum CPU tally was vice-versa. However, while prudent to use, squareroot filters may not even be needed [359, Sec. 9.5] for tracking RV's since a specific designated target is only in view for less than 30 minutes and the discrete measurements received may be somewhat sparse and relatively fewer than is usually the case for comparable Navigation applications, where frequent periodic measurements warrant use of computationally stable squareroot filters to compensate for the round-off error incurred with the relatively more frequent opportunities for measurement incorporation into navigation filter updates.)

[6]My perspective, which aligns with what Prof. Sanjoy J. Mitter (MIT) has publicly said, is that the theory of nonlinear filtering, in general, requires familiarity with Ito, Stratonovich, and McShane integrals as well as an understanding of measure-theoretic probability (including nested expanding sub-sigma algebras and conditional expectations with respect to them, martingales and their associated inequalities, and the *law of the iterated logarithm*). For more detail, see [125]. However, my article here is kept at a higher level.

[7]The CPU burden for sequential implementation is used merely as a baseline cross-check for eventual parallel implementation in modern day hardware consisting of either embedded processors or powerful parallel processing mainframes, where it is expected that considerable speed-up should accrue (but does not always initially occur because of processing bottlenecks that first need to be identified and then removed until the expected speed-up is achieved). Historically, many so-called fast parallel versions of famous algorithms were initially unexpectedly slower than their predecessor sequential counterparts until fixed.

[8]While Prof. Thomas Kailath (Stanford) is almost always in the right, a notable exception was [263] vs. [264].

[9]A new approach for computationally processing gravity mesurement data was recently availed [384] and we offered two counterexamples to the proposed procedure [385]. A way of squeezing more information out of a GPS receiver by utilizing more of its existing frequencies has also been offered [386] and we streamlined and simplified the calculations [387]. Both of our improvements were unsolicited.

[10]The main focus in [35] is the structure and performance of an IEKF vs. an EKF for RV tracking, even though the oversimplified model common to both is acknowledged to not include J_2 (which, when present, accounts for the earth's oblateness) nor is Doppler compensation in the face of the antenna present, as is now known to be desirable to include both of these terms in a tracker's model since they are needed for more realism in modeling Missile Defense situations [14], [15], [53] because greater tracking accuracy is reaped as a consequence of their presence in the model. These modeling simplifications were corrected later in [14].

Profs. R. E. Mortensen [355], [202], H. W. Sorenson, E. B. Stear, A. B. Stubberud, and R. C. Kolb hosted the (U.S. Air Force sponsored) Nonlinear Estimation and Its Applications Conference from 1970 until it ended in 1975. Although many innovative, well-funded researchers were working hard on the nonlinear filtering problem, this conference was canceled because significant new results usually do not accrue yearly [11] for the hard problem of nonlinear filtering. Even today, some of the best insights and early leads for tackling nonlinear filtering are found in these past proceedings (e.g., [129], [130]). Recommendation for future work in Secs. X and XI logically continue where our predecessors left off by emphasizing existing barriers that still need to be tackled and conquered rather than suggesting a search for new approaches just because they are new (while apparently ignoring past problems already identified as needing solving before further progress could be made). A brief summary and further perspective on goals accomplished here are provided in Sec. XII. For those readers desiring a refresher or more of an introduction, a brief summary of Kalman Filtering is in the Appendix, which emphasizes software architecture needed to appropriately match the structure of a particular application.

II. Why Covariance Intersection?

A counterexample is presented to a result claimed in a proof in [6], pertaining to using this new approach to Covariance Intersection (CI). Other researchers have already demonstrated certain problems that exist with earlier versions of CI, as summarized from a survey [74] of the previous CI approaches encountered in Target tracking applications. We alert readers to investigations along similar lines from the field of navigation that were apparently overlooked in [74] that convey similar but different results for ascertaining ellipsoidal overlap and for combining the estimates from two or more Kalman filters, each representing a different sensor's output that has been previously processed. In all cases, CI is practically useless despite having beautiful analytic proofs.

[11] With the notable exception of F. E. Daum's new nonlinear filtering results in '86 (IEEE AC), '86 (ACC), '86 (20^{th} *Conf. on Inform. Sciences and Systems*, Princeton), '87 (IEEE AC), '88 (Ch. 8, ed. by J. C. Spall, Marcel Dekker) '94 (SPIE, Orlando, FL), '97 (SPIE, San Diego, CA), '01 (*Proc. of Tribute to Y. Bar-Shalom*), '03 [8] but none yet applied (as of 2006) except for an application with only 4 planar states [311] that, perhaps, may be considered too overly simplistic to be practical and Daum's later important results are still infinite dimensional (viz., in general, requiring full integration of a non-Gaussian pdf constituting a Propagation Step up to the time anticipated for the next Update Step) so they are not computable in real-time (unless computed beforehand and stored off-line, a procedure found undesirable in the 1970's, by Sperry Systems Management, when measurements did not arrive exactly when expected, as had been previously planned). An exception is that all of these results simplify to a tractable real-time Kalman filter when both process and measurement noises are Gaussian and the system is merely linear, as do several other existing historical 40+ year-old nonlinear filtering approaches as a precedent, notably (1) solving the Fokker-Planck Partial Differential Equation (also known as the forward Kolmogorov equation) for the evolution in time of the conditional pdf of the system state directly or, equivalently, (2) taking its Fourier Transform to obtain a conditional Characteristic Function, from which the random processes' conditional moments can be generated via differentiating it a requisite number of times (where both these approaches benefit from the additional structural simplification of encountering certain noises from an *exponential family* possessing Gaussian conditional pdf's [180]).

The CI-based sensor fusion methodology of [6] usually degenerates to cases where $\mu = 0$ or $\mu = 1$, rather than the more useful situations where $0 < \mu < 1$. Some of the blame should be shared by IEEE reviewers of [6], who allowed it to be published even though there were no numerical parameters specified for the single diagram that appeared within, which was merely conceptual, and the single numerical example only illustrated the degenerate case of $\mu = 0$ or $\mu = 1$ (and not the useful case where $0 < \mu < 1$, which is seldom met)[12].

A. A more recent approach to CI

This technical note offers a counterexample to the use of the results of [6] in this new Covariance Intersection (CI) approach. An expression for the estimate that results from combining two prior (assumed) independent estimates consisting of $(\hat{x}_1, P_{aa})$ and $(\hat{x}_2, P_{bb})$ is of the following well-known form ([165], and as summarized from Eq. 2 to the end of Sec. II of [6]):

$$\begin{aligned} \hat{x}_c = K_1\hat{x}_1 + K_2\hat{x}_2 &= P_{bb}P_{cc}^{-1}\hat{x}_1 + P_{aa}P_{cc}^{-1}\hat{x}_2 \\ &= P_{aa}^{-1}P_{cc}\hat{x}_1 + P_{bb}^{-1}P_{cc}\hat{x}_2. \end{aligned} \quad (1)$$

The corresponding exact covariance for the above, with the *assumption of possessing unbiased estimates* throughout, is:

$$\tilde{P}_{cc} \stackrel{\triangle}{=} E[\tilde{x}_c\tilde{x}_c^T], \text{ where } \tilde{x}_c \stackrel{\triangle}{=} x_{\text{true}} - \hat{x}_c. \quad (2)$$

The above expression of Eq. 1, consisting of the indicated weighted combination of the two prior linear estimates and utilizing the accompanying covariances P_{aa} and P_{bb}, seeks to use an acceptable approximate covariance P_{cc} that conservatively suffices in its role of making Eq. 1 be a useful single combined estimator if and only if P_{cc} is a consistent covariance (in the matrix positive semi-definite sense) by satisfying the following required upper bound criterion ([6, Eq. 4]):

$$P_{cc} \geq \tilde{P}_{cc}, \quad (3)$$

(notice the distinction made using the tilde on the right in Eq. 3) and the quest for a satisfactory *consistent covariance* upper bound motivated use of this particular expression:

$$P_{cc}(\omega) = [\omega P_{aa}^{-1} + (1-\omega)P_{bb}^{-1}]^{-1}, \quad (4)$$

(advocated for use in Eqs. 9 and 10 of [6]), which when substituted back into Eq. 1 yields:

$$\begin{aligned} \hat{x}_c &= K_1(\omega)\hat{x}_1 + K_2(\omega)\hat{x}_2 \\ &= \omega[\omega P_{aa}^{-1} + (1-\omega)P_{bb}^{-1}]^{-1}P_{aa}\hat{x}_1 \\ &\quad +(1-\omega)[\omega P_{aa}^{-1} + (1-\omega)P_{bb}^{-1}]^{-1}P_{bb}\hat{x}_2. \end{aligned} \quad (5)$$

Ref. [6] then advocates optimizing ω in the above to minimize the trace of Eq. 4 (cf., [6, Eq. 14]):

$$\text{tr}\left(P_{cc}(\omega)\right) = \text{tr}\left([\omega P_{aa}^{-1} + (1-\omega)P_{bb}^{-1}]^{-1}\right), \quad (6)$$

and goes further to provide Theorem 2 [6, p. 1881] that claims the global minimum occurs for $\omega^* \in [0, 1]$. The resulting optimized ω^* is then substituted back, respectively,

[12] Analogous to illusionist Uri Geller's claim to the "Great Randy" that situations where Uri's magic did not work proved that it was *real*. It was **not**.

into the expressions of Eqs. 4 and 5 (even when the constituent component estimates are no longer independent and the cross-covariance P_{ab} may be unknown or inaccessible) to be the best fused estimate of the form:

$$\begin{aligned}\hat{x}_c &= K_1^*\hat{x}_1 + K_2^*\hat{x}_2 = \omega^*[\omega^* P_{aa}^{-1} + (1-\omega^*)P_{bb}^{-1}]^{-1}P_{aa}\hat{x}_1 \\ &\quad +(1-\omega^*)[\omega^* P_{aa}^{-1} + (1-\omega^*)P_{bb}^{-1}]^{-1}P_{bb}\hat{x}_2, \end{aligned} \tag{7}$$

with the corresponding accompanying associated covariance:

$$P_{cc}^* \stackrel{\triangle}{=} P_{cc}(\omega^*) = [\omega^* P_{aa}^{-1} + (1-\omega^*)P_{bb}^{-1}]^{-1}. \tag{8}$$

Since the original two estimates and accompanying covariances are all real quantities, clearly, the two expressions of Eqs. 7 and 8 also need to yield exclusively real results. If one were to obtain a complex answer for ω^* as the solution that globally minimizes the criterion of Eq. 6, this would constitute a counterexample to what is claimed and ostensibly proved in Theorem 2 [6, p. 1881], namely, that the optimizing ω^* either lies on the two boundary points 0 or 1 or lies within the interior of $[0, 1]$. Once this was *definitively established* according to [6], they could then turn their attention in [6] by just searching over the compact interval $[0, 1]$ for the minimum that is guaranteed to exist from first principles of real analysis for this continuous cost criterion of Eq. 6 (as the composite of the trace and the matrix inverse) since, from elementary mathematical analysis, scalar continuous functions always achieve both a maximum and a minimum on a compact set. However, only Theorem 2 of [6] asserts that such a minimum is also the global minimum (otherwise it would not be of interest since this criterion of using the trace of the associated covariance was specifically chosen within [6] to be consistent by dovetailing with what was already correspondingly used in the derivation of the underlying Kalman filters from which the prior constituents (x_1, P_{aa}) and (x_2, P_{bb}) were obtained). If this local minimum were indeed also the global minimum, we would have no further objections here. However, example 1 below serves as a counterexample to the global optimization assertion [6, Thm. 2] since it yields a complex answer for ω^*.

Similarities and connections to other tests for ellipsoid overlap and pre-existing warnings regarding other earlier Covariance Intersection approaches are discussed in Sec. II.D.

B. A numerical counterexample

A closed-form evaluation will now be provided for this new version of CI [6] for the simple numerical example below that exposes a difficulty with using this CI approach that has not been previously publicized.

Example 1:

$$\begin{aligned} &P_{aa} = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}; P_{bb} = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}; \\ &P_{bb} - P_{aa} = \begin{bmatrix} 1 & 1 \\ 1 & 7 \end{bmatrix} > 0, \text{ and } [P_{bb} - P_{aa}] \text{ has } \lambda = 2,\ 6. \end{aligned} \tag{9}$$

Notice that both P_{aa} and P_{bb} above are positive definite, as with all non-degenerate covariances.

To explicitly demonstrate this new CI technique of [6], we seek to apply the solution of Problem 2 (of [6]) to Eq. 9 above yielding the following abbreviated intermediate steps as we seek to directly solve for the optimizing ω^* that should minimize the trace of P_{cc} below (according to the procedure of [6, Eq. 16] using the derivative convention stated in the footnote below):

$$P_{aa}^{-1} = \begin{bmatrix} \frac{8}{15} & \frac{-2}{15} \\ \frac{-2}{15} & \frac{8}{15} \end{bmatrix}; P_{bb}^{-1} = \begin{bmatrix} \frac{12}{33} & \frac{-2}{33} \\ \frac{-2}{33} & \frac{4}{33} \end{bmatrix}; \tag{10}$$

$$\begin{aligned} P_{cc} &= \begin{bmatrix} \frac{8\omega}{15} + \frac{12(1-\omega)}{33} & \frac{-2\omega}{15} - \frac{2(1-\omega)}{33} \\ \frac{-2\omega}{15} - \frac{2(1-\omega)}{33} & \frac{8\omega}{15} + \frac{4(1-\omega)}{33} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{84\omega+180}{495} & \frac{-36\omega-30}{495} \\ \frac{-36\omega-30}{495} & \frac{204\omega+60}{495} \end{bmatrix}^{-1} = \frac{165\begin{bmatrix} (68\omega+20) & (12\omega+10) \\ (12\omega+10) & (28\omega+60) \end{bmatrix}}{\text{DENOM}}; \end{aligned} \tag{11}$$

$$\text{tr}[P_{cc}] = \frac{165[(68\omega+20)+(28\omega+60)]}{\text{DENOM}}, \tag{12}$$

where DENOM$\stackrel{\triangle}{=} (28\omega+60)(68\omega+20) - (12\omega+10)^2$. The critical points of the above trace are obtained from setting $\frac{\partial}{\partial\omega}\text{tr}[P_{cc}] = 0$ and, after simplifying, solving for the zeros of:

$$0 = 42,240\omega^2 + 70,400\omega + 61,600, \tag{13}$$

a quadratic equation; with solutions being:

$$\begin{aligned}\omega^* &= \frac{-70,400 \stackrel{+}{-} \sqrt{(70,400)^2 - 4(42,240)(61,600)}}{2(42,240)} \\ &= \frac{-70,400 \stackrel{+}{-} \sqrt{-5,451,776,000}}{2(42,240)}. \end{aligned} \tag{14}$$

The above Eq. 14 possesses no solutions [13] over the real field and, in particular, has no solution within the predicted interval $[0, 1]$ and so the new CI approach of [6] is apparently stymied here and can proceed no further for this numerical example corresponding to $[P_{bb} - P_{aa}]$ being strictly positive definite. We did not anticipate that this new CI approach of [6] would exhibit such problems when the containment condition demonstrated in Eq. 9 (i.e., $P_{bb} - P_{aa} > 0$ or, equivalently, $P_{bb} > P_{aa}$) was strictly met (a condition that was present but down played in the proof of [6]); so we were surprised when it failed to yield an adequately real solution for ω^*.

While it is indeed true that a continuous function of ω (such as the matrix inverse, constituting the RHS of Eq. 4, composed with the trace operation of Eq. 6) over a compact interval like $[0, 1]$ achieves its minimum there, we reject the suggestion that we merely confine optimization to be over $[0, 1]$ since such a constraint would, in general, only yield a local minimum. The proofs of [6] supposedly guarantee that by merely optimizing the expression of the RHS of Eq. 6 over just the interval $[0, 1]$, the result would also be the global minimum. However, this non-pathological Example 1 above demonstrates this claim of [6] to be false.

According to [6], only after minimizing the above Eq. 12 can the two optimal gains and resulting associated optimal

[13] We formed $\frac{d}{dw}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dw} - u\frac{dv}{dw}}{v^2}$ and set $v\frac{du}{dw} - u\frac{dv}{dw} = 0 \Leftrightarrow u\frac{dv}{dw} - v\frac{du}{dw} = 0$, and so Eq. 13 here was effectively multiplied throughout by -1, but that still preserves the location of the roots of the resulting quadratic equation.

covariance P_{cc} be explicitly evaluated (by substituting the result of Eq. 14 back into Eqs. 7 and 8, respectively, where Eq. 8 has already been simplified to be Eq. 11) using CI. The above numerical example exhibits a result that is therefore inconsistent with what the CI approach of [6] asserts (contrary to what is expected as supposedly proved in [6, Thm. 2]) so [6] appears to not work as it should in all cases.

By insights availed from [67, p. 1141], it is recognized that for two ellipsoids sharing a common center, the covariance inclusions (such as that depicted in Eq. 18) serve as a test for full containment of one ellipsoid within another if and only if the matrix difference between two covariance matrices is positive definite. Numerical tests for positive definiteness/semi-definiteness are well known [48] and can serve as a warning of this same condition where the approach of [6] will likely fail, as depicted here in the numerical example above.

While it can be argued that, initially, there is no apparent physical reason why these initial covariance matrices should exhibit any partial ordering between them. Two synchronized decentralized estimates of the same target state vector, as viewed from different sensors with, perhaps, different perspective views, different segments of the electro-magnetic spectrum utilized (to exploit inherent target characteristics), and different noise contamination intensities is but one example of why unaltered initial covariances would not necessarily exhibit such a partial-ordering in a completely general sensor fusion application but, instead, likely be skewed off from each other in tilt and overall size. However, use of conventional radar along with a collocated laser radar may yield one target ellipsoid contained entirely within another due to the greater resolution (due to shorter wavelength) and generally smaller azimuth error incurred for laser optics.

C. A simpler CI interpretation based on a different inequality

A simpler approach is now explored here, based on convexity of the matrix inverse over positive definite matrices [66], as:

$$[\omega A + (1-\omega)B]^{-1} \leq \omega A^{-1} + (1-\omega)B^{-1} \quad \text{for all } 0 \leq \omega \leq 1. \tag{15}$$

When this result is applied to the expression of Eq. 4 above in seeking a covariance upper bound as in Eq. 3, the following results, requiring no matrix inversions at all for the RHS vs. a LHS (from [6, Eq. 4]) that does:

$$P_{cc} \overset{\triangle}{=} [\omega P_{aa}^{-1} + (1-\omega)P_{bb}^{-1}]^{-1} \leq \omega P_{aa} + (1-\omega)P_{bb} \overset{\triangle}{=} P'_{cc} \text{for all } 0 \leq \omega \leq 1. \tag{16}$$

Notice that P'_{cc} on the RHS represents an upper bound that is easier and more convenient to obtain and, moreover, by performing trace operations throughout Eq. 16, also yields a corresponding simple upper bound on the trace of P_{cc} as:

$$\mathrm{tr}[P_{cc}] = \mathrm{tr}[\omega P_{aa}^{-1} + (1-\omega)P_{bb}^{-1}]^{-1} \leq \omega \mathrm{tr}[P_{aa}] + (1-\omega)\mathrm{tr}[P_{bb}] = \mathrm{tr}[P'_{cc}] \text{for all } 0 \leq \omega \leq 1. \tag{17}$$

However, although it is rigorous, this path is not a panacea since the resulting bound is likely to be slightly coarser (i.e., larger), in general than what would be provided by the optimizing CI approach of [6] (in situations where the approach of [6] in fact works, which is exceedingly rare). The benefit of this alternate approach is that (1) it requires no matrix inversions at all in its numerical evaluation and (2) it is always true for all without any qualifications. The convexity property itself delineates the interval of primary interest to be $[0, 1]$ and not because of some tenuous auxiliary theorem, as with [6]. The procedure of [6] cannot be applied for this example since [6, Thm. 2] is evidently violated.

D. Status of CI: insights & conclusions

Uncertainty being summarized as covariance ellipsoids normally only rigorously arises for the case of standard linear systems with Gaussian initial conditions independent of the additive Gaussian process and measurement noises (with known covariance intensities) and outfitted with a pure Kalman filter as an optimal linear estimator, mechanized either in a decentralized or centralized manner. Ellipsoidal confidence regions of constant pdf would also reasonably represent the class of elliptical distributions [14] and the conditional and marginal distributions of the exponential family of distributions but they typically do not arise (so far) in the standard estimation and filtering context of most normal target tracking or navigation applications.

Caution is conveyed here regarding the result of [6] apparently not applying when one ellipsoid is wholly contained within the other. An apparent hole in the applicability of the Covariance Intersection (CI) approach of [6] was illustrated here using an explicit numerical example. We were so adamant about pointing out the inherent problem with CI because at least two researchers [358], [389] built their circa 2006 approach to Particle Filter implementation upon intermediate internal use of the CI approach (being disputed here in perhaps more detail than otherwise warranted for this topic but, in our opinion, no one else had yet hit CI-revisited hard enough as a necessary warning).

A coincidence is that the two participating covariances being related as

$$P_1 < P_2 \tag{18}$$

was historically encountered by this author in [43] as a necessary condition that had to be satisfied before being able to specify a test for ellipsoid overlap (in n-dimensions) when the centers of the respective ellipsoids differ, where the particular covariance matrix, P_1, in [43], the solution of the Riccati equation is so related to the other covariance matrix, P_2, in [43], the solution of the Lyapunov equation. Remarkably, the result of [43] parallels (but is not identical to) what is done in [6]. However, the proof of Eq. 18 holding for the application of [43] was easily accomplished in Lemma 5.1 of [43] by just taking the synchronous difference of the two respective matrix differential equations that describe their evolution in time (in either continuous- or discrete-time) by demonstrating that the difference is always positive definite (as it evolves for

[14] *Elliptical distributions* have recently been used by Muralidhar Rangaswamy (IEEE Fellow, AFRL/RYRT, WPAFB, OH) in attempting to compensate for the ground clutter seen by airborne radar.

all time steps $k > 0$) as the positive definite matrices, as pre- and post-multiplied by an indicated non-singular matrix and its transpose (yielding a positive semi-definite intermediary matrix) and added to a positive definite matrix yielding a positive definite matrix result as in [43, Lemma 5.1] and in [245] (cf. [288, Lemma 2]).

The associated optimization problem in [6] has great similarity to that in [43] since the associated Lagrange multiplier was also merely a scalar. In the case of posing the simpler problem of a one dimensional test for the overlap of scalar Gaussian confidence intervals in [55] to show how the same test then generalizes to n-dimensions in [43], as a test for the overlap of Gaussian Ellipsoidal Confidence Regions, the version of the test in [55], [140] (being simpler than that in [43]) reveals other aspects that are similar in form to the structure encountered in [6] in enabling a closed-form answer to the optimization that also proceeds in both [43] and [55], after just optimizing the selection of λ^* along a scalar direction (i.e., the essence of the main result of [6]). When the containment condition is strictly satisfied, the numerical example of Sec. II.B failed to satisfy the expected condition on the optimum value of ω^* that it fall somewhere within the real interval $[0, 1]$. (Ref. [6] also lacks any corresponding numerical description or, alternatively, any explicit reference for the illustrative planar examples presented in Figs. 1 and 2 of [6], respectively, of the intersection inscribing and circumscribing ellipses whose purpose is to motivate how their approach should behave.)

A scalar example is now offered that should convince the reader of the inherent problem with CI. For a scalar case situation, the ideal formula for the associated covariance of two fused estimates (when the two underlying constituent estimates are independent) looks like the familiar formula from electric circuit theory for combining two resistances in parallel (and is known to result in an answer that is less than the smaller of the two). This result is intuitively appealing and consistent with the tenets of Kalman Filtering. The following two algorithms: (1) the CI covariance of [6] using any ω, and (2) the alternative expression for the covariance, offered in Eq. 17 as a new result (using convexity of the matrix inverse over positive definite matrices) both yield a covariance that is larger than the smallest of the two original covariances unless ω is either 0 or 1, in which case it has a resulting covariance that is identical to the smaller one.

Case I: For two ideal independent estimates, the resulting covariance for the combined estimates would be:

$$\tilde{p}_{cc} = \frac{1}{\frac{1}{p_1}+\frac{1}{p_2}} = \frac{1}{\frac{(p_2+p_1)}{p_1 p_2}} = \frac{p_1 p_2}{(p_2+p_1)} = \left\{ \begin{array}{l} \frac{p_2}{\left(\frac{p_2}{p_1}\right)+1} \leq p_2, \text{ for } p_1 > 0,\ p_2 > 0 \\ \frac{p_1}{\left(\frac{p_1}{p_2}\right)+1} \leq p_1, \text{ for } p_1 > 0,\ p_2 > 0 \end{array} \right\} \leq \min\{p_1, p_2\} \tag{19}$$

Case II: For the new CI algorithm of [6] in Eq. 4 for $0 < \omega < 1$, the covariance for the fused estimates would be:

$$\begin{array}{l} \tilde{p}_{cc} = \frac{1}{\frac{\omega}{p_1}+\frac{(1-\omega)}{p_2}} = \frac{1}{\frac{(\omega p_2+(1-\omega)p_1)}{p_1 p_2}} = \\ \left\{ \begin{array}{l} \frac{p_1 p_2}{(\omega p_2+(1-\omega)p_1)} \leq \frac{p_1 p_2}{(\omega p_2+(1-\omega)p_2)} \leq p_1, \text{ when } 0 < p_2 \leq p_1 \\ \frac{p_1 p_2}{(\omega p_2+(1-\omega)p_1)} \leq \frac{p_1 p_2}{(\omega p_1+(1-\omega)p_1)} \leq p_2, \text{ when } 0 < p_1 \leq p_2 \end{array} \right\} \\ \leq \max\{p_1, p_2\} \end{array} \tag{20}$$

Case III: For the CI alternative path of Eq. 16, based on convexity of the matrix inverse over positive definite matrices, for $0 \leq \omega \leq 1$, the covariance for the fused estimates would be:

$$\begin{array}{l} \tilde{p}_{cc} = \frac{1}{\frac{\omega}{p_1}+\frac{(1-\omega)}{p_2}} \leq \omega p_1 + (1-\omega)p_2 = \\ \left\{ \begin{array}{l} \omega p_1 + (1-\omega)p_2 \leq \omega p_2 + (1-\omega)p_2 = p_2, \text{ when } 0 < p_1 \leq p_2 \\ \omega p_1 + (1-\omega)p_2 \leq \omega p_1 + (1-\omega)p_1 = p_1, \text{ when } 0 < p_1 \leq p_1 \end{array} \right\} \\ \leq \max\{p_1, p_2\} \end{array} \tag{21}$$

So both these latter two CI covariance calculation paths yield the same answer for the scalar case! (Convexity of the inverse for positive definite matrices is rigorously established elsewhere, with many precedents in the engineering and mathematical literature cited in [66].)

From the above Cases II and III with similar manipulations, it is also easy to show that $\min\{p_1, p_2\} \leq p_{cc} \leq \max\{p_1, p_2\}$ and, likewise, that $\min\{p_1, p_2\} \leq p'_{cc} \leq \max\{p_1, p_2\}$. Please compare the last two results above to the tighter result of Case I above for 2 independent Kalman filter estimates. This is not a very satisfying outcome of applying either of these two CI strategies. Thus we illustrate the lack of appeal of either of these CI expressions to data fusion.

Ref. [74] is an excellent overview, in depth numerical ranking, and clear interpretation of all of the various approaches to sensor fusion that have occurred in the target tracking literature over the two decades rhat preceeded it and culminates in an algorithmic improvement to CI that [74] attributes to the pioneering work of the late Fred C. Schweppe's "unknown but bounded" approach [77], where, instead of embracing the assumption of Gaussian noises being present throughout, uses an entirely deterministic approach of circumscribing the set of potential outcomes arising at each discrete-time step of a linear system's output within an ellipsoid. Schweepe's bounding ellipsoid approach, although creative, was notoriously conservative and was never used in applications nor was it admired as frequently as what is reported on in [42], as more representative of Schweppe's genius. (Overlooked in [74], parallel developments were occurring in navigation, as similar techniques were being examined, e.g., [68], [75], [76]; some criticized by Larry Levy [JHU/APL] [78]; but refuted in [52] as pertains to Jason Speyer's exact 1979 version of decentralized estimation [193] (also see [79]). Ref. [74] demonstrates that the approximate algorithms of the CI approach should only be used with extreme caution since uncertainty increases as more measurements are used (and fused) as a counterintuitive, extremely unsettling result stated and proved simply and convincingly in [74, following Eq. 22]. Apparently CI also has round-robin Web sites [5] in 2005, where one site refers to the other for missing details and vice versa, yet the specific details are never supplied.

Please see Ref. [80] for an elegant solution to the problem of assessing whether two 3-dimensional ellipsoids overlap that is both easy to understand and apparently straightforward to test for numerically. [15] Problems with [80] and a more straightforward approach to applying its results to determine overlap are in follow-up discussions in [245] and [246].

Before proceeding, a distinction is made here between what is offered in [80] and what is offered in [66] as a test for ellipsoid containment before making other historical connections and observations. Ref. 2 provides a test for full containment of one ellipsoid within another only when they share a common center, $\bar{x}$, as between $\frac{(x-\bar{x})^T P_1^{-1}(x-\bar{x})}{2} \leq 1$ and $\frac{(x-\bar{x})^T P_2^{-1}(x-\bar{x})}{2} \leq 1$ and the second is fully contained within the first if and only if the condition of Eq. 18 holds as a strict positive definiteness condition on matrices which themselves are each positive definite (as are all well behaved, non-degenerate covariance matrices [48]).

The overlap test of Ref. [80] needs matrix positive definiteness/semi-definiteness tests along with an implied eigenvalue-eigenvector calculation. Ref. [80]'s test is obtained by exploiting features of a (3-D)-ellipsoid translation represented as a rotation in 4-D space, a technique familiar in computer graphics applications [166, pp. 479-481].

By not needing any condition of Eq. 18 to be satisfied, Ref. [80] is for a more general case than treated in [43], [55], [140], [167]; however, the numerical calculations of [43], [55] are tailored for a stand-alone real-time decision (that was used aboard U.S. submarines). If one were to attempt to generalize the results of [80] beyond 2- and 3- to n-dimensions (as already done in [169] for just the theory and proofs), a modified version of the computational approach of [43], [55], [245] may be useful in this endeavor (and perhaps even for 2- and 3-D as well) since the iterative algorithm used is a contraction mapping with a provably geometric rate of convergence (but needing double precision for all matrices and vectors involved). The use of an iterative solution technique is not necessarily at odds with providing real-time answers and may be the simplest path to follow. A navigation application using an even easier criterion of ellipsoid containment in dimensions higher than 3-D is discussed in [10]. *Cross-sectional processing* [175] may likely be useful as another technique to use for sensor fusion.

III. Unscented Filters, Particle Filters, and Cramer-Rao Lower Bounds

We are aware of the following approach that evolved (Nahi '69, Jaffer and Gupta '71, Hadidi and Schwartz '79, Monzingo '75, '81, Askar and Derin '84, and Tugnait and Haddad '75) to handle situations where there is data dropout or missing data but we will not dwell on it here (other than to point out its pitfall) because it sought to complicate the situation well beyond what was needed. The above seven references attempted to fit an inappropriate computational architecture that sought to force use of a constant uniform step-size and anticipated periodic measurement availability throughout the implementation [16], where any lack of measurement returns at an anticipated time-step k would be modeled using a scalar independent multiplicative random variable γ_k that takes two possible values, either 0 or 1, within the standard expression for the received measurement data:

$$z_k = \gamma_k h(x_k) + v_k. \tag{22}$$

In the above expression, the missing measurements correspond to $\gamma_k = 0$ for only noise being received. When $\gamma_k = 1$, the desired signal is present in the measurements. The problem with this formulation is that no real structure is available for predicting the behavior of γ_k as a function of discrete-time index k and, because of its presence, even applications possessing linear systems and linear measurements become horribly nonlinear and intractable as an infinite dimensional nonlinear filtering problem [17] even though it originally appears to only be finite dimensional since the system dynamics model is linear and it was hoped, at that time, to be a slight perturbation of a standard Kalman filter. Use of a more appropriate architecture that just processes measurements after they have been detected to be present by the received signal exceeding the prescribed mandatory detection threshold avoids these problems and is more straightforward to implement [174] (by not assuming or relying on having a constant step-size between measurements but just using the existing time-tag [18] of the last received measurements to propagate to the next available one). When handled this way, no measurement is ever "missing" but merely postponed by just having to wait for the next available one to arrive (as a perfect architecture to match the radar target tracking problem as an easy solution).

The above approach to estimation somewhat parallels what is attempted at first in the beginning of [9], as sketched for optimal Cramer-Rao Lower Bound calculation (but is subsequently dropped in [9] after it was realized that it would be much too unwieldy to figure out beforehand which combination of times will actually exhibit radar returns received). Moreover, the $P_d < 1$ situation is not needed since the absolutely lowest CRLB evaluation is obtained when all radar

[15] Observe that a solution to the well-known generalized eigenproblem $\lambda Ax = Bx$ [168, Sec. 7.7] is also a solution of the fundamental Eq. 12 of [80] since $\lambda Ax = Bx \Leftrightarrow [\lambda A - B]x = 0 \Rightarrow x^T[\lambda A - B]x = 0$. Use of *Choleski factorization* and the *symmetric QR algorithm* are offered in [168, Sec. 8.7.2] as a stable solution for the case of A, B being symmetric and A being positive definite, as is in fact the case for the matrices encountered in [80] and herein. Observe that [80] deduces overlap by focusing and dwelling on how pairs of the eigenvalues of non-symmetric $A^{-1}B$ behave. This extra step in [80] of decomposing into cases was not needed and actually complicates the problem unnecessarily (as pointed out in [245]). Symmetric matrices have all real eigenvalues and just a consideration of symmetric matrices apparently suffices in a complete test while non-symmetric matrices frequently have complex eigenvalues and lead to more structural considerations that "muddy the water" in deducing whether there is overlap or not.

[16] This architecture arises in academic simulations so the problem occurring was in not recognizing at the time how to properly transition from the construct currently being used for INS navigation, with its well known periodic measurements, to the proper construct to be used for radar, where the time of actual measurements being available for inclusion within the KF as an update are not known beforehand.

[17] Exact solutions in nonlinear filtering are infinite dimensional, in general, although certain extremely simple finite dimensional special cases exist [192] (1979) and as extensions to those of Beněs ('81, '85, '87); as Daum ('86); and as Tam, Wong, and Yau [120] (and their many further subsequent generalizations [126]); also see Stafford ('84) and [51, Sec. 2.3], [190], [191, Chap. 5 & Appen.], [196], [197], [233, Chap. VII], [235].

[18] For perspective, National Instruments' commercially available recent M-series© Data Acquisition devices all routinely provide such time-tags or time-stamps automatically. Military turnkey applications have done so for decades.

measurement returns are utilized (i.e., $P_d = 1$). When any measurements are missing because they were not detected, then the evaluation error incurred is worse (i.e., bigger). This evaluation represents the best that can be achieved with the existing geometry consisting of radar orientation of faces, distances away from target, and sensor location on the earth as all affect entire geometric perspective of target-to-radar. Actual estimation algorithms can now be evaluated by how close their performance accuracy gets to this CRLB representing "ideal behavior". If even the ideal situation is unsatisfactory, we then immediately know that no estimation algorithm will suffice. Improvements are only possible if illumination power is increased or if pulse repetition frequency (PRF) is increased, or radar scheduler allots more radar resources in order to illuminate the target more frequently. A tractable approach is instead invoked later in [9] (as in [17], [18, Lemma 6.5, Ex. 7.2, Sec. 7.10]-[20, Sec. 6.2], [22]-[24], [53]) of using the known times at which radar returns were actually received before calculating the CRLB for that situation. This simpler approach is believed to be quite acceptable since CRLB calculation is usually done a posteriori and super-imposed on performance graphs afterwards. Using the analogy of the Fisher Information Matrix in the "denominator" before matrix inversion flips it up so it may be directly compared for proximity to computed covariance evaluations obtained by Monte-Carlo sampling, this inverted denominator is used in the same role as Kalman filter covariance analysis for the linear case (where the post- and prior covariances are identical for stipulated times of sensor measurement receipt). Comparisons of proximity of aggregate covariance statistics to the corresponding CRLB is also the accuracy evaluation technique used throughout [177] for gauging how well the various approaches accomplish their goal of adequately tracking their respective targets

On-line covariances can be calculated a priori if one has a perfectly known linear model, knows the covariance of the noises, and knows the exact times at which the measurements are received without actually needing the measurement realizations themselves. Since radar RV target tracking is nonlinear, we need to know the actual measurement realizations in performing the EKF estimation in order to estimate the state, about which the linearization is to be performed at each time step. However, in evaluating CRLB's given the parameter (being the state, which is perfectly known in simulations), there is no need to know the measurement realizations too but just the times at which they are received and the corresponding covariance of the noises at those times. This was ultimately done in [9] as it was done earlier in [17]-[24]; however, [9] can handle the case of the truth model having non-zero process noise based on the theoretical result of [25]. Other historical attempts to handle non-zero process noise in decentralized filtering have their own drawbacks [68]. The four different decentralized filtering architectures for possible sensor fusion developed by the SDI Panel 25+ years ago (by (the late) Dr. Oliver Drummond and also presented at his 1997 short course at SPIE) were all predicated (as he rigorously acknowledged) on there being zero process noise present, otherwise these structures are useless or detrimental in situations where process noise is present (so they are inappropriate to use for Indo-atmospheric tracking). Ref. [68] discusses a decentralized version of filter fusion where the presence of non-zero process noise covariance was not well accommodated because the same small fraction of Q, being $\frac{Q}{M}$, was apportioned to each of the M participating filters (thus causing each participating filter to perceive the system as being much more benign than actual and be untuned to the true underlying situation) and likely to fail in its goal of adequately tracking for each, which then adversely affects the collated whole (since each of M subset filters will fail to track a single target system's plant that is much noisier than each individual filter expects) the aggregate can not be much better yet the computer burden is M times larger than would be the case if just one correctly modeled filter were used (thus this particular approach defies the reason for seeking a decentralized implementation in the first place: for the sake of redundancy consisting of more than one Kalman filter that tracks sufficiently well).

Dr. Dana Sinno (Lincoln Laboratory of MIT) has investigated *self-organizing* networks of Kalman filter-based sensors (not unlike the electronic acoustic sensors dispensed into the jungles of Viet Nam in the late 1960's and 1970's in a failed attempt to monitor Viet Cong activity in the vicinity). Now the sensors have a degree of intelligence and an ability to hierarchically self-organize (like the 1970's vintage **JTIDS RelNav** did for the U.S. Navy) and automatically turn-off to conserve power when noisy activity is absent. These smart sensors can be interrogated and the master sensor reports back results to a command center (which can be one of several to avoid a single point vulnerability). Other related activity along these lines is reported in [273]-[276], [289].

Only ideal exact initial conditions were presumed for each target model for each local Kalman-filter-model-based sensor, so results may be less encouraging when practical initialization is eventually invoked (a candidate being [224]). Recall that the local sensor models are differential equation based and apparently assume only constant velocity. Once ideal exact initial conditions are inserted in such a differential equations-based or difference equation-based model ($Q = 0$), the target's trajectory is completely determined precisely without any measurements being available. Subsequently providing sensor measurements is just "icing on the cake" by improving the on-line computed covariance but the target locations are already known precisely without ambiguity. Such would not be the case if the targets were allowed to accelerate later after tracking had been initiated as long as it was not the same constant acceleration as provided in the initial conditions. Even the use of available (but noisy) According to Dr. Sinno, Doppler measurements from the radar sensors degrade the accuracy available from using mere position measurements alone in this overly idealized experimental situation (as a first step). A real difficulty at present is how to handle multi-target tracking at the sensor level. It is likely that this aspect will only be handled at the level of the supervising Command Center(s) as master station(s). Our theoretical observation here is consistent with what was reported in an **IR&D** project at the 2004 MIT Lincoln Laboratory **ASAP Workshop**. Dr.

Sinno later performed follow-on activities under a DARPA contract at Lincoln Laboratory ($\tilde{2}$004-5) that, hopefully, is more realistic. The late Prof. George N. Saridis (RPI) had published a book on self-organizing systems [194]. The novel *MaxPlus* formulation, as reported in [250], [269], posses considerable potential for further revolutionary developments in estimation for networked systems described by Petri Nets.

Within [220], the IMM Filtering approach is again implemented only for targets described by linear systems, even under maneuvers. Strategic Reentry Vehicles (RV's) are targets known to be adequately modeled only as being nonlinear in both the dynamics and in the algebraic measurement model [218]. Ref. [220] also considers *Particle Filtering* (PF) and Probabilistic Data Association (PDA) approaches as well. As with almost all PF examples to date (i.e., 2006), the state space target dynamics model used within the calculations is planar and consists of only 4 states. The complexity of PF implementation as a CPU computational burden increases drastically (exponentially) as the state dimension increases. As already stated, practical Early Warning Radar target models must be at least six states, and must be nonlinear to realistically reflect the action of an inverse square gravity and the oblateness of the earth (effects that cannot be ignored in a realistic EWR application). Unfortunately, Ref. [220] does not reflect Bit Error Rates (BER) or possible request for retransmission within the communications channels as a consequence of BER. Ref. [220] does not consider having adequate processor capability at each sensor site, as would be the case for EWR with two way communication links present to Colorado Spring's Cheyenne Mountain (or to the newer command center location), and which is more compatible with methodology of decentralized filtering, reported on herein.

Particle Filtering utilizes numerous "mini-simulation" trials [243], [299]. That is a big computational burden and Gordon et al did not hide that fact in 1993 [7] when they came up with the original concept of *Particle Filtering*. They acknowledged that they could do many things in a better way to reduce the computational burden (selective sampling and re-sampling, invoking Metropolis sampling (1953)[302], Metropolis-Hasting's sampling method, invoking Metropolis-Gibbs' sampling method, Rao-Blackwellization techniques, etc.) and [7] predicted future reductions in computational burden. Sure enough, later versions of Particle Filtering, as obtained 9 years later by the same authors (and by others, such as Rudolph van der Merwe's *ReBEL* and Frederick Daum and Jim Huang [8]), made more efficient use of computations and the computer burden was significantly reduced. However, the CPU burden is still considerably larger than is likely tolerable for EWR. Fred Daum (Raytheon) compared the old and new versions of Particle Filtering (denoted by Daum as being *Plain Vanilla* and as *Bells and Whistles*, respectively, as PF-PV and PF-BW) to verify the reduction and improvements and also offered theoretical results that serve as a bound on the likely computational burden of future improvements in Particle Filtering. Refs. [207], [208] have imaging applications of the PF's as well!

At present, PF is too big a CPU burden to be practical (except [219]) and likely will remain so for the immediate future (cf., [256]). Larger state sizes, as needed in 6 state RVCC target tracking, would constitute a larger burden than quantified in Farina's paper [10] for an unrealistically benign situation of using only a 4 state filter for endo-atmospheric tracking, with target trajectory confined to a known plane [19] also containing the observing radar, and with ballistic coefficient (associated with atmospheric drag) being presumed completely known. Farina's Particle Filter was 440 times the computational burden of his EKF of the same state size (see Table 2). Because the number of particle necessary to support a Particle Filter increases exponentially with the state size, realistic filter dimensions of seven states would likely cause the corresponding Particle Filter to be an even larger computational burden than 440 times the computational burden of the corresponding 7 state EKF [20]. We await and encourage further innovative constructions of proposal densities (when needed) that should help concentrate the intermediate Monte-Carlo simulations [302] routinely present in PF to occur in more useful areas of state space so less of the simulation effort is wasted than as now occurs.

While the Unscented Kalman Filter (UKF) [1] is now portrayed in a better light in [2] [21] and in [4], [164], [278], the UKF is still a bigger computational burden but not as large as a standard "particle filter" (PF) [7], as aptly illustrated in [10] by comparing the performance of 4 different estimators: EKF, *Statistical Linearization* (KADET), UKF (Julier-Uhlmann), and PF (with n=25,000) for an over simplified version of endo-atmospheric radar target tracking (where the nature of the over simplifications are examined more closely herein in Sec. IV). Taking the computational burden of the EKF to be unity, the relative computational burden of the four filters was tallied in [10], as summarized here in Table II.

The study in [270] stacks up benefits and drawbacks to

[19] Motion along a conic section is exhibited by satellites and by reentry vehicles. For both, object position and velocity are governed by the nonlinear dynamics of body motion in a central force inverse squared gravity field. Even more unique target behavior can be gleaned by including consideration of the second harmonic J_2 for realism to account for the oblateness of the earth (and its presence induces two more characteristic motions known as the *regression of nodes* and the *rotation of Apsides* [297]. [328]).

[20] CPU times or flop counts of EKF's implemented on sequential Von Neumann machines go as a multiple of n^3.

[21] In 1997 during **HAVE GOLD**, TRW was funded to implement the Unscented Filter a.k.a. the Oxford Filter, but failed to do so. Our apprehension of UKF is because of: presence of an unexplained factor (or unconstrained non-integer free parameter, possibly positive, negative, or time-varying at the whim of the analyst) that can serve as an expanding or contracting twiddle factor in the denominator of the gain expression that is consequentially inherited by the covariance equations; numerical comparison in [1] of UKF vs. EKF performance appear contrived since real EKF practitioners would either take more frequent measurement fixes or better pose the target model to take into account its anticipated motion about a circular track by merely posing the problem in ρ, θ coordinates as the state (thus avoiding literally going off on tangents); lack of usual local *Lipschitz* assumption that would indicate an awareness of minimum conditions needed for solution to exist for the underlying stochastic differential equation model (but do invoke conditions that are impossible to check beforehand e.g., [1, Eq. 2] since probability measure for $x(k)$ is unknown); unconventional use of calculated covariance to account for nonlinear measurement equation and associated unconventional assumption of mean being zero and unconventional proposed handling if mean is not zero (saying it can be shifted, but mean is in fact unknown so one can not know beforehand how much it should be shifted by so user is stymied in trying to proceed [1, Sec. 4]); UKF also utilizes "mini-simulation" trials before each measurement incorporation step (but not as many as a PF would require).

TABLE II
RELATIVE CPU BURDEN OF 4 ALGORITHMS [10]

EKF	KADET	UKF	PF (N=25,000)
1	300	3	440

conclude that both EKF and UKF performed well for integration of MEMS-based IMU with GPS, with and without GPS blockage. However, Unscented Kalman Filter (UKF) was deemed superior in [270] for seamlessly handling situations where large initial attitude errors are present without needing to initiate special in-motion alignment procedures. An observation here is that this post-processing analysis study neglected to include a consideration of total operations counts incurred or the ability of a particular algorithm to keep up with processing demands in real-time navigation applications, where timeliness of computed results should be a primary concern. This aspect is where use of the Unscented Kalman Filter is less satisfying and where use of an Extended Kalman Filter (EKF) wins the contest hands down. Our last comment is based on our prior experience in both INS/GPS [266], [57] and in EKF. Another study [272] directly compares the performance of the UKF to that of the EKF to also favor UKF use as being more accurate (but also apparently failed to emphasize the important hard real-time constraints present in the application that UKF violated). See recent updates in [350], [351]. One should always ask the question "is it real-time yet?" since these researchers address every conceivable aspect except that limitation (or failure of PF being suitable for EWR).

With the passing of time, further refinements have been made and approximate approaches have been developed [254] that trade-off computational complexity against optimality (and associated accuracy) of the above algorithms, as has been quantified across the board for the same identical example application problem. Frederick Daum's (Raytheon) Mar. 2003 IEEE Aerospace Conference paper on Particle Filtering [8] demonstrates that, although the Particle Filter is easier to code, in some simple situations, the Particle Filter can be 10^5 to 10^8 more computationally burdensome than an Extended Kalman Filter for comparable accuracy. This is generally consistent with what Farina et al reported. Ref. [177] is an excellent, well-written yet concise book on the many aspects of radar tracking, as it affects algorithm selection; but ultimately admits significant PF drawbacks in the *Epilogue* [177, p. 287] which is directly relevant to many practical applications such as EWR tracking, paraphrasing: "when good mathematical models already exist for the system of concern (as they definitely do for EWR) and the noise present is Gaussian (also true for the measurement noise present in EWR and during mid course portion of trajectory and for background objects in earth orbit [in MEO and HEO to be considered in updating the Space Object Catalog] are devoid of any process noise) then use EKF instead of the more CPU burdensome PF" [22]. Moreover, the many recent creative, innovative, useful and impressive PF results (whittling down the CPU burden [that blossoms exponentially as a function of state size] by several orders of magnitude for implementation) appear to parallel or mimic the same general techniques that already arose elsewhere for routinely handling partial differential equations (PDE) as solutions are pursued using identical techniques (e.g., such as invoking a convenient homotopy, interpreting or visualizing the nature of a solution in terms of "particle flow", using meshfree techniques [cf. [320] vs. [379]], etc.).

I inadvertently uncovered an incompatibility in current hopes for future parallel implementation [23] of a Particle Filter as a further inherent barrier to PF ever being real-time. [Inadvertent since, at the time, I was merely summarizing why only one random number generator had historically been used within rigorous MIMO Monte-Carlo system simulations to avoid improper cross-correlation of noise realizations generated and to maximize the period before any RNG outputs repeat [357]. The relevance of these same observations to PF thus become obvious because PF's utilize numerous "mini-simulation" trials (that invoke use of a RNG within them), being a huge CPU burden, somewhat ameliorated by further performing sophisticated variants of the "Metropolis-Hastings-Gibbs" sampling/re-sampling to squeeze the most use out of random samples actually generated. Others had speculated that this aspect could be implemented in parallel. My aforementioned insightful connection now bashes this hopeful speculation by reminding of another practicality constraint needed to avoid premature repeating (and unacceptably high cross correlations of noise realizations) that arises unless restricted to use of only one instantiation of the linear congruential generator method for RNG of the necessary intermediate uniformly distributed variates before ultimately converting them to the necessary Gaussian variates that are needed from this well-known two step procedure.] While parallelization is only being attempted for Linear Congruential Generators approaches to date [366], [367], the drawbacks are many [368]. There are alternative, more recent approaches for generating sample realizations of uniform variates other than use of the aforementioned linear congruential technique, but there is no evidence (yet) that the handful of alternatives are any more amenable to eventual parallelization without possibly repeating the sample variates generated.

Mini-simulations may possibly be performed in parallel in the future but calculation of the statistical outcome (i.e., estimate of mean and variance [24] from averaging particles) will still need to be performed sequentially and its necessary entire operations count will likely not ever be sufficiently improved to be real-time (as is needed for PF use in EWR and/or maintaining an up-to-date catalog of background objects in space). For any estimation algorithm to be used for EWR target tracking, it needs to be real-time enough to keep up with anticipated rate of useful radar returns. The aforementioned practical "criteria of usefulness" has yet to be strictly applied to current state of PF evolution [350], [251]. Moreover, the specific number of simulations needed to have a specified confidence in the results is only conveniently (rigorously)

[22] A similar disclaimer also occurs on [177, Last para. of Sec. 3.7].

[23] Another barrier is mentioned in [177, p. 43, following Table 3.3] that "limits the opportunity to parallelize the implementation since all the particles must be combined".

[24] Recall that the variance is needed within the gating step of Fig. 1.

available for the case of Gaussians throughout (associated only with the case of the system dynamics and measurement models being strictly linear) and explicit knowledge of an appropriate number is, in general, unavailable for nonlinear systems and its associated nongaussian noises, thus defying the very situations where PF and USF claim to seek improvements to conventional EKF approximations. So PF and USF both incur this uncalibrated approximation.

Ref. [335] uses the latest in rigorous real-time estimation algorithms (neither a PF nor an Unscented/Oxford/Sigma-Point filter) for enabling accurate pointing (precise pitch and roll) within an aircraft, as reported within a high dynamics operating environment. While it does utilize rate integrating gyros, it also utilizes 2D accelerometer arrays and compares to an on-board gravity map to achieve its accuracy. Following reasonably large occasional offsets, it got back to within 0.1 degree pointing error within 10 seconds but results were much worse with turbulence present. This is not an EWR per se but EWR may be able to use some of these results in the situation described herein in Sec. IX.

The benefits of possessing a better (second order) approximation to any nonlinearities present in the system or measurement model, that is touted for PF and UKF (at the expense of incurring multitudinous “mini-simulations” at each measurement update step of the filter), would be an important consideration and lucrative aspect if the dynamics and/or measurement models were not well known. As we reminded above, this is definitely not the case for Exo- and Indo-atmospheric radar tracking where good models have been known and documented for decades. A 2^{nd} order filter variant or extension of an EKF (also known as a Gaussian Filter) achieves a second order accuracy in its approximation of these well-known and carefully modeled and documented nonlinearities (e.g., [59, p. 33]) by retaining the first three terms in the Taylor series representation of each using the corresponding Jacobian (1^{st} derivatives) and Hessian (2^{nd} derivatives), which need be computed only once as an off-line a priori analytic exercise, as discussed further in Sec. IX.

The Interactive Multiple Model (IMM) bank-of-Kalman-filters approach arose and is apparently supplanting the original 1965 Multiple Model approach of Magill (MMM) [360], which is architecturally similar but lacks IMM sojourn times (of an associated underlying Markov Chain) as the mere contrivance that keeps the several filter options alive as viable alternative filter models being continually actively entertained as possibilities as different operating regimes of the system are encountered. Such IMM approaches are being considered for tracking maneuvering targets and missiles that are boosting but exact nonlinear IMM probability calculations are currently impossible to compute in real-time [223] (also see [307]). Theoretical underpinnings of MMM are provided in [359, Sec. 9.3]. An early warning was that PF could not yet handle multi-target tracking [177, Epilogue]. Subsequently, ref. [156] combines both IMM and PF (as in [177, Sec. 10.4]) for Multi-target tracking [25]. Concerns regarding IMM may be found in [307], [58, Sec. 12] along with a list of 6 other cautions in Sec. IX herein relating to the vagaries of approximate solutions to nonlinear filtering.

[25]Please examine closely for likely departure from being real-time.

IV. A Lack of Realism in Particular Tracking Accuracy Evaluations

Further elucidating our concern regarding lack of realism by Farina et al's reentry model by its being completely planar in [10], central forces in 3-D give rise to trajectories that are confined entirely to be within a specific plane (known, historically, as the *osculating plane*). From a *physical mechanics* course, one learns about *central* force motions and associated properties. Such studies reveal that, for a central force field (like inverse squared gravity), the following cross-product $r \times \dot{r}$, where r is the position vector from the origin of a coordinate system erected at the center of gravity of the earth as focus to the location of the projectile, defines the normal to the plane in which the motion of the projectile is confined. In actual radar applications, the ideal behavior is not precisely obtainable because of the *range-Doppler ambiguity* encountered, as associated with use of practical radar measurements; so there are slight errors present between measured range and its associated range-rate to some degree which, further, corrupts the accuracy of the effective estimates $\hat{r}$ and $\hat{\dot{r}}$ that together degrade the estimate of the normal to the *osculating plane* to which the projectile is confined, to the degree of departure from the ideal as indicated from the calculation of $\tilde{r} \times \tilde{\dot{r}}$, where the individual contributing errors are $\tilde{r} \triangleq \hat{r} - r$ and $\tilde{\dot{r}} \triangleq \hat{\dot{r}} - \dot{r}$. The effect of *regression of nodes* and *rotation of apsides* (mentioned earlier) aggravates the problem of instantaneously estimating the correct plane of projectile confinement even more since, instead of being merely constant and fixed, the osculating plane now moves due to the oblateness of the earth. While Farina's tracker is being treated as a problem that is entirely planar in [10], the real world problem in Missile Defense is to actually figure out what plane the trajectory is situated in and how it is posed. By ignoring these real world effects, Ref. [10], as a consequence reaps greater accuracy than is likely in practice, where four other out-of-plane errors arise (that are ignored by definition in [10]). In like manner, Paul Zarchan (Lincoln Laboratory) et al used this same contrivance in evaluations presented at Colorado Springs at 1997 *AIAA/BMDO Symposium and Workshop* for RV target tracking via radar. Zarchan and Jesionowski [314] used a 5 state Extended Kalman filter (but also used inverse square gravity instead of Farina et al's constant gravity assumption) so Zarchan et al obtained even better results than [10] (because inverse square law gravity, although nonlinear, sets up a gravity gradient that varies with altitude and observed target behavior and better pin-points actual altitude and associated location as a consequence). Results were better in both these simulations than can be obtained in practice since both assume the plane of motion was already precisely known (and consequently optimistically treat the component of out-of-plane errors as being nonexistent and zero in the tally of total error incurred). Part of the real problem is to actually deduce in what plane the target is traveling. In some cases such as in benign pre-planned test shots from Vandenberg AFB in CA toward the

Kwajalein Atoll [26] in the Marshall Islands using our own cooperative target Reentry Vehicles (RVs), one may know the launch point precisely as well as the aim point and our own missiles may have the reentry angle completely known to us for tracking purposes (but for actual missile defense, the defender's tracking radar usually only has an unobstucted view of the target during a portion of mid-course through reentry and we know that anticipated launch points can be varied via use of wheeled or rail vehicles or submerged submarines and that a sophisticated enemy can also suddenly vary several parameters relating to reentry drag during end-game). Farina et al also assume that the observing stationary ground-based radar is at the launch point (and located exactly within the plane of the target trajectory), as a considerable advantage of having almost perfect initial conditions for the target tracking algorithms almost from the start, which is an unrealistic assumption for actual Missile Defense [27]. In reality, a lack of sufficiently accurate initial conditions is a big handicap that arises in actual target tracking.

Moreover, when atmospheric drag enters into the picture, as it did for both [10] and Zarchan and Jesionowski's Reentry tracking examples [314], the problem is no longer an exclusively *central force problem* (guaranteed to be planar). Any tilt of the reentry body forces the trajectory out of the expected osculating plane, which both simulation studies treat as being perfectly known (but did not list among their assumptions). The much more sophisticated follow-up investigation of [151] does not have any of the questionable aspects mentioned above and the trajectory analysis is first rate; however, there are still some concerns:

1) An $\alpha - \beta$ filter is historically well-known to be merely a special case of a Kalman filter under the *assumption that velocity* [28] *is constant* and the filter is run to *steady-state* to get the corresponding constant gains [154, p. 23]; why draw a big distinction now after four decades of *Moore's Law* being in effect [282] to guarantee ample computer capacity and speed now being available? Although [151, p. 621] is correct in its tallies, nickel and dime operation count bookkeeping comparisons of these steady-state-only algorithms vs. use of a Kalman filter (which can track through the transient regime) are less pressing than they once were 40 years ago, when computing resources were scarce. Most EWR applications nowadays use parallel processors and have blazingly fast operating speeds. However, such tallies for a Von Neumann sequential machine are still a useful comparison to check as parallel implementations are pursued that <u>should</u> be faster.
2) The derivation of $\alpha - \beta$ filter implementation equations, as spelled out in detail in [151], have apparently been obtained previously by A. W. Bridgewater, as reported in the proceedings of an earlier AGARD Conference [152, p. 38], [153] within the context of automatic track initiation and, in like manner, using decoupled components too.
3) It is stated in two places within [151, ff Eq. 31, ff Eq. 38] that the elements of the appropriate Q (expected to be the compensating fictitious process noise covariance) comes from Eq. 48, yet [151, Eq. 49] is the expression for the initial covariance P_o to be used for filter initialization and is apparently not for Q. (Perhaps it was a typo that occurred twice?)
4) Radar at sensor location 1, depicted in Fig. 2 of [151], is essentially broadside of the target trajectory (an extremely favorable position for tracking the target but not a likely position for land-based EWR already deployed at fixed known locations). Will resulting target tracking accuracy evaluations be representative of likely EWR performance (or, instead, be representative of only ideal behavior in the situation of best case geometry like this)?
5) Radar at sensor location 2, depicted in Fig. 2 of [151], is still essentially also broadside but now extremely close to the target and will naturally reap the benefits of an extremely large SNR detection and tracking signature. (Perhaps this is a subtle advocate for sea-based EWR by "stacking the deck" in this manner, without indicating this *assumption* in [151]).
6) The good aggregate performance observed (i.e., from averages of 100 Monte-Carlo runs) for three of the four older more conventional historical algorithms under consideration in [151] being close together in accuracy may have been the consequence of the close proximity of the radar locations relative to the target and almost ideal constant periodic measurement availability (exclusively at rates of, first, every 0.25 sec., then later at rates of every 4 sec.) utilized (initially without any data dropout gaps that would otherwise be present in practice with realistic radar schedulers that manage their finite antenna beam and signal processing resources between threat target tracking and surveillance fence monitoring in also performing the background bookkeeping to properly account for orbital debris and existing satellites in the 13,000+ item Space Track Catalog).
7) To be fair, data measurement gaps were later introduced in [151] but were extremely regular by periodically missing a whole batch of measurements (at the aforementioned rates) for 20 seconds then receiving all for the next 20 seconds and continuing to alternate in that fashion. Another situation in [151] looked at just one data dropout segment of 50 seconds duration arising 250 secs. after tracking had commenced and already settled out. Targets were non-maneuvering and so coasting with the inertia of the previous target history still gives pretty good target tracking accuracy as merely an extrapolation step since atmospheric drag likely had not kicked in yet as being significant. (Ideally, accompanying algo-

[26] The Tradex radar used for tracking these RV test shots is located there at KREMS (along with Altair, Alcor, and MMW). The L-band Tradex radar has MTT for up to 63 simultaneous targets appearing within the same mechanically scanned 0.61^o, 6 dB_m beam-width pencil-beam of the 25.6 meter diameter antenna, but has a 600 meter blind zone behind the primary target cluster grouping [150]. Actual EWR usually uses phased arrays and electronic scanning.

[27] Where initial conditions for each target must be deduced from detection and confirmation waveforms, available pulse patterns [296], or from other supplementary information [54].

[28] Heroic and novel compensation techniques were used to compensate for acceleration and velocity not being constant. A recent improvement to mere $\alpha - \beta$ is [306].

rithm self-assessment on-line covariances should have been large when drop outs occurred but no associated covariances were reported in [151].)

One could question why it appears that "the deck is always stacked in their favor" in [10], [151] when they evaluate the performance of their own algorithms? Was it merely a coincidence?

Official NMD evaluation procedures reduce this obvious leaway available to "cheat" in the ways described above by requiring ensemble averaging of N trials to access the aggregate mean square tracking accuracy over M different trajectories of high military interest using only radar sensors at actual current EWR locations (or newly postulate locations) at their known power levels thus maintaining realistic aspect angles throughout as performance is evaluated for several different target scenarios that are highly likely to occur (based on known or hypothesized enemy launch sites and known or hypothesized enemy missile characteristics as launched at our high value targets, expected to be of interest to an enemy).

A varying ballistic coefficient under enemy control but unknown to the defender can be a very challenging problem to track. George Souris (AFIT, retired) had an interesting approach for handling the radar tracking of an RV with unknown ballistic coefficient [26] by treating the ballistic coefficient within the "theory of Interval Matrices", where all entries of the matrix were explicitly known except the entry associated with the ballistic coefficient. This model matches the true situation in EWR for indo-atmospheric targets. This unknown entry was known to be confined to within a reasonable range or "interval", hence the name. The "theory of Interval matrices" has recently had some interesting and useful results [27] that make it even more appealing.

V. CRLB Evaluation Precedents for Exo-atmospheric Targets

Prescribed Measures of Effectiveness (MOE) for EWR applications currently exist as a gauge of tracking algorithm effectiveness, including widespread use of 97% Spherical Error Probable (SEP). However, there is a problem with this highly touted MOE, as demonstrated in Sec. V.A. We offer the Cramer-Rao Lower Bound (CRLB) as a less ambiguous MOE and discuss its theoretical basis in Secs. V.B-V.D and provide a representative numerical example for EWR in Sec. V.E.

A. Why use CRLBs for evaluating EWR Target Tracking Efficacy?

Simple transparent examples of 97%SEP behavior (or, more precisely, 97%CEP behavior which identically parallels in one lower dimension for simplicity the 97%SEP situation) will be presented here. Initial Tracker performance evaluations frequently consist of 250 trials as a goal. While 250 is a fairly large number and is generally obtained with fairly high computational expense incurred in obtaining the requisite realistic Monte-Carlo simulation runs of a particular tracking estimation algorithm under scrutiny, it still does not yield infallible results. To illustrate this claim, please consider the four different histogram diagrams depicted in Fig. 5 (a), (b), (c), and (d) for a uniform histogram . bucket size (not mandatory).

Consider that for x_1 and x_2 being zero mean independent Gaussian random variables with the same magnitude variance [29], the following miss distance, as the square root of the sum of the squares of the constituent position components: $y = \sqrt{x_1^2 + x_2^2}$, is well known to be Rayleigh distributed [305, p. 195]. For 250 trials, we have that from a true Rayleigh distribution, $\frac{y}{250} = 0.97 \rightarrow y = 242.5$ so by having the results of 242.5 trials to the left of a particular value on the abscissa corresponds to the critical value being sought (deemed to be 97%CEP for 2-D, as depicted here in Fig. 5 (a), (b), (c), (d), and (e). The same results would be obtained using the alternate cumulative distribution form of these same histograms. (The corresponding 3-D miss distance corresponding to 97%SEP having three independent Gaussian constituents (each component assumed to have an identical variance) would have a Maxwell distribution [305, Ex. 8-5, p. 273].)

These simulations were simply obtained from the "Histogram Generator" within the demonstration programs for the *Statistics Toolbox* provided by The MathWorks that may be used within MatLab©. The four runs depicted above were for a Rayleigh distribution with the B parameter being 4 (constrained to be between 1 and 6). Notice that for each of these four trials of 250 samples each, a different value of 97%CEP was obtained of 11, 12, 13, and 14 units for, respectively, cases a, b, c, and d. A less ambiguous MOE for gauging the efficacy of target tracking is discussed next.

B. Reviewjng a procedure for evaluating CRLBs for EWR target tracking

Under the standard assumption that the estimator is unbiased [30], then the familiar form of Cramer Rao inequality encountered or invoked most frequently is:

$$E[(x-\hat{x})(x-\hat{x})^T|x] \geq [-E\{(\frac{\partial}{\partial x})^T(\frac{\partial}{\partial x})\ln\{p(z|x)\}\}]^{-1} \equiv \left(1+\frac{\partial B}{\partial x}\right)^T \mathcal{I}^{-1}\left(1+\frac{\partial B}{\partial x}\right) \quad (23)$$

where the inequality here for these matrices is interpreted in the matrix positive semi-definite sense (i.e., $A \geq B \Leftrightarrow A - B \geq 0$) and $\mathcal{I}$ is the *Fisher Information Matrix*. Please see references cited in [21] for details. It is this form [31] (under

[29] In actuality, the variance would likely be different for each constituent component but still the miss distance is just as obviously non-Gaussian even when each component is Gaussian. The main point is that the miss distance is blatantly non-Gaussian in reality as well as for the ideal of all component variances being the same.

[30] The bias referred to here is inherent to a particular estimator and is generally not directly related to any underlying fundamental biases arising for reasons other than the structure of the estimator being employed within a particular application. The techniques for removing biasedness from estimators such as that by D. Lerro and Y. Bar-Shalom (1993) have been for situations where the system Dynamics are linear [288], [312], [313] (unlike the case for EWR).

[31] The summarizing notation $\mathcal{I}$ appearing on the Right Hand Side (RHS) in Eq. 23 is known as the Information matrix prior to matrix inversion, after which the entire expression (after $\frac{\partial B}{\partial x} \rightarrow 0$) is the so-called or so-designated Cramer-Rao Lower Bound (CRLB), which can be numerically evaluated.

Fig. 5. 97%CEP Results for 4 separate Histogram samples (N= 250) having a common underlying Rayleigh Probability Density Function (pdf)

the widely invoked assumption that the estimator bias is either non-existent or negligible) that has a RHS that is independent of the particular estimator being used and that may be compared to a wide variety of distinctly different estimators as a single relative gauge throughout. The CRLB methodology is used here to gauge the quality of filter performance in the tracking task.

The above Cramer-Rao inequality arises in seeking to estimate an unknown parameter x using any estimator $\hat{x}$ and the measurement $z(t)$, referenced above and available from the sensor as a time record, must be a non-trivial function of the unknown parameter x as

$$z(t) = h(x, t, v(t)) \ . \tag{24}$$

In the above, $p(x|z)$ is the conditional probability density function (pdf) of x given all the measurements z, and $v(t)$ is the measurement noise. In exo-atmospheric target tracking, x is deterministic and satisfies a known nonlinear ordinary differential equation and v is additive Gaussian white noise of known covariance intensity, hence $p(z|x)$ is known.

Although other bounds exist, like that of Barankin, the CRLB was selected for use as the familiar bound most appropriate for EWR application because it matches the situation and is tractable. A high-level overview of the CRLB methodology and its benefits and limitations may be found in [21] while it is specialized specifically for EWR exo-atmospheric tracking in [22]-[24].

The CRLB being achieved means that the error of estimation term on the LHS of Eq. 29 touches the CRLB term on the RHS by satisfying the indicated inequality as an exact equality. For EWR target tracking, the lower bound should not generally be achievable (hence this CRLB is **not** expected to **exactly** match the average sampled tracking error variance compiled from N Monte-Carlo trials).

This CRLB was derived by adapting a time-varying radar SNR to realistically correspond to fluctuating PRF and other underlying signal processing as an enhancement of the fundamental methodology that evolved in [17]-[20], as tailored to this EW radar application using the conventions laid out in [22]-[24]. The procedure of [22]-[24], [53] already considered $P_d < 1$ since it included explicit consideration of the detection threshold settings and, moreover, used measurement reception times that correspond to the time-tags for when measurements were actually received (so these CRLBs are a posteriori bounds). While [9] initially tries to tackle a more general case of a priori bounds, it found that approach to be intractable and so [9] then merely resorts to using the structure of CRLBs in [25] for handling process noise that is not zero (as occurs in Indo-atmospheric reentry tracking and not in exo-atmospheric tracking, where $Q = 0$). This is the big distinction between the CRLB approach of [9], [25] and that of [22]-[24], [53], where the latter constitute a much lower CPU burden.

C. Insights into where, when, and why CRLBs sometimes exhibit weird behavior

There is frequently a small initial time segment in the beginning of an estimation error plot when an estimator's covariance lies below the CRLB (as it should, considering the approximations that are usually invoked up to that point, as will be explained) before switching to the usual situation of the CRLB lying below. Sometimes the initial values are so large and far off that, by the automatically adjusted vertical scale inherent in many plot packages, this initial switch appears to be such a proportionately small segment of the figure that it does not raise suspicion or concern enough to be explained to an audience of readers. The same type of thing occurred for each of Farina's four estimators in [10]. To the present author, this is a mark of honesty in the preparation of the results" (but impedes a presentation somewhat in a situation where the speaker has to stop and explain why an apparent bewildering situation occurs of the direction of the expected inequality flipping around and the expected lower bound being above the sampled σ). An explanation was not given in [10] nor has it been given anywhere else to this author's knowledge so we will do so here now.

The explanation is because of that numerator factor $\left(1 + \frac{\partial B}{\partial x}\right)^2$ in Eq. 29 that is needed for CRLB to be below **all** the time, where B is the bias in the estimator. Since we do not usually have B explicitly available and even when we do, its sensitivity to the parameter (in this case the state vector) being estimated needs to be evaluated as the indicated derivative (which is usually not conveniently tractable) so it is usually ignored entirely since it can not be evaluated anyway and we focus instead on the denominator term (which is the inverse of the Fisher Information matrix), which we can evaluate numerically. The numerator term can be a magnifier or a minimizer, depending on whether it is greater or less than 1 and it changes with time. Since estimators frequently proceed to have a steady-state bias (where $\frac{\partial B}{\partial x}$ becomes zero), the exact CRLB expression (numerator and denominator) eventually converges to the approximate CRLB expression (involving denominator alone). Since we frequently have explicit access to only the denominator, we usually use just that and wait past the initial transient until it is appropriate to compare against because only then does the right hand side CRLB$\approx \mathcal{I}^{-1}$, where it is reminded that $\mathcal{I}$ here is the *Fisher Information Matrix* (c.f., [179]).

D. CRLBs for tracking targets devoid of process noise

Post-boost ballistic trajectories that are exclusively exo-atmospheric (within what is designated as the mid-course regime) correspond to targets with no process noise present (i. e., $Q = 0$). The CRLB that is treated here goes beyond just using the historically familiar per pulse CRLB angle measurement error [32] of Eq. 41 (in [22]-[24], [53]) since our CRLB goes further to additionally utilize:

1) information provided by the target dynamics model over time in an inverse square gravity field,
2) the initial (starting) covariance $P(0)$ of the tracking filter

[32]Receive sum-pattern beam-width: $\theta_3 = 3\ dB_m$.

as handed-over [33],

3) the structure of the radar as a measurement sensor/device having additive Gaussian measurement noise with parameters including:
   - (3a) explicit use of the radar range uncertainty due to resolution size of the range gates,
   - (3b) the monopulse SNR time-record with its adaptive step size (as a consequence of a realistically varying PRF) as it affects the corresponding angle uncertainty.

However, one CRLB version used the SNR records simulated by TD/SAT, as Government Furnished Equipment (GFE), and each sample function realization interpolated to common times throughout and then averaged (by Dan Pulido, General Dynamics) to obtain SNR values at designated periodic times, thus providing smooth CRLBs as an envelope for comparison to estimator performance at arbitrary step times.

For an exo-atmospheric EWR tracking application, which has additive Gaussian [34] white measurement noise $v(t)$, the radar measurements of Eq. 24 has this further benign and accommodating structure to be exploited:

$$z(t) = h(x,t) + v(t), \tag{25}$$

and since the equation for the system evolution is essentially deterministic (with $Q_c = 0$), then the pdf's of interest here (to be used in numerically evaluating the CR lower bound of Eq. 23) are of the form:

$$p(z|x) = \frac{e^{-\frac{1}{2}(z-h(x))^T R^{-1}(z-h(x))}}{(2\pi)^{n/2}|R|^{-\frac{1}{2}}}\ . \tag{26}$$

Now taking natural logarithms on both sides of the above pdf yields:

$$\begin{aligned} &\ln\{p(z|x)\} \\ &= -\tfrac{1}{2}(z-h(x))^T R^{-1}(z-h(x)) - \ln(2\pi)^{n/2}|R|^{-\frac{1}{2}} \end{aligned} \tag{27}$$

which upon taking the gradient is:

$$(\frac{\partial}{\partial x})^T \ln\{p(z|x)\} = \frac{\partial^T h(x)}{\partial x} R^{-1}(z-h(x)). \tag{28}$$

When the above expression is post-multiplied by its transpose and expectation taken throughout, the result is:

$$\begin{aligned} &E[(\tfrac{\partial}{\partial x})^T \ln\{p(z|x)\}\tfrac{\partial}{\partial x}\ln\{p(z|x)\}] = \\ &\tfrac{\partial^T h(x)}{\partial x} R^{-1} \overbrace{E[(z-h(x))(z-h(x))^T]}^{R} R^{-1}\tfrac{\partial h(x)}{\partial x} \\ &= \tfrac{\partial^T h(x)}{\partial x} R^{-1}\tfrac{\partial h(x)}{\partial x}. \end{aligned} \tag{29}$$

Finally, over corresponding discrete-time steps (not necessarily uniform in step size), the total pdf of the whole collection of independent (white) measurements is the product of each individual measurement pdf of the form of Eq. 26 as $p(z_1|x(0))p(z_2|x(0))p(z_3|x(0))\cdots p(z_k|x(0))$, where each pdf for each constituent measurement here focuses on or is conditioned on the initial condition for the deterministic system equation. Once the initial condition $x(0)$ is known with confidence [391], then the time evolution of the deterministic system is completely determined (as a consequence of initial condition observability). The corresponding information matrix for each of these measurement time points is of the form of Eq. 29 so the aggregate is of the form [35]:

$$\begin{aligned} &\mathcal{I}(k,0) = \\ &\textstyle\sum_{j=1}^{k} \Phi^{-T}(k,j)\frac{\partial^T h(x)}{\partial x}|_j R^{-1}(j,j)\frac{\partial h(x)}{\partial x}|_j \Phi^{-1}(k,j), \end{aligned} \tag{30}$$

for $k\ geq\ j$, where the transition matrix $\Phi^{-1}(k,j) \triangleq [\Phi(k,j)]^{-1} = \Phi(j,k)$ and, likewise, corresponds to an evaluation of the system matrix linearized about the true state. Now, when there is a finite initial covariance being utilized by the estimator as tracking commences, then there is an additional term [36] that should appear in the above Information matrix to properly reflect this situation, as depicted as the first term on the RHS here:

$$\begin{aligned} &\mathcal{I}(k,0) = \Phi^{-T}(k,0)P^{-1}(0)\Phi^{-1}(k,0) \\ &+\textstyle\sum_{j=1}^{k} \Phi^{-T}(k,j)\frac{\partial^T h(x)}{\partial x}|_j R^{-1}(j,j)\frac{\partial h(x)}{\partial x}|_j \Phi^{-1}(k,j), \end{aligned} \tag{31}$$

In either the case of Eq. 30 or Eq. 31 holding, the Information matrix can be interpreted or formulated as evolving recursively with each received measurement arrival time as:

$$\begin{aligned} &\mathcal{I}(k,0) = \Phi^{-T}(k,j)\mathcal{I}(j,0)\Phi^{-1}(k,j) \\ &+\tfrac{\partial^T h(x)}{\partial x}|_k R^{-1}(k,k)\tfrac{\partial h(x)}{\partial x}|_k \end{aligned} \tag{32}$$

and, as such, may be implemented within software as merely a loop (but by observing all the constraints and coordinate conventions, where $\frac{\partial f}{\partial x}$ is evaluated within the ECI frame and $\frac{\partial h}{\partial x}$ is evaluated in the (E,N,U) frame [37] with corresponding

[33]Using a standard hand-over covariance of $(100km)^2$ for all three components of the position block and $(100m/sec)^2$ for all three components of the velocity block. Physically, this should originate and be communicated from Space Based InfraRed Satellites (SBIRS) for National Missile Defense (NMD) [54], [300]. The Levenberg-Marquardt method that has been advocated for use as cutting edge statistical curve fitting for SBIRS was ostensibly developed earlier by a numerical analyst at Dupont Laboratory in the 1960s. Unfortunately, although the source code may be found in [304, pp. 197-209], the reference cited there for Levenberg-Marquardt does not pertain to this particular algorithm at all. Suspecting a slight mix-up, we searched further for it in other publications by the same Dupont researcher that appeared at around the same time but to no avail. The Levenberg-Marquardt method probably should not be used for EWR without an explicit rationale being available.

[34]Gaussianity arises as a result of the *Central Limit Theorem* (CLT), which with weakened hypothesis, no longer requires that contributing constituents be independent and identically distributed (iid) and hypotheses have likewise been weakened to be more easily met so that conclusions can now be invoked from the *Law of Large Numbers* [84]. Our Appendix offers a quick review.

[35]After taking the natural logarithm of the aggregate pdf, the exponents in the Gaussian distribution correspond to the indicated sum, after performing a gradient and taking expectations, as illustrated in detail above in Eqs. 27 to 29 for just a single measurement for clarity.

[36]An additional "fictitious measurement" was called for in [17, following Eq. 4] as being needed to avoid encountering numerical difficulties but use of $P^{-1}(0)$ as suggested here appears to suffice as a remedy that arises naturally.

[37]A representation in sine space, centered within the antenna array, is recommended for consistency with EWR.

translation offset to the location of the tracking radar) [38]. We have particular interest in the total position error and the corresponding total velocity error to determine how well we are actually doing in tracking a target complex. To this end, we must rigorously contort the inequality of Eq. 23 to a form that we can use. This is accomplished by properly applying matrix operations that yield the expressions that we seek [39] as:

$$\begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \sigma_{13}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \sigma_{23}^2 \\ \sigma_{31}^2 & \sigma_{32}^2 & \sigma_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \cdot E[(x_t(t)-\hat{x}(t))(x_t(t)-\hat{x}(t))^T|\mathcal{Z}(t)] \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T \geq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \mathcal{I}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} crlb_{11} & crlb_{12} & crlb_{13} \\ crlb_{21} & crlb_{22} & crlb_{23} \\ crlb_{31} & crlb_{32} & crlb_{33} \end{bmatrix} \tag{33}$$

and

$$\begin{bmatrix} \sigma_{44}^2 & \sigma_{45}^2 & \sigma_{46}^2 \\ \sigma_{54}^2 & \sigma_{55}^2 & \sigma_{56}^2 \\ \sigma_{64}^2 & \sigma_{65}^2 & \sigma_{66}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot E[(x_t(t)-\hat{x}(t))(x_t(t)-\hat{x}(t))^T|\mathcal{Z}(t)] \cdot \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T \geq \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathcal{I}^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} crlb_{44} & crlb_{45} & crlb_{46} \\ crlb_{54} & crlb_{55} & crlb_{56} \\ crlb_{64} & crlb_{65} & crlb_{66} \end{bmatrix}, \tag{34}$$

and then by taking the trace of a matrix throughout [40], respectively, yields **radial position error variance**:

$$\sigma_{position}^2 = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 = \mathbf{tr}\begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \sigma_{13}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \sigma_{23}^2 \\ \sigma_{31}^2 & \sigma_{32}^2 & \sigma_{33}^2 \end{bmatrix} \geq \mathbf{tr}\begin{bmatrix} crlb_{11} & crlb_{12} & crlb_{13} \\ crlb_{21} & crlb_{22} & crlb_{23} \\ crlb_{31} & crlb_{32} & crlb_{33} \end{bmatrix} = crlb_{11} + crlb_{22} + crlb_{33} \tag{35}$$

and **total velocity error variance**:

$$\sigma_{velocity}^2 = \sigma_{44}^2 + \sigma_{55}^2 + \sigma_{66}^2 = \mathbf{tr}\begin{bmatrix} \sigma_{44}^2 & \sigma_{45}^2 & \sigma_{46}^2 \\ \sigma_{54}^2 & \sigma_{55}^2 & \sigma_{56}^2 \\ \sigma_{64}^2 & \sigma_{65}^2 & \sigma_{66}^2 \end{bmatrix} \geq \mathbf{tr}\begin{bmatrix} crlb_{44} & crlb_{45} & crlb_{46} \\ crlb_{54} & crlb_{55} & crlb_{56} \\ crlb_{64} & crlb_{65} & crlb_{66} \end{bmatrix} = crlb_{44} + crlb_{55} + crlb_{66}, \tag{36}$$

and, finally, by taking squareroots throughout [41], respectively, yields:

$$\sigma_{position} = \sqrt{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2} \geq \sqrt{crlb_{11} + crlb_{22} + crlb_{33}} \triangleq \mathrm{CRLB}_{position} \tag{37}$$

and

$$\sigma_{velocity} = \sqrt{\sigma_{44}^2 + \sigma_{55}^2 + \sigma_{66}^2} \geq \sqrt{crlb_{44} + crlb_{55} + crlb_{66}} \triangleq \mathrm{CRLB}_{velocity}. \tag{38}$$

Please notice in the above that we do not decouple position and velocity states but merely project both of the 6 x 6 matrices of Eq. 32, respectively, into the position subspace (as Eqs. 33, 35, 37) and into the velocity subspace (as Eqs. 34, 36, 38) for viewing in a plotter display. These instantaneous inequalities are now the theoretically justified comparisons that we invoke in monitoring performance of a target tracking algorithm as a function of time.

E. Assessing BASELINE Performance: an existing standard EKF vs. CRLB

We illustrated the CRLB calculations relative to ensemble sampled Monte-Carlo results for the BMEWS radar: Thule [42] tracking an RV on a ballistic trajectory (post cut-off) having the following position and velocity states at cut-off time normalized to $t_o = 0$ seconds:

$$x^T(t_o) = [-3217302.678, 3527834.349, 4535013.695, -767.670, -2520.638, 5065.414]^T \tag{39}$$

where in the above, the units are in meters for position and meters/sec for velocity, respectively. The simulations of the radar case, using known BMEWS published Cobra Dane measurement covariance's for range and angle being [43]

$$\sigma_{\mathrm{range}} = 30 \text{ meters (per pulse)}; \tag{40}$$

$$\sigma_{\mathrm{angle}} = \frac{2.2}{1.6\sqrt{2\cdot \mathrm{SNR}(t)}} \text{ degrees (per pulse)}, \tag{41}$$

[38]Notice that nothing was presumed of the estimator in deriving and evaluating Eq. 29 beyond the underlying measurement structure of Eqs. 25, 26 and the availability of all measurements **up to the current time** k. Alternative estimators that "smooth" by estimating the state x_k using measurements beyond k may violate this assumption and this CRLB but they are not real-time. The appropriate CRLB to correspond to an estimator that uses measurements beyond the current time of interest (such as in "sliding window" smoothing or in "fixed point" smoothing, or BLS) should just have the additional corresponding terms beyond the current time also included in Eqs. 31 and 32.

[39]Pre- and post-multiplying $A \geq B$ by the same matrix L yields $LAL^T \geq LBL^T$.

[40]The matrix inequality $A \geq B$ implies that $\mathrm{tr}[A] \geq \mathrm{tr}[B]$.

[41]Scalar $a \geq b \geq 0$ implies that $\sqrt{a} \geq \sqrt{b}$.

[42]This 10 MHz bandwidth Thule radar (AN/FPS-123V5), with a beam-width of 1.8^o is located in Greenland at Latitude = 76.56^o N, Longitude = 297.70^o E. The actual range resolution is determined by beam forming to reduce side-lobes and assumptions on range accuracy of from as little as 15 meters (for the 10 MHz signal) up to more than 30 meters (for the 5 MHz signal) should not significantly alter the subsequently computed results since sensitivity to the range uncertainty parameter is low as compared to the effect of the more dominant angle uncertainty.

[43]Expressed within the software in MKS units with angles in radians, respectively.

respectively [44], appear to be performing properly, as depicted in Fig. 5 for the case of a nonlinear target (corresponding to use of the system truth model used for simulating the trajectory, but linearized about the estimate within the EKF) while both situations utilized the same nonlinear measurement model. Both parameters in Eq. A.26 of [53] (with SNR varying with time) are used in Eq. A.69 of [53] with $\sigma_E \equiv \sigma_{\text{angle}}$. For position error at time t (and similarly for corresponding velocity with obvious direct replacement substitutions in the LHS of Eqs. 37 and 38), calculated as

$$\sqrt{(x_t(t) - \hat{x}(t))^2 + (y_t(t) - \hat{y}(t))^2 + (z_t(t) - \hat{z}(t))^2}, \quad (42)$$

and the corresponding ensemble sampled variance over N trials (N=250) being [45]:

$$\begin{aligned}\Sigma_N = &\left[\frac{1}{N}\sum_{i=1}^{N}(x_t(t) - \hat{x}_i(t))^2 + (y_t(t) - \hat{y}_i(t))^2 + (z_t(t) - \hat{z}_i(t))^2\right] \\ &- \left[\frac{1}{N}\sum_{i=1}^{N}\sqrt{(x_t(t) - \hat{x}_i(t))^2 + (y_t(t) - \hat{y}_i(t))^2 + (z_t(t) - \hat{z}_i(t))^2}\right]^2,\end{aligned} \quad (43)$$

were depicted for UEWR as diagrammatic plots in [22]-[24], [53] (N=1,000 in 1997 results). The subscript t appearing in both of the above equations denotes the available "truth" that is unabashedly known in simulations.

Fig. 6. CRLB on radial velocity accuracy (for Threat 1)[53]

VI. BLS vs. EKF Trade-Offs

We now discuss trade-offs of Batch Least Squares (BLS) vs. Extended Kalman Filter (EKF) as these two algorithms affect radar target tracking efficacy. First, the Batch Least Squares Maximum Likelihood algorithm is familiar from being present at the core of many diverse yet familiar estimation approaches [58, Sec. 11] such as:

- the Prony method of power spectral estimation,
- some approaches to GPS Local Area Augmentation Systems (LAAS),
- within input probing for improved parameter identification [16].

The BLS that is present in all these situations has the following fundamental structure and characteristics in common, as discussed below.

BLS use (which, as an algorithm, harkens back to Karl Frederick Gauss himself) incurs a larger computational burden than an EKF by needing a larger (and growing) CPU memory allocation to accommodate all the available sensor measurements for a particular candidate target track that are to be iteratively processed in one fell swoop over the entire time interval over which the available measurements have been accumulated and, consequently, BLS incurs more associated senescence (computational delay time that is not fixed but is also growing) than exhibited or needed by an in-place EKF (which has a delay time for computed output that is fixed and known to be on the order of n^3, where n is the state size of the EKF). Since BLS processes all the available measurements en masse and is solved iteratively over all the measurement sensor data it is provided with, the BLS algorithm may converge if the measurement data are consistent with its internal model; but if not consistent enough (as with cross target measurement mis-associations caused by crossing targets or with anomalous radar propagation characteristics due to an atmosphere that is disturbed by sunspots or by other more ominous causes), may fail to converge (a situation prudently handled by specifying a parameter LMAX as the maximum number of allowable iterations, above which BLS is treated as having NOT converged and therefore stopped; thus being prevented from running away).

The EKF immediately avails outputted estimates in a more timely fashion and tends to, more or less, follow any measurement data that it is provided with. The EKF appears to be more appropriate to use with an MTT data association algorithms [46] because it is a fixed CPU burden, which is much less than that of a BLS. On the other hand, the BLS algorithm [301] provides more accurate estimates with a higher fidelity (i.e., being more trustworthy) on-line computed covariance accompanying its estimates for the same data segment length. EKF estimation errors obtained from the on-line prediction of 1-sigma bounds were observed to be 8 times higher than the actual value (gauged against truth) for the representative scenarios that were investigated [14], [15]. The BLS on-line calculation predicts 1-σ errors of a similar magnitude but paid off by actually realizing estimation errors in the same vicinity and so possesses greater veracity in its covariance computed on-line than the standard EKF candidates discussed above (see Fig. 6).

Analyzing a variation on standard BLS use involves a slightly more complicated expression and corresponds to when BLS is called repeatedly at a known, fixed periodic rate. In this situation too, there is an upper bound worse case (conservatively arrived at to be when the BLS fails to converge) as:

$$\begin{aligned}&\text{LMAX} \cdot U \cdot \left(\sum_{i=1}^{\left[\frac{m}{r}\right]} i\right) \cdot r \\ &= \text{LMAX} \cdot U \cdot \left[\frac{m}{r}\right] \cdot \left(\left[\frac{m}{r}\right] + 1\right) \cdot \frac{r}{2} \text{ flops.}\end{aligned} \quad (44)$$

[44] The radar's intrinsic range gate size dictates the effective range resolution, which is a constraint that is less restrictive than the angle acuity. The structure of the phased array radar as a measurement sensor/device having additive Gaussian measurement noise with parameters including **(a)** explicit use of the radar range uncertainty due to resolution size of the range gates and **(b)** the monopulse SNR time-record (as deduced from sum and difference channels) with its adaptive step size (as a consequence of a realistically varying PRF) as it affects the corresponding angle uncertainty.

[45] Notice that this is of the form $E[(W - E[W])^2] = E[W^2] - (E[W])^2$.

[46] Examples being *Munkres* algorithm, *generalized Hungarian* algorithm, Multiple Hypothesis Testing (MHT), *Murty's* algorithm, *Integer Programming* approach of Morefield, *Jonker-Valgenant-Castanon*, all of which either assign radar-returns-to-targets or targets-to-radar returns, respectively, like assigning resources to tasks as a solution to the *Assignment Problem* of Operations Research. Also see [85].

Fig. 7. Position & Velocity Track Estimation Errors vs. Time for Batch and EKF Simulations[15]

In the preceding expression, m is number of measurements and r is the period at which BLS is automatically invoked and the brackets in the upper limit of the finite summation and within the expression on the RHS (where the prior sum is simplified) denotes the smallest integer portion of the resulting division indicated to be performed within the brackets. U is the processing time required for the BLS to handle a single measurement (a per unit value).

The per measurement normalization for BLS utilized above appears to be appropriate and consistent with numerical analysis theory. The main problems being solved at the heart of each BLS iteration is the solution of a system of linear equations (the array of regression equations). Recall that this is the crux or fundamental kernel and the Householder transformation is usually used to solve it (as the algorithm of least computational complexity, which accomplishes the task at hand). Operations counts are available for a perfectly implemented sequential version of the Householder transform from [168, p. 148] ($n^2m - n^3/3$) and the associated back substitution step is O(mn), where n is the state size and m is the total number of measurements from the particular target available at that time. The CPU burden of BLS is merely linear in the number of measurements being processed.

To see how the expression of Eq. 44 was obtained, first consider the case for measurements being processed by BLS at a periodic rate where BLS is invoked after every 10 measurements (where at each invocation, all the measurements logged since the beginning for this object ID are reprocessed by BLS). For the first 40 data measurement points, where BLS was invoked after 10, after 20, after 30, and after 40, the total number of data points processed after 40 is 10 + 20 + 30 + 40 = 100. This is 100 times the measured individual per measurement CPU times discussed above. At the 47^{th} measurement, the remainder now processed is nominally no more than at 40 since the big burden of BLS processing is not invoked again until at 50 data points as:

$$\left(\sum_{i=1}^{\left[\frac{47}{10}\right]} i\right) \cdot 10 = \left[\tfrac{47}{10}\right] \cdot \left(\left[\tfrac{47}{10}\right] + 1\right) \cdot \tfrac{10}{2} = 4 \cdot 5 \cdot 5 = 100, \quad (45)$$

where a useful formula is

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}. \quad (46)$$

Using this result in the above CPU loading factor at time step k for a periodically invoked BLS yields a loading of:

$$\left(\sum_{i=1}^{\left[\frac{k}{c}\right]} i\right) \cdot c = \left[\frac{k}{c}\right] \cdot \left(\left[\frac{k}{c}\right] + 1\right) \cdot \frac{c}{2} \text{ flops.} \quad (47)$$

Ideally, parallel implementations should be no slower than these estimates for a Von Neumann sequential machine and parallel multi-threaded implementations may be considerably faster; however, there are considerable parameter options and variations in a parallel mechanization such as number of processors, compatibility of algorithm architecture to parallelization, characteristics of the operating system, compiler switches invoked, automated automatic distribution of algorithm over available processors (versus use of more careful manual specification and tailoring) that may jeopardize expected ideal CPU time incurred (also see [279] for more aspects).

This operations count goes as m (the dominant power) and again just grows linearly with m. Averaging by dividing the previous expression by m to obtain an expected per measurement normalization yields a constant based on this numerical analysis theory. A similar invocation of a Householder transformation per a measurement depicted on [168, p. 252] also obtained a constant that is a cubic in the remaining fixed variable, being n^3. All theoretical CPU flop time estimates reported here are consistent among themselves and with what was observed within the numerical computations.

In general, the more accurate 1-σ bounds from BLS help better constrain the region of space to be searched by a multi-target tracking (MTT) algorithm than would otherwise be provided by an EKF (see Fig. 6). Use of optimistic (smaller or tighter than true) 1-σ bounds usually provided by an EKF as a practical real-time sub-optimal estimator usually causes search to be more limited than prudent in Fig. 2, although supporting theoretical calculations may falsely assure success (if they expect the 1-σ available from the EKF to be trustworthy, which is usually not the case for the sub-optimal covariances provided from an EKF) while providing adequate state estimates of the target is an EKF's primary goal as a successful tracking filter (and providing covariance veracity is only secondary and is willingly sacrificed for the primary goal (notice that Zarchan et al did not even report covariances or show them in Ref. [151]). However, it is desirable to have both (and BLS does a better job at this) but EKF's are more expediently efficient.

Historically, a finite dimensional filter was sought for estimation and tracking so that it would not grow without bound as the length of the measurement data record got longer. Otherwise, the computer implementation code might overwrite itself or overwrite other critical functions (also required for mission success) that may reside on the same computer. As such, the KF processes a single measurement at a time. A trend that has been observed in investigating actual parallel implementations of other assorted algorithms over the last 20 years is that the ideal expected speed-up is seldom achieved, where a prior well-known sequential CPU time loading is anticipated to be scaled down by a factor of N or merely by a more modest $\sqrt{N}$ (where N is the number of processor resources available in the particular application situation). Actual behavior also depends on the vagaries of the operating system (OS) at hand. Some OSes support parallel processing by allowing the user to stipulate where a particular module thread would reside and run. As an example, for Interactive Multiple Model (IMM) with two comparably sized filters and three available processors, having one filter per processor and the probability calculations/collations (weighted estimates and covariances) IMM output performed on its own processor, all synchronized, would be a natural fit. Other OSes no longer leave it for the user to specifically partition his own algorithm as he sees fit but instead performs this partitioning exercise for the user automatically (so the resulting fit may not be as nice as previously described). Having it done automatically is not always desirable especially if pieces of each IMM filter are distributed over all three available processors. The end result may exhibit worse CPU time than a sequentially implemented version. Performance also depends on the switch settings during compile time. See good tidings in [131] as compared to [129], [130]! [47]

In lieu of not comparing to a prior Von Neumann bound for sequential machines as a parallel implementation is sought, the alternative is to run "open loop" without any comparison of this sort. In such a case, it may be tempting to accept any performance on the parallel processors as "the best that can be done" (whether it really is or not). According to [155], the structures of "interpolating loops" more readily reap the benefits of a likely speed-up in a parallel processor implementation than "integration loops" would. Apparently an early versions of BLS used only interpolation.

As an aside, while U.S. SSBN's (a.k.a., "boomers") possess the world's most accurate Inertial Navigation Systems (consisting of gyros and accelerometers) to support their 3 month long missions at sea, there is also a non-real-time estimation problem associated with their use. Of course, there is a need for maintaining a certain prescribed degree of real-time Navigation accuracy to support fire control in case these submarines are called upon to launch their missiles. However, less publicized is that in the 1970's after each 3 month mission, a massive least squares fit was historically performed on all the stored data to better represent the exact track, from beginning to ending at the same port for assessing an "error of closure" (as also occurs in surveying). The human navigators on board are rated by how well the real-time portion matches the "actual" position while they were on duty but are also checked to make sure that they were sufficiently conservative and parsimonious in their use of alternative external navaids (which instantaneously improve the INS position accuracy but, unfortunately, at a cost of exposing the ship to greater enemy surveillance, to varying degrees, each time they are used [44], [56]).

[47]Refs. [129], [130] were constrained to use only existing parallel processors that had architectures that were optimized for calculating FFT's without any special further modifications to handle parallelizing estimation.

VII. The Need for One Shot Success

What is needed in Missile Defense is an ability to achieve one shot *success with probability one* (instead of in terms of Mean Square Monte-Carlo averages). Researchers should pursue the work [48] of the late Frank Kozin (Brooklyn Polytechnic Univ.) [137],[160] (also see [202], [203]). Kozin sought to make strong proofs about adequate results being reaped from each and every single sample function (i.e., as probability one arguments and not just as mean square argument [49]). According to Kozin, earthquake resistant buildings were a consequence of Kozin's work as it related to stochastic stability. Since much of current tracker filter evaluation of utility is from Monte-Carlo runs (e.g., Farina et al use 100 trials for their accuracy and consistency evaluations in [10]; others may use 250 or 1,000), a worry is that real missile interception possesses the characteristic of being a one shot Monte-Carlo trial. There are no averages available [50] from the target tracking filter in this real-world test, just the conclusion to answer the question: did it work or did it not? Did the tracking filter convey the correct (or adequate) coordinates of the target to the intercepting missile or not?

VIII. Potential for Lambert Solution in NMD

While (the late) Richard Battin (emeritus MIT Aero & Astro and past Deputy Director at Draper Lab. and hero of Apollo navigation and guidance for NASA) was cognizant of most numerically efficient and accurate ways to solve the Lambert Problem [328] (and lead an MIT Ph.D. thesis student [352] and Masters student [353] through this research area), many others have stepped in to fill the void [329]. Dick Battin was primarily interested in this particular problem for NASA space missions regarding orbital transfers of space craft.

Others (notably, Xontech) got involved in this particular version of the Lambert Problem for National Missile Defense (when I also became aware of Battin's earlier precedents [330]) as associated with Early Warning Radar (EWR) detecting enemy Reentry Vehicles (RVs) launched at us. This particular approach for NMD tracking of RV's was investigated via simulations and eventually abandoned since the initial early radar measurements of RV velocity apparently lack sufficient accuracy. For updating the catalogue, the results of [353] may still be useful.

[48]Early on, Prof. Kozin referenced these ideas to the late Prof. J. Clifton Samuels (Purdue University, Howard University).

[49]That there are some unsavory aspects associated with mere averages, please consider that "an engineer with his feet in an oven and his head in an icebox is comfortable on the average!"

[50]Regarding averages, if 50% of the intercepts ended up 20-σ ahead of the target at detonation time and 50% ended up 20-σ behind it at detonation time, is that considered, on the average, right on target? Recall that in seeking to approximate a periodic square wave by its *Fourier* series representation, even if an infinite number of terms are retained in the approximation, there would be no ringing at the jump points but, despite the fact that its *Fourier* series converges in mean square to the piecewise constant periodic idealization, the well known *Gibbs phenomenon* at the jump is a spike of 30% beyond the target goal. Such are the frailties of mean and mean square convergence behavior, respectively.

IX. AOT When Escort Jamming Denies Range

Escort jammers (accompanying RV's as countermeasures within the same threat complex) typically seek to deny target range to the observing radars but the well known and documented locations (and altitudes) of all participating EW radars can be used to an advantage to compute the baseline length between pairs of radars that can now simultaneously obtain almost synchronized views of the same targets from different perspectives and use the known observed angles to explicitly triangulate and deduce the target's range using the law of cosines (i.e., of known angle-side-angle in Fig. 7). Even better is to implicitly triangulate and calculate target range within a single angle-only tracking (AOT) filter (because it uses the history of previously observed target locations) but in order to do so, measurements from two (or more [51]) observing radars must be used simultaneously to update the filter. Historical consensus is that updating with the same measurement results from one-radar-at-a-time [52] simply will not resolve the ambiguity of many different target ranges corresponding to the same observed angle. MTT for AOT is more challenging and apparently still needs to be worked out for this situation and [216] and [244], [303] are good starts. AOT is even more sensitive to initial conditions than with range measurements being directly available. Target observability considerations for AOT are more challenging and are still evolving [146]-[149], [325]. Although some analysts claim that AOT can be performed with just one sensor, the surface ship experience (Hammel and Aidala, 1981) is that with use of just one AOT sensor, the sensor platform is required to make a controlled maneuver in order to do so (while the target is assumed to be moving in a straight line at constant velocity). Alternative angles-only (a.k.a., bearings-only) filter formulations are discussed next.

Fig. 8. Synchronous target triangulation from two (or more) radars [33]

Angle-only tracking (AOT) results are reported by J. R. Sklar '69; C.-B. Chang '73, '80; E. Tse, R. E. Larson, Y. Bar-Shalom, '73; R. W. Miller '78; C.-B. Chang and K.-P. Dunn '79; M. R. Salazar '81 [284]; C.-Y. Hsiao '88; F. D. Gorecki '91; D. V. Stallard '91; J. R. Guerci et al '94; L. G. Taff '97; and by J. P. LeCadre et al '97. An error that occurred in one of the above cited original 1973 AOT formulations, which persisted into its updated 1980 installment in IEEE AES, was discovered and corrected in [33] and the missing associated covariance (when the velocity constraints are active) was also obtained (otherwise the usual Kalman covariance should be used). The approach of [33] provides a Kalman filter formulation that removes considerable ambiguity in an RV target's motion by constraining the computed angles-only solution to lie within an acceptable range for RV velocities (similar to those mentioned in [107]). Alternative AOT approaches also exist [28]-[32] but a worry is that the batch approach of [31] may not be real-time enough. The handling of multi-targets [53] is also aggravated in situations where just angle-only information is available; however, inroads are apparently being made [280]. Other defense applications exhibiting a similar angle-only tracking geometry are: passive sonar/sonobuoy tracking using several participants, acoustic tracking of air breathing cruise missiles, and Space Based InfraRed Satellites. Once a good AOT solution is obtained, "one size should fit all!" Relatively new filter structures that can process several sensor directional measurements simultaneously (in order to point at or track the escort jammer as the source of the wide band WGN jamming signal in an EWR scenario) even though the sensors are not colocated (but are rather distantly distributed geographically and auxiliary joint processing is speculated to reasonably take place at a designated central location such as within the [prior] Cheyenne Mountain control center proper)

[51]In planar multi-target situations, many ghost targets arise from intersecting lines-of-sight [LOS] (as the angles of actual targets are viewed by different sensors) but can be distinguished from actual targets by using more than just two simultaneously observing sensor's measurement with their more varied perspective views. Indeed, ghost targetss are less likely to occur in 3-D where the variously skewed lines-of-sight are less likely to intersect than in 2-D (as with sonobuoy DIFAR detecting enemy surface ships).

[52]I made this mistake 25 years ago, without benefit of a warning from any mentor. However, except in simulations, it is impossible to obtain simultaneity of RV jammer target measurement reception even for bistatic situations. Simultaneous fix updating has a technical structure analogous to what arises in submarine navigation [44], [56] when 2 simultaneous navaid fixes of opportunity are taken together via different antennas, such as from both GPS and Loran-C [103].

[53]The Kalman filtering technology of either a standard Kalman Filter or an EKF or an Interactive Multiple Model (IMM) bank-of-filters appear to be more suitable for use with Multitarget Tracking (MTT) data association algorithms (as input for the initial stage of creating *gates* by using on-line real-time filter computed covariances [more specifically, by using its squareroot or standard deviation] centered about the prior *best* computed target estimate in order to associate new measurements received with existing targets or to spawn new targets for those measurements with no prior target association being "close enough") than, say, use of Kalman smoothing, retrodiction, or Batch Least Squares Maximum Likelihood (BLS) curve-fits because the former are a fixed, a priori known and fixed in-place computational burden in CPU time and computer memory size allocations, which is not the case with BLS and the other "smoothing" variants. Examples of alternative algorithmic approaches to implementing Multi-target tracking (MTT) in conjunction with Kalman Filter technology (in roughly historical order) are through the joint use of either (1) Munkres algorithm, (2) generalized Hungarian algorithm, (3) Murty's algorithm (1968), (4) zero-one Integer Programming approach of Morefield, (5) Jonker-Valgenant-Castanon (J-V-C), (6) Multiple Hypothesis Testing [MHT], all of which either assign radar-returns-to-targets or targets-to-radar returns, respectively, like assigning resources to tasks as a solution to the "Assignment Problem" of Operations Research. Also see recent discussion of the most computationally burdensome MHT approach in Blackman, S. S., "Multiple Hypothesis Tracking for Multiple Target Tracking," Systems Magazine Tutorials of IEEE Aerospace and Electron. Sys., Vol. 19, No. 1, pp. 5-18, Jan. 2004. Use of track-before-detect in conjunction with approximate or exact GLR has some optimal properties (as recently recognized in 2008 IEEE publications) and is also a much lesser computational burden than MHT. Also see Miller, M. L., Stone, H, S., Cox, I. J., "Optimizing Murty's Ranked Assignment Method," IEEE Trans. on Aerospace and Electronic Systems, Vol. 33, No. 7, pp. 851-862, Jul. 1997. Another: Frankel, L., and Feder, M., "Recursive Expectation-Maximizing (EM) Algorithms for Time-Varying Parameters with Applications to Multi-target Tracking," IEEE Trans. on Signal Processing, Vol. 47, No. 2, pp. 306-320, Feb. 1999.Yet another resurgence: Buzzi, S., Lops, M., Venturino, L., Ferri, M., "Track-before-Detect Procedures in a Multi-Target Environment," IEEE Trans. on Aerospace and Electronic Systems, Vol. 44, No. 3, pp. 1135-1150, Jul. 2008.

are discussed next.

Ref. [334] explores two variations on JMEKF formulations that properly handle higher order moments (that lurk in the background while trying to get good estimates and covariances from EKFs). Approximations utilized are acknowledged and properly handled (rather than be ignored, as is usually the case). Resulting estimation errors are reduced by several orders of magnitude within 5 sec., but results are expressed in normalized units (for comparisons to ordinary EKF approach, which it beat by a wide margin). Down side is its larger CPU burden that is yet to be completely quantified and reported.

Ref. [336] achieves a big breakthrough by providing a proof that this particular EKFs possesses global stability as a consequence by stating that it possesses almost" global asymptotic stability; however, the term almost" is required terminology to keep probability theorists and purists happy with the wording of his claim. Author Jensen attains his results by utilizing appropriate stochastic Lyapunov functions (proper handling of such due to Prof. Emeritus Harold J. Kushner, Brown Univ.). Proof of Stability for EKF's as formulated for specific applications [331], [332]. Now worries about EFK divergence evaporate for these particular applications. Ref. [337] provides a proof of Stability for use of Luenberger Observers too (ẼKF).

Ref. [333] is an excellent survey on the subject of attitude estimation. It provides insights into what is important in estimation algorithms. It is a more practical and rigorous addendum to NASA's many earlier surveys,concerned with utilizing alternative EKF's or Nonlinear Luenberger Observers (as alternatives to Extended Kalman filter-based approaches). They admonish to "stick with EKF".

Matrix KF material that I summarize here was originally developed primarily by Daniel Choukroun, B. S. (Summa cum Laude), M.S., Ph.D. (Technion 1997, 2000, 2003, respectively), post-doc (UCLA), currently an Assistant Professor at Delft University of Technology, Netherlands. Requisite operation steps for implementation somewhat resemble those of a conventional Kalman filter and are concisely summarized in terms of Kronnecker sums. Others contributed to further refinements: Ref. [338] provides a linear Matrix Kalman filter for DCM (DCM refinement #1). Ref. [339] offers a linear Matrix Kalman Filter for DMC using either vector or matrix measurement updates (DCM refinement #2). An alternative viewpoint is offered in Refs. [340], [341] as Quaternion refinements #1 and #2, respectively. Refs. [342] and [344] are Quaternion refinements #3 and #4, respectively. Ref. [343] is DCM refinement #3. Refs. [345]-[349] are careful NASA investigations, updates, and summary assessments of current alternative approaches to Spaceborne estimation for attitude determination. Ref. [248] is a critical and thorough analysis of 3 different EKFs vs. use of Technion's Matrix Kalman Filter (MKF) which ultimately availed further improvements to the MKF. Recapitulating, the inherent nonlinear angular complexity associated with triangulation to figure out the direction to the target jammer is instead replaced with a need for simultaneous linear Matrix measurements (within a linear system structure) both independently pointing at the target from the known location of each sensor and the Matrix filter algorithm internally figures out the appropriate resulting pointing direction to jammer (within the relatively tight target complex that can still be intercepted since the jammer is screaming "here I am" even though explicit radar range is denied). Implicit range can be cooperatively deduced (as internally calculated) in this manner.

X. A Need for Further R&D

In the early 1970's, many researchers from the University of Washington (e.g., Alfred S. Gilman, K-P. Dunn, Prof. Ian B. Rhodes) investigated approximate nonlinear estimation in the presence of so-called "Cone Bounded" nonlinearities so that the resulting mechanizations are tractable. Dunn and Gilman later worked at Lincoln Laboratory after obtaining their Ph.D.'s but unfortunately these nice results apparently were not relevant enough to EWR target tracking, which does not exhibit these characteristics.

For the case of an **ideal linear possibly time-varying system** with additive Gaussian white process and measurement noises of known covariance intensities, with Gaussian initial condition [54], independent of the aforementioned noises and of specified mean $\bar{x}_o$ and initial covariance, P_o, and satisfying certain technical regularity conditions (of being *Completely Totally Observable* and *Controllable* [99], [201] or satisfying less restrictive, more generally met, technical conditions of being merely *Detectable* and *Stabilizable*), the following 6 properties listed below are associated with the ideal KF filter:

1) the finite dimensional n-state Kalman filter is an **optimal linear** estimator and is also the overall **optimal** estimator (according to five different statistical criteria of goodness or measures of effectiveness (MOE) listed in [18]) for tracking the state of the n-state *linear* system;
2) the estimation problem is completely solved using just the conditional mean and variance available on-line in real-time from the Kalman filter estimate and its associated Riccati equation solution, respectively. (Conditional refers to being conditioned on the sensor measurements received). Everything is Gaussian, so merely means and variances suffice;
3) there is a guarantee that the Kalman filter is stable and will converge to the true state (even if the underlying system being tracked is unstable), as has been proved using Lyapunov functions (see detailed references in [76] which explain how it was done);
4) the Kalman filter will converge exponentially asymptotically fast (this is darn quick) to the true state [115];
5) even if the initializing estimate x_o and P_o are way off (incorrect) but P_o is still *positive definite*, then the Kalman filter will still converge quickly to the **right answer** (independent of how bad the initial guess or starting values were) [115];
6) the on-line computed covariance (from the Joseph's form of the Riccati equation) is an excellent gauge or measure

[54] Strictly speaking, even for a linear system with all the other usual *regularity* conditions being satisfied, the filtering problem is infinite dimensional if the initial conditions are not Gaussian but instead belong to some other arbitrary known distribution [138].

of how well estimation is proceeding and is even better (more accurate) in fact than statistics computed from any finite number of Monte-Carlo simulations or mission time records.

The above 6 statements are true only for the **linear** case with known Gaussian white noise statistics and only if the filter model appropriately matches the underlying linear system model (including correctly accounting for any biases present), unlike in [116]. For nonlinear systems or non-Gaussian noises, all six [55] of the above bets are off! All are violated in general! Strategic target tracking arising in EWR typically employs nonlinear system and measurement models, as discussed in Sec. I. Historically for EWR, EKF's have been used almost exclusively except for the $\alpha - \beta$ filters of an earlier era.

To explicitly distinguish between what to expect of a tracker for the above postulated ideal linear case and the more realistic nonlinear case encountered in practice [73], the status of approximate nonlinear filtering is discussed by paralleling the format of the previous 6 item list immediately above:

1) the **optimal nonlinear** filter is **infinite dimensional**, in general [56], and therefore not practical to attempt to compute (otherwise, taking possibly an infinite amount of time to do so) while a reasonable engineering approximation is to, instead, employ an Extended Kalman Filter as a **best linear** estimator (but not expected to be an optimal estimator $\hat{x}(t)$ but, hopefully, adequate for tracking the state of the nonlinear system);
2) the estimation problem is **not** completely solved using just the conditional mean and variance available on-line in real-time from the Extended Kalman filter estimate and its associated Riccati solution, respectively. Hopefully, such an estimate will be adequate but its intermediary variance usually is not. Unlike the situation for the linear case, where everything is completely characterized by just the estimator mean and variance, the actual optimal estimator needs all higher moments specified as well [64, Ch. 1], [65] (or, equivalently, specification of the conditional pdf or of its Fourier transform, being its characteristic function, from which moments may be generated). The on-line variance can be optimistic (smaller than actual) or pessimistic (larger than actual) and may crisscross several times over a tracking time interval between being one or the other. The primary focus is usually on the adequacy of just the state estimate as the major consideration. However, there are situations where the variance needs to be of comparable quality (see Sec. I);
3) there is **no longer any analytically provable general guarantee** that the EKF is stable and will converge to the true state. Unfortunately, EKF's sometimes diverge [255];
4) the EKF **does not** converge exponentially asymptotically fast to the true state. We are happy if it gets there fast enough to be useful;
5) when the initializing estimate x_o and P_o are way off (incorrect) but P_o is still positive definite, the EKF **may diverge** away from the right answer at an exponential rate [116]. (EKF performance can be highly dependent on how good or bad the initial guess or starting values are);
6) the on-line computed covariance (from the Joseph's form of the Riccati equation) is a **lousy** gauge or measure of how well estimation is proceeding and is **never** better (or even as accurate) as the off-line statistics computed from an adequately large finite number of Monte-Carlo simulations or mission time records. (Employing a 97% histogram-based Spherical Error Probable [SEP] from as many as 250 Monte-Carlo run evaluations is not atypical in some EWR applications. Perhaps the number should be much larger.)

A desirable goal would be for researchers to eventually achieve as pleasant a resolution in the above 6 categories for handling and tracking nonlinear systems as currently exists for handling the tracking of linear systems. Despite the plethora of new estimation algorithms offered and discussed in the Mar. 2004 Special Issue of the *IEEE Proceedings* dedicated to "Estimation and Tracking" topics, it appears that not enough attention is given to the above 6 topics for the nonlinear case! Control theorists have also compiled a list of unsolved problems for esoteric abstract situations [234] yet have missed mentioning these 6 more mundane bread and butter issues that face estimation practitioners. New approaches should at least remedy one or more of the 6 problems listed above or what use are they? This same gap existed between theory and practice in 1965 and some remedies are in [242]. However, Refs. [34], [70], [81], [105], [109] [57], [121], [127] (fulfilling the promise of [128]) do appear quite lucrative and especially the milestone accomplishment of [108] for their particular imaging-based tracking solution. Ref. [108] has apparently constructively exploited every major ground breaking result in novel cutting edge random process theory that has occurred in the last 30 years (reaping structural benefits of martingale inequalities being available as associated with the greater rigor of using a Brownian motion process interpretation over merely

[55] There was actually a seventh concern expressed in [58, Sec. 12] questioning the applicability of IMM for nonlinear system models and noting the apparent lack of prior precedents of IMM use with such nonlinear systems. The utility of IMM over purely KF's were recently demonstrated in [101] for certain linear systems but no nonlinear systems are treated in these comparisons of KF and IMM performance that favor IMM use. Similarly, [156] only uses linear system models for IMM even with PF that can ostensibly handle nonlinear systems and non-Gaussian noises. Also see [307].

[56] There are some limited nonlinear special cases that have finite dimensional optimal filters (as characterized by Beněs, Daum, Tam et al, Stafford [op. cit.]), where the distributions encountered within the system proper are of the *exponential family* [124, Chapts. 1-4] yet the marginal or conditional distributions will still be tractably Gaussian.

[57] While it is a laudable evolutionary service to collect and analyze the various models available for describing maneuvering targets, as done in [109], some may disagree with the remark on page 1349 that the models of Eqs. 79, 80 there are highly nonlinear. In fact, they are *bilinear* and, as such, are slightly less tractable than purely linear systems [110] (so there is no need to "pretend" in obtaining the results of Eqs. 81 and 82). Prof. Roger Brockett (Harvard) helped pioneer how to get such nice results for these almost linear systems. Alan Willsky (MIT) and David Kleinman (NPS) have stability results for these too (circa 1974) and Willsky and J.T.-H. Lo have estimation results for similar systems that have associated *Lie Algebras* that are finite dimensional (as do Beněs, Daum, Tam, Wong and Yau, Mahler [191, Chap. 5], [196], [197] also see [190], [233]). Ref. [109] missed including [111], which creatively utilizes the *Maximum Principle*. Newer approaches also exist [310].

a "Gaussian white noise" viewpoint, use of Poisson point process to capture realistic imaging aspects, use of Girsinov's transformations of measures to a computational advantage [125]) to push the envelop and reap landmark results (thus taking to fruition what Donald Snyder, Terrence McGarty, Moshe Zakai, David Sworder, Yaakov Bar-Shalom, and so many others may have had in mind as an ultimate goal for this evolving theory). This approach could eventually be as beneficial as Irving S. Reed's *streak processing* is for a similar space target application (but now no longer constrained to handling merely straight line tracks)! The new results of [172] appear to be very similar to those of Frederick E. Daum (1986, 1987). The results of [173] appear to be directly applicable to further performance analysis of a BLS batch filter (viz., CRLB). The results of Refs. [254], [354] nicely augment the approach of [251]-[253].

For EWR, there is still room for improvement of the EKF itself including automated "process noise covariance filter tuning" [34] (recently developed and applied to GPS), [70]; possible use of iterated EKF's (of two different flavors) [35], [36]; or possible use of more terms in the Taylor series approximations of the significant nonlinearities present (constituting use of a Gaussian 2nd order filter) [3], [18], [19], [37]-[39]; and possible parallel mechanizations [40], [41] A recent evolutionary change is to use polynomial interpolations via Stirling's formula [71] or via [72] instead of evaluating any higher derivatives. All these strategies should improve the accuracy of the measurement linearization with but a slight increase in the computational burden. Manually calculating the first derivative and second derivative Jacobian [58] and Hessian matrices, respectively, was challenging 30 years ago but is quite tractable now with the advent of symbol manipulation software like Maple©, MacSyma©, or Mathematica©so [71], [72] are less enticing for this application than perhaps for others where derivatives are less readily available or nonexistent. Recent innovative results in contraction mapping analysis [157], [158] should be explored for likely relevance in seeking to improve theoretical underpinnings of EKF for nonlinear applications.

New exact and approximate solutions have been obtained for incorporating out-of-sequence measurements into Kalman filters [174]. This will likely be useful to compensate for transport delay incurred in cross-communications if several EWR participate to jointly track the same targets as seen from different geographical perspectives or even when augmented with the output of other types of sensors to enhance target-tracking capabilities beyond that availed from each alone. A caution is that innovative researchers sometimes use a definition of stability that differs from that classically and historically agreed upon and consistently used for decades. As a consequence, new results can validly claim to yield "stable systems" or to yield an "algorithm that converges" even though limit cycles are present (that historically would be viewed as being unstable or as an algorithm that did not converge). An earlier precedent for this situation occurring arises with use of the Min-H technique, as discussed in [44], (that is guaranteed to converge in-the-sense-of-orthogonal-search-algorithms) yet can actually vacillate and never settle down completely but, instead, can continue to hop around forever between a small finite number of equally valid options as solutions. Occasionally, the Min-H technique converges to a single unique answer and only then is its output useful as a solution to the problem at hand (as used in [44], [56]).

Rudolf Kalman used the *Hilbert Space Projection Theorem* [59] in originally deriving the Kalman filter in 1960 (cf. [327]). The norm in L^2 is also an inner product, which is what one needs in a *Hilbert Space* (along with the space being *complete* by containing all its *limit points*). Other researchers, such as Ruth Curtin, have pursued use of *Banach space* techniques for obtaining Optimal filters for systems whose dynamics are described by Partial Differential Equations (PDE's) and whose corresponding observations constitute natural boundary conditions [267, Chap. 5]. Kalman filters for such PDE systems are also found in Andrew Sage's 1968 textbook, as identified (with applications) in [52], [58]. Randall V. Gressang and Gary B. Lamont submitted "Observers for Systems Characterized by Semi-groups," to IEEE Automatic Control in 1977 but it was rejected (not because it was wrong but because it was so far ahead of its time). Gressang and Lamont's paper posed the problem (arising for infinite dimensional systems described by PDE's) and solved it using only corresponding *Banach space* techniques (rather than use the Hilbert Space techniques that were prevalent at the time and familiar to the reviewers who failed to recognize that the Hilbert space techniques were inappropriate to use for this particular infinite dimensional situation). There is also an existing mathematical theory for handling unbounded linear operators that some believe is appropriate to use in this context when the operator at hand involves derivatives. Ref. [195] uses *Banach space* techniques when needed to handle Riccati Equations.

XI. PRECEDENTS FOR USE OF NEWER ESTIMATORS WITHIN STOCHASTIC CONTROL IMPLEMENTATIONS? ANSWER: NONE ARE SUFFICIENTLY REAL-TIME!

We now discuss an open question that remains to be addressed for the new alternative estimation approaches. But first a short review is needed to set the context and define terms: the term LQG represents the feedback control strategy obtained by concatenating two back-to-back ideas of using a Kalman filter in conjunction with use of a Linear Quadratic (LQ) feedback control. The LQ regulator is the feedback control for driving a noise-free linear system to the zero state (termed "regulation") that minimizes or optimizes an associated convex Quadratic integral cost function (i.e., quadratic in both the state and the

[58]Calculating the Jacobian for a 6 state filter corresponds to forming $6^2 = 36$ derivatives that can be fairly challenging. Seeking to calculate $2^{\rm nd}$ derivative Hessians can be very taxing unless symbol manipulation software is utilized (e.g., MacSyma, Maple).

[59]Prof. Thomas Kailath alerted the estimation community to a precedent by some Japanese researchers that posed linear estimation within a Krein Space instead of within a Hilbert Space and apparently obtained faster convergence as a consequence. While Matrix Positive definiteness plays a prominent role within all the analytic proofs supporting the usual Hilbert Space-based derivation of Kalman filters, the Krein Space approach frequently involves matrices that are indefinite. The tool in common is still projections onto linear subspaces [104]. The Krein Space approach decomposes the problem into two Hilbert Spaces.

control), with such an endeavor being achieved over a finite time interval $[t_0, t_1]$ (for a finite planning horizon) or over an infinite interval $[t_0, \infty]$ (for an infinite planning horizon) and thus minimizing or optimizing the total energy expended in each case.

When the linear system to be regulated via a feedback control is noise corrupted, the **Separation Theorem** allows us to validly decompose the problem into the two parts mentioned above by first obtaining an estimate of the state in lieu of not having the actual noise-free state available for multiplying by the LQ feedback gain as feedback control: $u(t) = M(t)x(t)$; then we, instead, use the best available estimate of the state (being the output of the Kalman filter) in forming the corresponding LQG feedback control: $u(t) = M(t)\hat{x}(t)$. This approach is straightforward but requires solving two similar looking Matrix Riccati equations:

1) one solved forwards in time for the KF covariance used in computing the KF gain, $K(t)$, which, in turn, is used in obtaining the state estimate $\hat{x}(t)$;
2) one solved backwards in time to obtain the matrix subsequently used in computing the (possibly time-varying) LQ gain $M(t)$.

If there is negligible noise present in both the plant and in the measurement sensors, then a Luenberger Observer is utilized instead of the Kalman filter to reconstruct any unavailable states (i.e., states that are not directly accessible) for use in the feedback control.

The **Separation Theorem** (for linear, possibly time-varying, systems) supports the above described strategy by guaranteeing that the Optimal Control, which minimizes the expected value of the scalar Quadratic Convex cost function, can validly be separated into two sequentially applied parts. Unfortunately, such an easy-to-obtain solution lacks a reasonably conservative phase margin to guard against instability of this LQG control result ([88], [89]). Pure LQG solutions have a paucity (as in zero) in phase and gain margins! (As previously observed with the calculus of variations for obtaining the elusive optimum solution, it had already been observed by earlier generations of researchers that the (piece-wise continuous) time-optimal bang-bang control is also on the cusp of being unstable since a drastic instability occurs if any of the indicated "switching instants" actually implemented are even slightly offset from the ideal switching goals, and these systems can similarly go unstable even with the associated smooth LQG control strategies (cf. [119]). Loop Transfer Recovery (LTR) [90] is a further slight modification of the basic LQG methodology to force a practical solution that does have the necessary margins for safety's sake so that the resulting total feedback control solution of LQG/LTR is more robust in a changing environment (of aging hardware components resulting in slightly changing parameter values, possible presence of unmodeled high frequency dynamics unaccounted for because they did not reveal themselves as being present during the original data reduction, where the test stimulus may have been of a lesser bandwidth than needed in implementation) and, as a consequence, the LQG/LTR feedback control strategy is no longer on the cusp of going unstable, as use of LQG alone would be.

Richard Gran (retired from Grumman Aerospace and later from The MathWorks) authored [91]. W. H. Wonham (Brown University, now with Univ. of Toronto wrote in the same proceedings [92]. Many others have also participated in this quest for nonlinear separation [93]-[96], [367] [60].

A famous counterexample, where nonlinear separation fails, was published by H. S. Witenhausen (Bell Labs) [97]. It reveals the fallacy of attempting nonlinear separation and dashed hopes (for awhile anyway) for complete generality in the nonlinear case but engineering approximations frequently invoke this Separation procedure anyway by separating the problem of nonlinear optimal control with noise being present into two distinctly different sub-problems that are treated and solved separately, in isolation, by first performing nonlinear estimation followed by nonlinear optimal control. This two-step technique can sometimes still be useful by treating the total problem sequentially in this way although, in reality, the problem of nonlinear optimal estimation and nonlinear optimal control is inherently mixed together. Extensive simulations of the resulting algorithms are used to gauge whether it is adequate for the application at hand. More research is needed to "crack this nut" and reduce reliance on mere simulation (especially since no directions arise pointing to a better solution strategy as the next step to pursue if the results of the initial simulation of the separated and later combined strategy are disappointing) except [123] as a recent suggestion.

Appealing to use of the more recent so-designated H^∞ or Robust Control methodology will not likely take up the slack! (See further confirming revelations in [117, Epilogue], [118] cf. [119]) Although an H^∞ approach may perhaps be useful for process control applications, where possessing a rapid response time is not an issue because it is not sought as a goal in process control; by assuming a worse case situation for its implementation, it usually has a conservative response that is notoriously sluggish (analogous to the situation of being over-damped in 2^{nd} order linear time-invariant systems). An example supporting this assertion is that the useful and very familiar Least Mean Squares (LMS) algorithm that is known to converge [308], but sometimes slowly [309], [315], has been shown in [161] to be H^∞-optimal [61]. More to the point, the Robust Control methodology does not yet handle general time-

[60] Unlike what the title says, this is for feedback stochastic control using an EFK within the loop and, in my opinion, uses contorted notation. For $T_1 \leq T_2$, then $L^p[0, T_1] \subseteq L^p[0, T] \subseteq L^p[0, \infty]$, (with p a positive integer) as standard existing results of *functional analysis* instead of making up their own nonstandard notation. For arbitrary integer $p > 0$, each is a Banach Space. For $p = 2$, each is a Hilbert Space, with all the properties thereof. This structure exists for the Lebsque measure (for integrals) and for the Counting measure (for sums). Despite the claims of generality in the conclusion, it does not overcome Witenhausen's famous counterexample.

[61] Over 12 years ago, some Japanese researchers reported getting better estimation accuracy for a somewhat uncertain system model when they used covariances, P(t), that were **not** positive definite but instead were indefinite. While such results initially appeared to be counter to prior analytic intuition, considering the important role that positive definiteness was known to play in estimation theory and LQG control and in its underlying proofs, the subsequent analysis of Refs. [162], [163] explains how this can occur when estimation is posed in a *Krein Space* rather than in a *Hilbert space*. Application results are reported in [163] but are applicable only to systems with linear time invariant (LTI) plant and measurement models but [163] provides further insight into the interrelationship with estimation and control.

varying linear systems, general nonlinear systems, nor systems with noises present for MIMO except in some heroic cases for a single isolated scalar system component. This is especially unsettling upon recalling that when general nonlinear systems are linearized, the result is a time-varying linear system! Recently, H^∞ control techniques have been used to detect the event of hardware component failures in systems, where the simulated failures were 1,000 times nominal; thus, use of such gigantic failure magnitudes merely conveys the impression of good performance (since results are less impressive for more reasonably sized failure magnitudes). Such demo tricks were historically warned about in [75], [141]-[143] but, evidently, still occur. Do H^∞ approaches offer solutions to any problems that could not already be solved more conventionally?

Although the late George Zames is credited in a moving (and extremely informative) tribute on pp. 590-595 in the May 1998 issue of IEEE Trans. on Automatic Control with, essentially, single-handedly bringing mathematical functional analysis to the aid of control and system theory via use of the contraction mapping principle (CMP) in [136] (see [265]), please peruse the earlier contribution by Jack M. Holtzman's (Bell Telephone Lab., Whippany, NJ) [132], which also has the use of CMP as its main theme in such systems. However, Holtzman worked everything out in detail in [132] so that his results were on a platter in such a form that they could be easily understood and conveniently applied immediately to practical system design by engineering readers faced with real applications and who may not necessarily be interested in abstract results in a technical paper whose significance is not known until several years later. Charles Desoer's and M. Vidyasagar's (U. C., Berkeley) textbook came out several years earlier than Zames too and also had a functional analysis bent. A. V. Balakrishnan (UCLA) has also been an avid practitioner of functional analysis in analyzing the behavior of systems and in understanding optimal control (including numerical solution algorithms) since the early 1960's. Ref. [43, App.] even uses CMP in its convergence proof as does [55].

What about the use of *feedback linearization* or the use of neural networks for control? Answers to these questions appear in [106]. Apparently, we are still awaiting investigations [62] into the **downstream control impact of using the new alternative estimation approaches** of Sec. IV in place of EKF's as the first step for actively controlling noise corrupted nonlinear systems. (See Vol. II of [64] for an alternative approach as a precedent for handling or compensating for the effect of noise on relays and on the synchronization of oscillators.) Unlike the situation for EKF's, apparently none of the new (or alternative older algorithms) "yet play a role in Stochastic Control". Evidence confirming this assertion is available by perusing the recently published Ref. [159]. Indeed, Ref. [159] elucidates a new, well-funded application area in Stochastic Control yet nary a word is mentioned about using α-β filters in the pursuit of stochastic control algorithms, nor use of Particle Filters, nor Unscented Filters (= Oxford Filters [63] = Sigma Point Filters) even though the editors of this special issue are at Oxford. An even more challenging PDE arises within the context of stochastic control [93] as the Bucy-Mortensen-Kushner PDE (see [18, p. 176] for a clear concise perspective).

Realities of Nonlinear Filtering:

- The presence of Gaussian noises in nonlinear systems (described by nonlinear ordinary differential equations) yields outputs that are unlikely to be Gaussian and perhaps not even unimodal.
- Since non-Gaussian outputs and estimates are encountered, they need more than merely the estimated mean and variance for a full characterization of its tracking capability. In general, all the moments and cross-moments must be specified (up to a certain point before exceeding practicality) for a sufficiently complete characterization of non-Gaussians so the problem is generally infinite dimensional (but finite dimensional for an approximation that includes only higher moments up to a specified maximum number before ignoring all higher order moments [by assuming their effect to be of little consequence so treated as zero]). Differential equations can be specified for the time evolution of all higher moments but they are, in general, coupled with the time evolution of even higher order moments ([64], [65]) [which drop out when assumed to be zero].

Other alternative tractable approaches:

- Can use conjugate pdf's before incorporating a new measurement into the filter. The candidate pdf's are to be selected from the *exponential class* (which contains many familiar pdf's as members).
- Can approximate the unknown pdf using a hypergeometric series with parameters to be estimated. Other issues arise concerning when do pdf's even exist (relating to when the *Radon-Nikodym derivative* exists, and, in turn, relates to when the cumulative distribution function is an *absolutely continuous* function).
- Can pursue use of *alpha-stable filters* for "fat tailed" noise (i.e., outliers are prevalent) [a scalar case example was provided by Stuck '79 [370]).

XII. Summary

We encourage investigations into new estimation approaches (such as in [182]-[185], [222]) as a way to possibly get past prior fundamental barriers that nonlinear filtering practitioners had tripped up on in the past (that we reconnoiter about in Secs. IX and X). However, we also desire that realistic procedures be used to evaluate the suitability of the new as

[62] We are also awaiting investigations into why Space-Time Adaptive Processing (STAP) algorithms assume enemy threat is merely stationary WGN "barrage" jamming. STAP appears to be very vulnerable to nonstationary jamming [57]. Many STAP algorithms to date (e.g., [145]) have utilized Wiener filters (which only handle time invariant situations in the frequency domain). It is well-known that Wiener filters are a special more restrictive case of a Kalman filter [139, p.142, 242] and that MIMO Wiener filters incur the more challenging extra baggage of needing Matrix Spectral Factorization [47], [49] to take them to fruition instead of equivalently just needing to compute the more benign Matrix Riccati Equation solution utilized in Kalman Filters. In the early 1990's in an award winning paper [144], Prof. Thomas Kailath (Stanford) and his thesis students established that most so-designated adaptive filters in current use are in fact merely special cases of Kalman filters.

[63] Named for the affiliation of the original developers.

well as older conventional α-β filter and BLS algorithms for specific applications, where we are only specifically concerned herein with EWR.

Covariance Intersection was shown to be wanting in Sec. II and questionable aspects were identified in the derivation of the Unscented or Oxford filter in Sec. IV (footnotes). In Secs. III and IV, we were concerned with an apparent lack of realism relative to the EWR mission in the evaluations of [10] as it would severely affect tallies of absolute accuracy (that ignored the existence of out-of-plane errors by treating them as being essentially zero as also done in like manner in a 1998 AIAA/BMDO workshop presentation by Paul Zarchan and R. Jesionowski [314] as a possibly misleading precedent) and consequently treated by default as no longer contributing to the total error incurred in tracking. This is a serious oversight since a major part of the EWR problem, in attempting to track targets under central forces, is to identify what oscullating plane they occupy. However, since all 4 algorithms considered in [10] were evaluated under identical controlled conditions, the CPU timing and CPU loading studies of [10] for cross-comparing the algorithms were still appreciated and reported here as useful (but the state dimension of the tracker models were too low). For a higher dimensional system dynamics model of greater significance to EWR (of at least 6 states: 3 position and 3 velocity) instead of the 4 used in [10] and 5 used in what Paul Zarchan and R. Jesionowski presented [314] (in an indo-atmospheric situation where there should be at least 7 states), the CPU burden increases or blossoms nonlinearly and exponentially as the model state size increases and is affected by other parameters as well (as quantified in [8] for only a particular type of Particle Filter with Bells and Whistles, as a class). We also offered warning relative to the use of α-β filters for EWR at the end of Sec. IV.

In Sec. V, we revisited the appropriateness of an historical 40 year old approach to CRLB evaluation as still being germane for EWR. In Sec. VI, we obtained new original expressions for BLS CPU timing and loading bounds for a sequentially implemented version so that this bound may be used as a starting place gauge for comparison as more modern versions of BLS are implemented on parallel processing machines[64] to, hopefully, reap considerable speed-ups by scaling down the CPU operations times relative to this upper bound. In Sec. IX, we reviewed the various new approaches for improving the behavior of EKF's since EKF's have historically been the workhorse in EWR and are likely to remain so for the immediate future. In Sec. X, we noted what remains to be done before other estimation approaches fill the role that EKF's currently occupy exclusively within strategies for handling noisy control systems. Since the topics of "one-shot trials" and AOT in Secs. VII and VIII are likely of relevance to applications beyond just EWR (as identified), we rounded out our discussion by pointing to the future with desiderata. We hope that others find our insights and comments on the field to be useful and relevant to their own unique estimation applications.

[64]The original version of BLS, originally formulated by Dr. Peter Bancroft, creatively utilized interpolation, which is more amenable to benefit from parallelization.

As justified by the discussion in the latter part of the Appendix versus the statistical idealizations underlying Particle Filters (PF) offered as the descriptive equations throughout [177, Chap. 3], a disconnect is perceived to exist between the very sophisticated statistical arguments underlying both PF's "ideal" and PF's "many reasonable approximations" (based on Bayesian statistics) yet the pseudo-random variates that they are actually capable of numerically generating in implementation are currently not quite able to match the assumed analytical statistical structure as a practical issue. Pseudo-random noise generation does not yet computational match what they really need (even if they assume that it does) neither as a scalar single channel sequence of uniformly distributed variates nor as parallel sequences of the same (as the intermediate step). Failing this, the subsequent conversion of uniform variates obtained to Gaussian variates also misses the mark as a consequence. The variates actually generated computationally lack the expected properties and, as a consequence, the Particles generated for particle flow lack the expected properties as well. The number and nature of the statistical Bayesian arguments invoked in implementing either a KF or an EKF is far fewer than a PF within their supporting theory. Also, the KF mechanization can be derived by seven shorter more robust alternative derivation paths:

- orthogonal projection (onto a subspace spanned by the measurements) [18],
- recursive least squares [18],
- maximum likelihood [18],
- minimum variance [18],
- conditional expectations [18],
- matrix maximum principle [382],
- three martingales approach [382].

The first 5 derivation approaches listed above are demonstrated in [18, pp. 200-212] and the last two are as cited above; and all follow directly and simply without explicitly invoking Bayesian approaches (nor Fisher approaches, nor R. von Mises approaches) but still rely on the underlying rigor provided by the 1933 measure-theory-based approach [65] of A. Kolmogoroff (as mentioned by A. Papoulis in [305, footnote, p. 8]) to probability theory, as used by J. L. Doob (1953), M. Rosenblatt (1967), W. Feller (1966), and M. Loeve (1963). The EKF follows merely as a tractable linearized approximation beyond the ideal situation to approximately match the nonlinear problem being faced.

XIII. Stuff Appended

Various novel approximate approaches for handling nonlinear filtering by being alert to possible improvements to supplant, replace, or augment Extended Kalman Filters or Iterated Extended Kalman Filters, such as (please excuse the somewhat critical view point that I initially convey below as I get my licks in [after waiting for $\tilde{2}0$ years by intentionally delaying my criticisms until early 2019 so that I could not be accused of interfering with or blocking any potentially competitive algorithm developments or its subsequent evolution

[65]The cardinality of random processes and random variables for which pdf's do not exist is greater than the cardinality of those for which pdf's do exist!

after encountering normal, likely temporary roadblocks to later be circumvented]). My criticisms being conveyed here now for the specialists are both from my own past experience and from the past experiences of others earlier on, as identified. My tone will be more mellow further below (beyond this color) at the end of this critique of the 3 different current alternative estimation approaches, as I later also discuss Fred Daums very nice and clear tutorial and summary of these same 3 competitive alternative estimation algorithms within his own overview status-of-the-field discussion that is generally accessible to all, including the non-specialists. My own criticisms precede Daums discussion below (but Daums comments are not criticisms) while mine are definitely criticisms and follow next: -Particle Filtering (PF)-(only if they live up to their hype [which has not completely happened yet]) with careful assessment of their associated respective computational burdens. (PF provides very good tracking accuracy but can seldom be computed in real-time! Moreover, PF is not needed in situations where good mathematical models already exist for the system dynamics [such as objects acted upon solely by central forces, as with gravitational forces] and the process or plant noise is merely a minor consideration [especially when it is absent entirely, as is the case with radar tracking of RVs in the midcourse phase (where the RVs are to be kinetically intercepted) and for tracking all satellites, in general]. The strong suggestion that particle filters should only be used for difficult nonlinear/non-Gaussian problems, when conventional methods fail is made within the Epilogue on page 287, next to the last sentence of the 1st paragraph of the book: Ristic, Branko, Arulampalam, Sanjeev, Gordon, Neil, Beyond the Kalman Filter: particle filters for tracking applications, Artech House, Boston, 2004. (Moreover, pp. 271-283 in Section 12.5.2 discuss Rao-Blackwellized Particle Filters (for additional speeding up of the computations) as well and also discusses this topic further on page 287. Much benefit had already accrued by 2016 in use of Rao-Blackwellized Particle Filters.) Many researchers, like me, are somewhat suspicious when claims are made by other researchers that they used a Particle Filter for a particular application, when adequate linear Kalman Filters had been successfully used for that same particular INS/GPS airborne application for decades (prior to now being applied to an airborne drone, as recently claimed for a PF used by MIT/Draper Laboratory).

Since most recent so-called ground-breaking results for PFs claim orders-of-magnitude improvements over prior original PF implementation/ formulation, which itself increase exponentially in complexity with dimension; so a several orders-of-magnitude improvement/reduction is still an exponentially increasing computational burden overall. Richard Bellman identified a Curse-of-Dimensionally relating to the computational burden of his Dynamic Programming (DP) algorithm (a.k.a., a Viterbi algorithm equivalent), but Bellmans Dynamic Programming came first in 1953 Rand Report: http://www.dtic.mil/dtic/tr/fulltext/u2/074903.pdf) and Curse-of-Dimensionally was not claimed back then for Particle Filtering per se since Particle Filtering did not yet exist. Robert E. Larson (when he was a VP at Systems Control Inc. in Palo Alto, CA) published his approximate simplifications, in IFAC Automatica circa 1976, that Larson invoked for taming the Dynamic Programming CPU burden in order that its implementation would be tractable for practical applications. Since it was an algorithm that differed considerably from that of Particle Filters in structure, the same simplifications do not directly apply nor carry over and other simplifications for PF were needed, as sought by others within the last 20 years. That distinction was not originally clarified by those who were pursuing use of Particle Filters and sought to reduce the Curse-of-Dimensionally but needed to do so in different ways since the precedents used in reducing the computational burden for DP dont strictly apply for PFs.

When process noise is present (as well as the usual sensor measurement noise), because of the Central Limit Theorem and especially the Central Limit Theorem (with weakened hypothesis but similar strong conclusion), the corrupting noises are usually Gaussian in general, and consequently don't require anything special beyond an EKF for successful tracking. The methodology for determining what measurements are needed, as availed from the full rank condition being satisfied from an associated Observability analyses or the weaker Detectability analysis routinely associated with KF and EKF, apparently dont exist for PF since there is no system model specified beforehand for a PF for which these conditions can be tested for compliance. Similarly, full rank conditions for Controllability or the weaker Stabilizibility also cannot be tested for compliance since there is no system model specified beforehand for a PF to be used in such a test.) Without such conditions being satisfied, how can analyzers and implementers be assured of the stability of a PF filter estimator to be assured that it is not diverging from the true state. I have appealed here to the very familiar ample framework that has existed for 4+ decades pertaining to use of available Lyapunov functions to demonstrate stability of KFs (even if the underlying system is unstable, the KF estimator will still appropriately track it well) and approximately, through linearization, for EKFs (and now exactly for some particular special case EKFs using stochastic Lyapunov functions), where such a useful framework apparently does not yet exist for PFs.

Unlike the benign situation for a purely linear Kalman filter (KF) that allows use of a so-designated separate Covariance Analysis (without any system sensor measurements needing to be specified or collected nor any explicit KF estimates needing to be specified or calculated) to set system Error Budgets beforehand that serve as specifications on the actual hardware to be implemented later along with the software algorithms under consideration now so that system accuracy goals may be met [as discussed on pp. 260-266, Sec. 7.4 of Gelb, Arthur (ed.), Applied Optimal Estimation, MIT Press, Cambridge, MA, 1974 and a view also confirmed in Maybeck, P. S., Stochastic Models, Estimation and Control, Vol. 1, Academic Press, NY, 1979], the PF has no such capability since PF Covariances are not available in that same way without PF estimates being simultaneously calculated. However, for EKFs, the situation is similar to that for a PF since a linearization about an EKF estimate is usually needed at each time step in order to calculate the covariances used in a Covariance Analysis for an EKF in order that an approximate Error Budget can be obtained.

When having real-time estimates is not an issue or constraint on the utility or estimator usefulness, such as in some Data Analytics situations where underlying financial models may be completely unknown, a PF may be the best approach to use since there is plenty of data. Recall that Kalman smoothers (nowadays referred to as a Kalman retrodiction) are of three different forms: (1) fixed interval smoothing, (2) single point (in time) smoothing, or (3) fixed-lag smoothing are also NOT real-time algorithms but useful nonetheless such as in closely evaluating missile behavior at the particular event time when a later stage ignites and separates for a multistage rocket. Sometimes a KF smoother is implemented using two Kalman filters, one running forwards in time and the other running backwards in time (where initial conditions for the first and final conditions for the second are made to corroborate correctly (see Backwards Markov Models by Prof. George Verghese [MIT], who obtained his Ph.D. at Stanford University on this topic under Prof. Thomas Kailath, but the aggregate of these two KF's being a smoothing solution that is still NOT real-time). As with PF's, the Kalman filter has also been historically derived by James S. Meditch (Boeing) [Meditch, J. S., Stochastic Optimal Linear Estimation and Control, McGraw-Hill, New York, 1969] using Bayesian arguments similar to those used throughout for PFs, as well as yielding the exact same KF form as independently derived and shown to simultaneously satisfy other important optimality criteria, as discussed immediately below: (The image inserted above [and its references cited therein] is a screen shot from TeK Associates TK-MIP software product. Another missing detail is Ref. 29 Mendel, J. M., Lessons in Estimation Theory for Signal Processing, Communication, and Control, Prentice-Hall PTR, Englewood Cliffs, NJ, 1996.)

With hopes for benefits to PF in parallelization (multi-threaded parallel processing and/or embedded); I continue to follow recent developments in use of Particle Filters but also continue to have concerns about their failing to be real-time (except when they degenerate & essentially collapse into being merely EKFs or KFs [even if they are not explicitly acknowledged as being so]); I also noticed an incompatibility in current hopes for future parallel implementation of a Particle Filter as a further inherent barrier to PF ever being real-time: approaches currently being pursued to accomplish parallel implementation of pseudo-random number generators & maximizing the cycle before they repeat are based on use of Linear Congruential Generator (LCG) algorithmic structures & Mersenne primes to generate variates from a uniform distribution before converting to Gaussian, as needed for PFs to utilize within numerous mini-simulation trials (that invoke use of a RNG within them) before each measurement incorporation step, being a huge CPU burden, somewhat ameliorated by performing sophisticated variants of the Metropolis-Hastings-Gibbs sampling/re-sampling. Donald Knuth only showed what tests LCG passes in The Art of Computer Programming, Vol. 2, Addison-Wesley, 1969. The late George Marsaglia has warned for 30+ years that LCG yields variates that lie in planes, a weakness that has been verified by Profs. Persi Diaconis (Stanford Univ.), P. LEcuyer (Stanford Univ.), and many other current researchers in this area. Even if LCG were perfectly random (which it is not), by attempting a parallel implementation of it risks inadvertent early repetition by a 2nd , 3rd, or 4th LCG, etc. of being somewhere within the same sequence already initiated by an earlier LCG invocation, thus preventing the maximum cycle length from being attained for the following three integer parameters of the computer register: (a, b, and T) involved in the hardware implementation of an LCG [rigor regarding the proper selection of the above mentioned parameters, (a, b, and T), is provided near the bottom of the NEXT screen that pops up after the USER clicks the navigation button at the TOP of this screen labeled TK-MIP for the PC], before premature repetition of the series that is sought to be generated. This is the same reason why, as observed in the rigorous simulations of the 1970s and 1980s, only one LCG should be invoked (but repeatedly) in the implementation of LCG for a standard serial von Neumann machine. My long standing real-time PF concerns (stated above) are now somewhat mitigated in year 2018: (This link is accessible only from LinkedIn by registered LinkedIn users.) However, even this last hypothesized potential path offered from the link immediately above apparently does not yet exist as hardware (even though IBM [which claims to have achieved a working quantum computer in early January 2019], Google, some universities, and several smaller companies are working on it, as identified on the link just offered).

Prof. P. LEcuyer (Stanford Univ.) has proprietary improvements to eventual generation of Gaussian variates as his approach to a pseudo random number (prn) generator. (According to him, he has ostensibly provided these to The MathWorks and to other software developers for a hefty fee.) The conclusion to date is that the older approaches to generating Gaussian variates is not as good as the more recent approaches mentioned here. Apparently missing so far in PF considerations is any attention to the effect of less-than-ideal prn generation of uniformly distributed variates leading to less-than-ideal Gaussian generation. Since the Bayesian-based derivation of the PF strongly utilizes the properties of conditional probability density functions at several critical places, and, all the more, those of Gaussians; it is highly likely that the departure from ideal Gaussianess in what is actually used in implementing a PF will have a significant adverse affect in that PFs performance. One could postulate a type of sensitivity analysis that should be performed for this situation in order to quantitatively gauge the effect on the expected consequential PF performance associated with one or the other of the two approximate approaches to be used in PF implementation, neither being perfect.

–See M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking, IEEE Trans. Signal Processing, vol. 50, pp. 174188, Feb. 2002.

–See R. van der Merwe and E. Wan, Gaussian mixture sigma-point particle filters for sequential probabilistic inference in dynamic state-space models, in Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing (ICASSP), Hong Kong, 2003, pp. 701704.

–Also see http://ieeecss.org/CSM/library/2010/june10/11-

HistoricalPerspectives.pdf

–Please see the excellent discussion of how a PF was implemented for their application: Yozevitch, R., Ben Moshe, B., A Robust Shadow Matching Algorithm for GNSS Positioning, Navigation: Journal of the Institute of Navigation (ION), Vol. 66, No. 2, pp. 95-109, Summer 2015 [Notice that they did not say that their PF was real-time] and some of their pertinent references: (1) Crow, F. C., Shadow Algorithms for Computer Graphics, ACM SIGGRAPH Computer Graphics, Vol. 11, No. 2, pp. 242-248, 1977; (2) Bourdeau, A., Sahmoudi, M., and Tourneret, J. Y., Constructive Use of GNSS NLOS-MUltipath: Augmenting the Navigation Kalman Filter with a 3D Model of the Environment, 15th International Conference on Information Fusion (FUSION), pp. 2271-2276, IEEE, 2012; (3) Thrun, S., Burgard, W., and Fox, D., Probabilistic Robotics, MIT Press, 2005; (4) Muralidharan, K. Khan, A. J., Misra, A., Balan, R. K., and Agarwal, S., Barametric Phone Sensors: More Hype than Hope!, Proceedings of the 15th Workshop on Mobile Computing Systems and Applications, ACM, 2014, 12; (5) DeBerg, M., Van Kreveld, M., Overmars, M., and Schwarzkopf, O. C., Computational Geometry, Springer, NY, 2000.

–Please consider that Observability and Controllability yea/nay tests for linear systems with time-varying System matrix, Obsevation matrix, and System Noise Gain matrix are presented in Bucy, R. S., Joseph, P. D., Filtering for Stochastic Processes with Applications in Guidance, 2nd Edition, Chealsa, NY, 1984 (1st Edition Interscience, NY, 1968).

–A long view reveals that: tractable techniques for handling fractional derivatives for applications have been around for over 40+ years based on Cauchys integral theorem as a representation for derivatives in a Complex Variables comtext (where the order of the derivative is generalized using Cauchys theorem to no longer be restricted to being merely an integer) or by being based on a Fourier integral. Much of the theory and practical applications of fractional derivatives were worked out back then (40+ years ago), as pioneered and published in SIAM by Prof. Tom Osler: Another useful more recent source on this topic is: Kenneth S. Miller and Bertram Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, A Wiley Interscience Publication, John Wiley & Sons, Inc., NY, 1993.

–Other more mundane practical considerations: What will the practical challenges be for documenting Particle Filters for DoD applications in Principles of Operation (POPs) rationales and later in B1s, B2s,amd B3s or in C1s, C2s and C3s without a clear delineation of what the system dynamics matrices and sensor observation matrices and Noise Covariance Matrices are beforehand, as had been established as historical precedents in documentation for Kalman filter or for EKF tracking applications? [By the early 1980s, the aforementioned documentation for DoD tracking, Kalman filtering, and EKF applications had already standardized on conventions for state variable notation that TASC (as also utilized by Peter Maybeck [AFIT] in his 3 Volume textbooks, respectively, in 1979, 1980, and 1981, on this subject) had adopted and popularized as system: d[x(t)]/dt = F x(t) + B u(t) + w(t) and sensor measurements: z(t) = H x(t) + v(t), and independent zero mean white noise covariance matrices corresponding to w(t) and v(t) above, respectively, being: Q(t), R(t), and Kalman gain: K(t); the familiar TASC discrete-time notational conventions were also adopted.] Appropriate DoD documentation was indeed a challenge for Neural Network (NN) applications that still had to be trained to obtain the necessary weights for Perceptrons and multi-layer NNs. DoD documentation was also challenging for Fuzzy Neural Networks. Who or what organization is going to perform the necessary associated IV&V of PF documentation? I wish them good luck!

–The excellent and extremely readable book: Gelb, Arthur (ed.), Applied Optimal Estimation, MIT Press, Cambridge, MA, 1974 had a few errors (beyond mere typos); however, corrections are provided in Kerr, T. H., Streamlining Measurement Iteration for EKF Target Tracking, IEEE Transactions on Aerospace and Electronic Systems, Vol. 27, No. 2, Mar.1991 and in Kerr, T. H., Computational Techniques for the Matrix Pseudoinverse in Minimum Variance Reduced-Order Filtering and Control, in Control and Dynamic Systems-Advances in Theory and Applications, Vol. XXVIII: Advances in Algorithms and computational Techniques for Dynamic Control Systems, Part 1 of 3, C. T. Leondes (Ed.), Academic Press, NY, 1988 (as my expose and illustrative and constructive use of counterexamples).

–See Section 12 of: Kerr, T. H., Exact Methodology for Testing Linear System Software Using Idempotent Matrices and Other Closed-Form Analytic Results, Proceedings of SPIE, Session 4473: Tracking Small Targets, pp. 142-168, San Diego, 29 July-3 Aug. 2001 for some warnings and concerns regarding the direct applicability of Yaakov Bar-Shalom and William Dale Blair (Editors), Multitarget-Multisensor Tracking: Applications and Advances, Vol. III, Artech House Inc., Boston, 2000 for the challenging case of a system with nonlinear dynamics. While Section 12 of the above just cited Kerr paper above was true in 2001, my 7 item comparison then between what was possible for KFs for explicitly linear systems and what was possible for EKFs and IMMs for nonlinear systems now needs modification in 2018, since now a few special case EKFs can be shown to be stable using a stochastic Lyapunov function, as in: Jensen, Kenneth J., Generalized Nonlinear Complementary Attitude Filter, AIAA Journal of Guidance, Control, and Dynamics, Vol. 34, No. 5, pp. 1588-1593 , Sept.-Oct. 2011. [Jensen achieves the big stability breakthrough by providing a proof of this particular EKFs global stability but now states that it possesses almost global asymptotic stability; however, the term almost is required terminology to keep probability theorists and purists happy with the wording of his new claim. Author Jensen attains his new results by utilizing appropriate stochastic Lyapunov functions (proper handling of such is due to Prof. Emeritus Harold J. Kushner, Brown Univ.). I dont know whether Jensen was the first to achieve this new result?] Please forgive me as I use the following two images to explain and clarify the technical term almost that Jensen was obligated to invoke: (The two images inserted above [and its references cited therein] are screen shots from TeK Associates TK-MIP software product.)

–I am aware that Fred Daum, Jim Huang, and Mike Hough

(Raytheon) have jointly published a recent paper on the use of a PF for Strategic Early Warning Radar tracking of Reentry Vehicle targets but I have not yet seen it! I look forward to viewing it soon. Maybe it will calm my qualms.

-The solving of Partial Differential Equations (PDEs) has been described by practitioners and others as an infinite-dimensional problem because of its numerical and computational complexity. It is reasonably well known that PDEs describe the time evolution that is needed to specify the associated probability density function underlying continuous-time optimal estimation for both linear and nonlinear systems. The important fundamental PDE describing this is known as the Kolmogorov equation (where there is, in general, a forwards (in time) and a backwards (in time) Kolmogorov equation that describe the statistical estimation situation in continuous-time) and the former is also known as the Fokker-Planck equation arising in optimal statistical estimation for both the case of the system being linear and the noises Gaussian (which is very tractable since it degenerates and simplifies nicely to the standard Kalman Filter) and the general nonlinear case (usually very intractable and computationally tedious for all except the simplest of problems that, frequently, are neither realistic nor practical for most applications). Both of these PDEs deal with the time evolution of probability density functions (pdfs) or information flow. Also see Pavel B. Bochev, Max G. Gunzburger, Least-Squares Finite Element Methods, Applied Mathematical Sciences, Vol. 166, Springer Science + Business Media, LLC, NY, 2009. There is a PDE textbook that was published within the last 15 years that routinely invokes use of scalar homotopy and log-homotopy, and. further, has beautiful color images of associated particle flows, as are reminded there to be standard tools and methodologies for handling solutions of PDEs. However, in general, PDEs can not be solved in real-time! Sometimes speeded-up videos are shown to the convey the trend of the solution process to an audience. However, within the following paper: Daum, F. E., Exact finite-dimensional nonlinear filters, IEEE Transactions on Automatic Control, Vol. 31, No. 7, pp. 616-622, Jul. 1986, a novel, insightful, and creative method was developed for decomposing the solution of the important PDE, described above, into two parts: the 1st part was a large computational burden to be solved off-line beforehand and stored until needed; the 2nd part is to be solved on-line in real-time. The two parts, when put together, constituted a solution to the PDE described in the preceding paragraph and yielded the exact optimal estimator or optimal filter for the nonlinear case. However, a constraint on the 1st part is that the times at which the measurements arrive was needed beforehand too! That is usually only the case for navigation applications with periodic updates such as by Omega (now defunct), Loran-C (now defunct but maybe coming back as eLORAN to help GPS recognize and compensate for GPS spoofing), or GPS and/or GNSS satellites in an unjammed benign environment; otherwise, the aforementioned navigation aid (i.e., are not deterministic in the time at which they occur and the exact time of an external position fix is not known beforehand because of complicating factors such as atmospheric interference (e.g., atmospheric scintillation for EWR) ; thus computational calculation of the 1st part beforehand is stymied! Radar applications seldom involve radar sensor measurements arriving at a strictly periodic rate that is known beforehand since targets are in motion and sometimes the radar platform is too, consequently, the round trip time of the radar pulse varies from transmitter to receiver even if the transmitter rate is periodic at a constant Pulse Repetition Frequency (PRF). Moreover, there is an historical precedent in the 1960s and early 1970s to avoid pre-calculated KF gains, as performed by Dr. Hy Strell and Norm Zabb (Sperry Systems Management as SSBN navigation work for SP-2413) which found pre-calculated KF gains satisfactory for simulations and test of concept for the Ships Inertial System (SINS) utilizing a 7-state STAtistical Reset (STAR) Kalman filter on a surface ship used strictly for testing but not satisfactory for the real world application for SSBNs at sea because the external position fixes were seldom available exactly as pre-planned in attempting to synchronize to the pre-computed filter gains. Admittedly, this example is from a different application area entirely but it is more benign in general than that for radar applications. When there are problems within the more benign linear situation of navigation, the same problems will likely plague the slightly more challenging nonlinear situation of radar for the same reasons! However, more recent results: Schmidt, G. C., "Designing nonlinear filters based on Daum's theory," AIAA Journal of Guidance, Control, and Dynamics, Vol. 16, No. 2, pp. 371-376, Mar.-Apr. 1993, apparently offer a way around the limitation that I mentioned (see Schmidt's admission in his conclusion section of having obtained mixed results). It was expanded upon and generalized by others too and they cite his work as their starting point. [It is, perhaps, worth mentioning in passing that there are two special cases of nonlinear filters that have an optimal estimator that is finite dimensional and the mean and variance are sufficient statistics, as in the purely linear system and Gaussian noises case: that of Benes and that of Daum (and subsequently by many others), but there are no realistic applications yet to which these results apply. However, they are useful for the valuable insights that they provide.]

-Unscented Kalman Filter also known as (a.k.a.), the Oxford Filter, a.k.a. the Sigma-Point Filter): Our historical apprehension regarding the Unscented Kalman Filter (UKF) is because of the presence of an unexplained factor (or unconstrained free real scalar parameter [not necessarily an integer], possibly positive, negative, or time-varying, at the whim of the analyst/implementer) that can serve as an expanding or contracting twiddle factor in the denominator of the gain expression that is consequentially inherited by the covariance equations; which, for linear systems, still appropriately computes the exact covariance associated with any approximate Gain that is used in an accompanying estimation filter (even if it is not the optimal Kalman Gain), as clearly explained on page 234 in Eq. 5.4.18 and further emphasized in the last sentence following Eq. 5.4.22 in: Brown, Robert Grover, Hwang, Patrick Y. C., Introduction to Random Signals and Applied Kalman Filtering, 2nd Edition, John Wiley & Sons, Inc., New York, 1983. The numerical comparison in Julier, S. J., Uhlmann, J. K., and Durrant-Whyte, H. F., A New Method for the Nonlinear

Transformation of Means and Covariances in Filters and Estimators, IEEE Trans. on Automatic Control, Vol. 45, No. 8, pp. 477-482, May 2000 of UKF vs. EKF performance appears to be somewhat contrived since actual EKF practitioners would either take more frequent measurement fixes to supplement tracking the objects trend and/or better pose the target model in the first place to take into account its known anticipated planar motion about a circular track of constant radius about the origin by merely posing the problem in (rho,theta) polar coordinates [with known constant angular velocity] as the two states of interest, or if the constant angular velocity is unknown beforehand, then including this unknown parameter as an additional state to be estimated using parameter identification techniques or by using an approach that came later: Souris, G. M., Chen, G., Wang, J., Tracking an Incoming Ballistic Missile Using an Extended Interval Kalman Filter, IEEE Trans. on Aerospace and Electronic Systems, Vol. 33, No. 1, pp. 232-240, Jan. 1997, but Julier, S. J., Uhlmann, J. K., and Durrant-Whyte, H. F. do invoke conditions that are impossible to check beforehand e.g., [Julier, S. J. et al, op. cit., Eq. 2] since probability measure for x(k) is unknown); unconventional use of calculated covariance to account for nonlinear measurement equation and associated unconventional assumption of mean being zero and an unconventional proposed handling if mean is not zero (by their saying it can be shifted, but mean is in fact unknown so one can not know beforehand how much it should be shifted by, so user is thus stymied in trying to proceed as they recommend [Julier, S. J. et al, op. cit., Sec. 4]; UKF also utilizes mini-simulation trials before each measurement incorporation step (but not as many as a PF would require).

The above mentioned Global Lipschitz condition is much stronger in contrast to the mere continuity condition of the system dynamics being a sufficient condition on the nonlinear system dynamics for a deterministic nonlinear differential equation to have a solution. For uniqueness of the solution of the latter, only a local Lipschitz condition need be satisfied. The need for a global Lipschitz condition is discussed in Bucy, R. S., Joseph, P. D., Filtering for Stochastic Processes with Applications in Guidance, 2nd Edition, Chealsa, NY, 1984 (1st Edition Interscience, NY, 1968) (and is also discussed in: Kerr, T. H., Applying Stochastic Integral Equations to Solve a Particular Stochastic Modeling Problem, Ph.D. Thesis in the Department of Electrical Engineering, University of Iowa, Iowa City, Iowa, January 1971, where a detailed proof is provided on pp. 188-213 utilizing Ito integrals for stochastic integrands).

So the numerical comparison between Julier, S. J., Uhlmann, J. K., and Durrant-Whyte, H. F., A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators, IEEE Trans. on Automatic Control, Vol. 45, No. 8, pp. 477-482, May 2000 is less of how well the UKF filter performed (as they claimed) but more about how bad an EKF can perform if it uses an inappropriate or bad model for the system. This should be NO surprise! A more appropriate posing of the estimation problem on a circle is: Li, J. T.-H. Lo and A. S. Willsky, Estimation for Rotational Processes with One Degree of Freedom-Part 1, IEEE Trans. on Automatic Control, Vol. 20, No. 1, pp. 10-21, Feb. 1975. 5_pub_IEEE.pdf

Equal time here now for some views of others that are pro-use of Unscented Filter or Sigma-Point Filters in specific applications:

-Sigma-Point Filtering for Integrated GPS and Inertial Navigation:

-Sigma-Point Filters in Robotic Applications:

-Sigma-Point Kalman Filters for Nonlinear Estimation and Sensor-Fusion - Applications to Integrated Navigation:

-Robot Mapping Unscented Kalman Filter:

-Please see: Daum, F. E., Nonlinear filters: beyond the Kalman filter, IEEE AandE Magazine, Vol. 20, No. 8, pp. 57-69, Sept. 2005 for an excellent, clear discussion of the three estimation algorithms that I have just critiqued above. My only complaint here is that Daum seems to have overlooked or missed the earlier Lie Algebra results of : Li, J. T.-H. Lo and A. S. Willsky, Estimation for Rotational Processes with One Degree of Freedom-Part 1, IEEE Trans. on Automatic Control, Vol. 20, No. 1, pp. 10-21, Feb. 1975 as a precedent. [Willsky and Lo explicitly handle estimation on a circle, SO(2), rather than estimation on a sphere, SO(3), as NASAs F. Landis Markley, et al deal with in their extensive NASA survey and comparison between approaches and techniques. However, Willsky and Lo are particularly lucid in their development and exposition and, moreover, within the last sentence of their conclusion, provide specifics of their suggested generalization to estimation results on arbitrary Abelian Lie groups, such as SO(3).] 5_pub_IEEE.pdf Also see: Lo, J. T.-H. and Willsky, A. S., Stochastic Control of Rotational Processes with One Degree of Freedom, SIAM Journal on Control, Vol. 13, No. 4, 886ff, July 1975. Another aspect that Daum may have, perhaps, overlooked is the strong applicability of Lie Algebras well beyond mere separation-of-variables for PDEs or ODEs, as in: Wu, Y., Hu, X., Hu, D., Li, T., and Liam, J., Strapdown Inertial Navigation System Algorithms Based on Dual Quaternions, IEEE Trans. on Aerospace and Electronic Systems, Vol. 41, No. 1, pp. 110-132, Jan. 2005 and Savage, P. G., A Unified Mathematical Framework for Strapdown Algorithm Design, AIAA Journal of Guidance, Control, and Dynamics, Vol. 29, No. 2, pp. 237-249, March-April 2006 and Bernard Friedland, Analysis of Strapdown Navigation Using Quaternions, IEEE Transactions on Aerospace and Electronic Systems, Vol. 14 , No. 5, pp. 764-768, Sept. 1978 and Bell, D. J., Manifolds and Lie Algebras, in Mathematics of Linear and Nonlinear Systems: for Engineers and Applied Scientists, Clarendon Press, Oxford, UK, 1990.

JUST BECAUSE APPROACHES ARE NEW AND DIFFERENT FROM BEFORE DOES NOT MEAN THAT THEY ARE BETTER (& VICE-VERSA)!

Appendix
Summary of Kalman Filter-Like Algorithms

As promised, here is an overview summary of the salient aspects of a Kalman filter in Fig. 9, with structural details and consequences that can be exploited to an advantage in Figs. 10, 11. In my opinion, the best discussion of the order or sequence of major operations constituting a correct KF implementation is only in [359, Figs. 5.9, 6.1].

Fig. 9. Essential Aspects of using a Kalman Filter (as explained in detail in [35], [45], [51], [52], and, especially, as offered in [58] as benchmark validation cross-checks for software developers)

The optimal estimate is always the conditional expectation for both linear and nonlinear systems and with Gaussian or Non-Gaussian noises being present. Only for linear system structures and only for exclusively additive Gaussian noises being present does the Kalman filter exactly provide such an estimate (using only a finite dimensional linear filter structure with a time-varying gain) that processes the measurements as inputs and provides this optimal estimate as output (along with its computed covariance). An important aspect is that a Kalman filter can be implemented in real-time. When the underlying system and or measurement model is nonlinear or when the noises are non-Gaussian (or as innocuous as the initial condition not being Gaussian and independent of the other aforementioned noises [138] cf. [115]), then the structure of the filter needed to obtain an optimal estimate is, in general, nonlinear and infinite-dimensional. Since an infinite dimensional filter would be impractical to implement for real-time use, approximations are invoked and use of an Extended Kalman Filter is one such approach that yields a best linear estimate (i.e., a linear function of the measurements) that frequently provides enough accuracy to satisfy the application at hand and also involves only computations that are expediently real-time. A nonlinear estimator could ostensibly be more accurate (but would likely be impractical to pursue further because of a likely exorbitant computational burden that would preclude being real-time). Novel EKF generalizations for some non-Gaussian noises are [204]-[206].

From Fig. 9, it is seen that when the original system is linear (possibly time-varying) and all noises are WGN, then the linear system structure of the available measurements feeding into a Kalman filter (with time-varying gains) for further processing preserves Gaussianess throughout since linearity is preserved throughout. Gaussian processes are completely characterized by just their mean and variance. The Kalman filter is just an efficient computational algorithm for generating both of these two important monents in real-time. Use of Extended Kalman filters (EKF) is one way to attempt an approximation for handling nonlinear and or non-Gaussian applications in the same way. Many engineering refinements to a basic EKF are described in this paper as relevant to EWR use.

Both Refs. [175], [176] present insightful cutting edge results from probability and statistics, already laid out within a Kalman filtering context and tailored to estimation applications, that pertain specifically to extending KF applicability to situations involving noises from exponential families and in seeking out sufficient statistics that, by capturing the available information in the most compact way, minimizes the complexity incurred in algorithm implementation. See [180]. Jerry Mendel and Max Nikias trailblazed with many published papers and a book (along with a short course in the 1990's through their company: Circuits and Systems Inc.) on α-stable noises and "Stack filters" (and on sorting out bispectra and trispectra approaches for multidimensional/multichannel real and/or complex random processes in engineering systems and we observe here that are also relevant to RV target discrimination) yet their breakthrough work done decades earlier is not referenced in recent papers on these same topics [214], [215]. Other rigorous approaches that appear to be very useful arise in [236]-[241]. Ref. [362] is an excellent book [66] (with chapters coauthored by some of my TASC cohorts from the 1970's).

While Extended Kalman Filters are model-based (as is the Kalman filter for the purely linear case), both are generally applied to state variable representations of the system, which for radar target tracking, usually models the target dynamics (and any maneuvers anticipated for the particular application). A generalization of the standard state variable representation is to model the system in terms of so-designated *descriptor* systems (DS), where simplifications frequently accrue that reduce the computational burden associated with implementation of the appropriate estimator corresponding to the descriptor structure of the dynamics model [198]-[200], [227], [247]-[249], [277], [281]. When these types of descriptor model representations are applicable and are utilized as a more natural fit of the physical system to its software implementation with estimation algorithms, the block-by-block connection diagrams such as those used by The MathWorks' Simulink©(or IBM's CSMP©), as a throw-back to the technology of 40 years ago are no longer necessary, as was originally clearly explained in [209], [210]. Such a descriptor systems approach avoids or side steps the need for special high CPU overhead algorithms for integrating "stiff differential equations" (typically done for block-by-block representations via use of Gear's implicit integration routines [211]), as touted for MatLab©/Simulink©in [212]. Descriptor system representations decompose the short circuit-like "fast loop" or short time constant into an algebraic equation devoid of any dynamics (i.e., integrators) along with a lower dimensional residual dynamics representation. Both of these operations reduce the complexity in adequately representing such otherwise "stiff" systems and, moreover, frequently obviate any need to use special Gear-type implicit integration algorithms altogether in these particular situations since simpler Runge-Kutta predictor-corrector algorithms then frequently suffice. However, in systems that have a residual wide range of effective time constants (even after algebraic loops are removed) or fast inner and slow outer control loops present (as with fighter aircraft guidance laws) or because of

[66] In their the preface they poke fun at some of the current developers of PF, who spend an inordinate amount of time depicting pictures of famous mathematicians of the past rather than more thoroughly explaining what they have done and why, by saying "we have similar pictures too".

Fig. 10. Structural Aspects of a System's Model can be exploited to an Advantage-Part I

Fig. 11. Structural Aspects of a System's Model can be exploited to an Advantage-Part II

the presence of multiple sampling rates, Gear-type integration may sometimes still be needed, but now needed less frequently with descriptor system decomposition and may be invoked more parsimoniously (since it is a larger CPU-burden), as an exception rather than as the general rule.

Background:

- Kalman Filters (KF) are used in ALL modern navigation systems and in ALL modern radar target trackers and elsewhere.
- KF processing is needed for situations where sensor measurement data (and its associated system) are noise-corrupted.
- The Kalman Filter ameliorates the effects of the noises and significantly improves assessment of whats going on based on processing on-line sensor measurement data!
- KF methodology enables imposing rational system SPECS (before hardware is built) using KF to specify the "error-budget".
- KF is an in-place algorithm that is of order n3 in required processing time and n2 in memory size, where n is the state size of the state-variable mathematical model used to completely describe each application at hand.
- Most KF applications need to process in real-time to keep up with the stream of sensor measurement data.

Realities:

1) Kalman Filters (KF) are in widespread use in a variety of applications.
2) The Kalman Filter ameliorates the effects of noises and significantly improves users assessment of "whats going on" from sensor data!
3) Most Kalman Filter applications need to process data in real-time.
4) For KF applications: Linear Time Invariant (LTI) case = EASY!
5) For KF applications: Nonlinear or Time-varying cases = HARD!
6) Real-world applications are always nonlinear and usually time-varying! For nonlinear situations, KF must be generalized to an EKF (of same structure) merely by linearizing about previous time estimate.
7) For KF: merely LTI is what students usually learn and practice in school.

Significant Theoretical Advances in Probability and Statistics: The conclusion of the Central Limit Theorem (CLT) [and it's variations]: That sums of random variables that are independent and identically distributed (iid) with finite variance go to Gaussian in distribution (as number of terms increases)

To illustrate that mere sums of idd are not necessarily Gaussian, consider the sums of Cauchy variates: sums of Cauchy are always Cauchy (even an infinite number of them) and although it is bell-shaped, its tails are so fat (i.e., "platykurtic" [t.e., fat-tailed like a platypus]) that all its moments are infinite and so do not exist. Cauchy can arise physically as the ratio of independent Gaussians or as their arctan [305, p. 199]. CLT requires that the variance of the contributing variates be finite in order to invoke the desirable conclusion that the sum is Gaussian in distribution. Insights & pointers to the more recent CLT generalizations no longer requiring iid are available in [361, Sec. 9.3]. In particular, the Lindberg-Feller Theorem on [361, p. 239] does not require the contributing variates to be identically distributed (citing Woodroofe, 1975, p. 255) and [361, Sec. 9.3.2] reveals the "dependent variable case" (citing Moran, p. 403 & Serfling, pp. 1158-75, both 1968). Despite the pathologies of Cauchy, tractable approximate Kalman-like estimatators have been recently derived for situations when the additive noise is Cauchy [395].

We are aware of **recent thinking and explicit numerical comparisons regarding the veracity of uniform (pseudo-)random number generators (RNGs)** as, say, reported in [368] with prescribed remedies. (Please see L'Ecuyer's article [and Web Site: http://www.iro.umontreal.ca/ lecuyer] for explicit quantifications of RNG's for Microsoft's Excel ©and for Microsoft's Visual Basic ©as well as for what had been available in Oracle/Sun's JAVA ©.) Earlier warnings about the failings of many popular RNG's have been offered in the technical literature for the last 35 years by George Marsaglia, who, for quite awhile, was the only "voice in the wilderness" alerting and warning analysts and software implementers to the problems existing in many standard, popular (pseudo-)RNG's since they exhibit significant patterns such as "random numbers falling mainly in the planes" when generated by the Linear Congruential Generator (LCG) method of [371].

Prior to these cautions mentioned above, the prevalent view regarding the efficacy of RNGs for the last 35 years had been conveyed in [357], which endorsed use of only the linear congruential method consisting of a iteration equation of the following form: $x_{n+1} = ax_n + b(modT)$, starting with $n = 0$ and proceeding on, with x_0 at $n = 0$ being the initial seed, with specific choices of the three constant parameters a, b, and T to be used for proper implementation with a particular computer register size being specified in [357]; however, variates generated by this algorithm are, in fact, sequentially correlated with known correlation between variates s-steps apart according to: $\rho = [(1 - 6(\beta_s/T)(1 - (\beta_s/T))]/a_s + \mu$, where this expression along with the constant parameters appearing above are defined and explained on the first page of [371, Sec. 26.8] as matched to a specific CPU register bit size. Marsaglia had found this Linear Congruential Generator approach to be somewhat faulty, as mentioned above. The problems with existing RNG's were acknowledged publicly by mathematicians in a session at the 1994 meeting of the American Association for the Advancement of Science. A talk was presented by Persi Diaconis [67] (Prof. of Statistics and Mathematics, Stanford Univ.) "on a minor scandal of sorts" (this was their exact choice of words, as reported in the AAAS publication, **Science**) concerning the lack of a well-developed theory for random number generators. For most applications, random numbers are generated by numerical algorithms whose outputs, when subjected to various tests, appear to be random. Diaconis described the series of tests for randomness proposed by George Marsaglia in the mid-1980's; all existing generators

[67] Who has also published extensively on deep understandings of the Metropolis-Hastings-Gibbs sampling/resampling as well as on many other statistical topics that are relevant here.

of the time FAILED at least one of these tests. Also see [374]. Prof. L'Ecuyer offers improvements over what was conveyed earlier by Prof. Persi Diaconis and claims to be up-to-date with modern computer languages and constructs. Even better results are reported for a new hardware-based simulation approach in [369].

All sources recommend use of historically well-known Monte-Carlo simulation techniques to emulate a Gaussian vector random process that possesses the matrix autocorrelation function inputted as the prescribed symmetric positive semidefinite WGN intensity matrix. The Gaussianess that is also the associated goal for the generated output process may be obtained by any one of four standard approaches listed in [371, Sec. 26.8.6a] for a random number generator of uniform variates used as the input driver. However this approach specifically uses the technique of summing six independent uniformly distributed random variables (r.v.) to closely approximate a Gaussianly distributed variant. The theoretical justification is that the probability density function (pdf) of the sum of two statistically independent r.v.'s is the convolution of their respective underlying probability density functions. For the sum of two independent uniform r.v.'s, the resulting pdf is triangular; for the sum of three independent uniform r.v.'s, the resulting pdf is a trapezoid; and, in like manner, the more uniform r.v.'s included in the sum, the more bell shaped is the result. The Central Limit Theorem (CLT) can be invoked, which states that the sums of independent identically distributed (i.i.d.) r.v.'s goes to Gaussian (in distribution). The sum of just six is a sufficiently good engineering approximation for practical purposes. A slight wrinkle in the above is that supposedly ideal Gaussian uncorrelated white noise is eventually obtained from operations on independent uniformly distributed random variables, where uniform random variables are generated via the above standard Linear Congruential Generation method, with the pitfall of possessing known cross-correlation, as already discussed above. This cross-correlated aspect may be remedied or compensated for to an extent (since it is known) via use of a Choleski decomposition to achieve the theoretical ideal uncorrelated white noise, a technique illustrated in [372, Ex. 2, pp. 306-312], which is, perhaps, comparable to what is also reported later in [373].

An incompatibility in current hopes for future parallel implementation of a Particle Filter as a further inherent barrier to PF ever being real-time: all approaches currently being pursued to accomplish parallel implementation of pseudo-random number generators [68] and maximizing the cycle before the generated variates repeat are based on using Linear Congruential Generator (LCG) and Mersenne primes to generate variates from a uniform distribution before converting to Gaussian, as needed for PF's to utilize within numerous "mini-simulation trials" (that invoke use of a RNG within them), being a huge CPU burden, ameliorated by performing sophisticated variants of the "Metropolis-Hastings-Gibbs" sampling/re-sampling. Prof. Donald Knuth (Stanford Univ.) only showed what tests LCG passes in [357]. George Marsaglia has warned for 30+ years that LCG yields variates with a repetition pattern that "lie in planes", a weakness that has been verified by Profs. Persi Diaconis, P. L'Ecuyer, and many others [374]-[377].

[68] It has been observed that the code words or catch phrase that specialist in this area use is that parallelization of random number generators is "embarrassingly obvious". As discussed here (and elsewhere), it apparently is not.

I inadvertently uncovered an incompatibility in current hopes for future parallel implementation of a Particle Filter as a further inherent barrier to PF ever being real-time. [Inadvertent since I was merely summarizing why only one Random Number Generator (RNG) had historically been used within MIMO Monte-Carlo system simulations to avoid improper cross-correlation of noise realizations generated and to maximize the period before any RNG outputs repeat. The relevance of these same observations to PF thus become obvious because PF's utilize numerous "mini-simulation" trials (that invoke use of a RNG within them) before each "measurement incorporation" step, being a huge CPU burden, ameliorated by performing sophisticated variants of the "Metropolis-Hastings-Gibbs" sampling/re-sampling. Others had speculated that this aspect could be implemented in parallel. My insightful connection now bashes this hopeful speculation by reminding of practicality constraint (to avoid premature repeating of noise realizations) that arises unless restricted to use of only one RNG.]

Acknowledgment

Research was funded by TeK Associates' IR&D Contract No. 07-102 associated with the development of its PC-based commercial software product: $\mathbf{TK-MIP}$© for Kalman filter estimation/Extended Kalman Filter/Iterated Extended Kalman Filter/Least Mean Square/Linear Quadratic Gaussian/Loop Transfer Recovery optimal feedback control, and its related built-in tutorials and illustrative numerical examples on the relevant supporting technology, both within the software and as discussed at www.TeKAssociates.biz.

References

[1] Julier, S. J., Uhlmann, J. K., and Durrant-Whyte, H. F., "A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators," *IEEE Trans. on Autom. Contr.*, Vol. 45, No. 8, May 2000, pp. 477-482.

[2] Lefebvre, T., Bruyninckx, H., De Schutter, J., "Comment on 'A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators'," *IEEE Trans. on Autom. Contr.*, Vol. 47, No. 8, Aug. 2002, pp. 1406-1408.

[3] Ito, K., Xiong, K., "Gaussian Filters for Nonlinear Filtering Problems," *IEEE Trans. on Autom. Contr.*, Vol. 45, No. 8, May 2000, pp. 910-927.

[4] Van Zandt, J. R., "A More Robust Unscented Transform," *Proceedings of SPIE, Session 4473: Tracking Small Targets*, San Diego, CA, 29 Jul.-3 Aug. 2001.

[5] Covariance Intersection Web Sites:
- http://www.ait.nrl.navy.mil/people/uhlmann/CovInt.html
- http://www.cse.sc.edu/research/dag/html/cs_scendf/sld001.htm
- http://www.fusion2002.org/nielson_document.htm

[6] Chen, L., Arambel, P. O., Mehra, R. K., "Estimation Under Unknown Correlation: Covariance Intersection Revisited," *IEEE Trans. on Auto. Contr.*, Vol. 47, No. 11, Nov. 2002, pp. 1879-1882.

[7] Gordon, N. J., Salmond, D. J., Smith, A., "Novel Approach to Nonlinear-Gaussian Bayesian State Estimation," *Proceedings of the IEE*, Part F, Vol. 140, No. 2, Apr. 1993, pp. 107-113.

[8] Daum, F., Huang, J., "The Curse of Dimensionality for Particle Filters," *Proceedings of IEEE Aerospace Conference*, Big Sky, Montana, 8-15 Mar. 2003.

[9] Farina, A., Ristic, B., Timmoneri, L., "Cramer-Rao Bound for Nonlinear Filtering with $P_d < 1$ and its Application to Target Tracking," *IEEE Trans. on Signal Processing*, Vol. 50, No. 8, Aug. 2002, pp. 1916-1924.

[10] Farina, A., Ristic, B., Benvenuti, D., "Tracking a Ballistic Target: Comparison of Several Nonlinear Filters," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 38, No. 3, Jul. 2002, pp.854-867.

[11] Fitzgerald, R. J., "Effects of Range-Doppler Coupling on Chirp Radar Tracking," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 10, No. 4, Jul. 1974, pp. 528-532.

[12] Fitzgerald, R. J., "On Reentry Vehicle Tracking in Various Coordinate Systems," *IEEE Trans. on Autom. Contr.*, Vol. 19, No. 5, Jul. 1974, pp. 581-582.

[13] Daum, F. E. and Fitzgerald, R. J., "Decoupled Kalman Filters for Phased Array Radar Tracking," *IEEE Trans. on Autom. Contr.*, Vol. 28, No. 3, Mar. 1983, pp. 269-283.

[14] Kerr, T. H., and Satz, H. S., "Evaluation of Batch Filter Behavior in comparison to EKF," TeK Associates, Lexington, MA, (for Raytheon, Sudbury, MA), 22 Nov. 1999. http://http://www.tekassociates.biz/bmdo.pdf

[15] Satz, H. S., Kerr, T. H., "Comparison of Batch and Kalman Filtering for Radar Tracking," *Proceedings of* 10^{th} *Annual AIAA/BMDO Conference*, Williamsburg, VA, 25 Jul. 2001 (Unclassified) http://www.dtic.mil/cgi-bin/GetTRDoc?Location=U2&doc=GetTRDoc.pdf&AD=ADP011192

[16] Kerr, T. H., "A New Multivariate Cramer-Rao Inequality for Parameter Estimation (Application: Input Probing Function Specification)," *Proc. of IEEE Conf. on Decision and Control*, Phoenix, AZ, Dec. 1974, pp. 97-103.

[17] Taylor, J. H., "Cramer-Rao Estimation Error Bound Analysis for Nonlinear Systems," *IEEE Trans. on Autom. Contr.*, Vol. 24, No. 2, Apr. 1979, pp. 343-345.

[18] Jazwinski, A. H., *Stochastic Processes and Filtering Theory*, Academic Press, N.Y., 1970. (CRLB for $Q = 0$.)

[19] Maybeck, P. S., *Stochastic Models, Estimation, and Control*, Vol. 2, Academic Press, N.Y., 1982.

[20] Balakrishnan, A. V., *Kalman Filtering Theory*, Optimization Software, Inc., NY, 1987. (CRLB for $Q = 0$.)

[21] Kerr, T. H., "Status of CR-Like Lower bounds for Nonlinear Filtering," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 25, No. 5, Sept 1989, pp. 590-601 (reply in Vol. 26, No. 5, Sep. 1990, pp. 896-898). Also see: http://www.tekassociates.biz/branko_05.pdf on p. ? for historical perspective.

[22] Kerr, T. H., "Cramer-Rao Lower Bound Implementation and Analysis for NMD Radar Target Tracking," TeK Associates Tech. Rpt. No. 97-101 (for MITRE), Lexington, MA, 26-30 Oct. 1997.

[23] Kerr, T. H., "Cramer-Rao Lower Bound Implementation and Analysis: CRLB Target Tracking Evaluation Methodology for NMD Radars," MITRE Tech. Report, Contract No. F19628-94-C-0001, Project No. 03984000-N0, Bedford, MA, Feb. 1998.

[24] Kerr, T. H., "Developing Cramer-Rao Lower Bounds to Gauge the Effectiveness of UEWR Target Tracking Filters," *Proceedings of* 7^{th} *Annual AIAA/BMDO Technology Readiness Conference and Exhibit*, Session 9: Innovative Science and Technology, paper No. 09-04, Fort Carlson, Colorado Springs, CO, 3-6 Aug. 1998 (Unclassified). [This provides details of the underlying theory and implementation (but the data initially supplied by MITRE from TDSAT in 1997 for our input was faulty). Please see [53] for proper resolution of this input data issue.]

[25] Tichavsky, P., Muravchik, C., Nehorai, A., "Posterior Cramer-Rao Bounds for Adaptive Discrete-Time Nonlinear Filtering," *IEEE Trans. on Sig. Proc.*, Vol. 46, No. 5, May 1998, pp. 1386-1396. (First rigorous CRLB for $Q \neq 0$.)

[26] Souris, G. M., Chen, G., Wang, J., "Tracking an Incoming Ballistic Missile Using an Extended Interval Kalman Filter," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 33, No. 1, Jan. 1997, pp. 232-240.

[27] Chen, G., Wang, J., Shieh, L. S., "Interval Kalman Filtering," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 33, No. 1, Jan. 1997, pp. 250-259.

[28] Payne, A. N., "Observability Problem for Bearings-Only Tracking," *International Journal of Control*, Vol. 49, No. 3, 1989, pp. 761-768.

[29] Zhou, Y., Sun, Z., "Observability Analysis of Single Passive Observer," *Proceedings of the 1995 IEEE National Aerospace and Electronics Conference*, NY, 1995, pp. 215-219.

[30] Murphy, D. J., "Noisy Bearings-Only Target Motion Analysis," Ph.D. Dissertation, Dept. of Electrical Engineering, Northeastern University, Boston, MA, 1969.

[31] Kirubarajan, T., Bar-Shalom, Y., Lerro, D., "Bearings-Only Tracking of Maneuvering Targets using a Batch-Recursive Estimator," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 37, No. 3, Jul. 2001, pp. 770-780.

[32] Kerr, T. H., "Angle-Only Tracking," slide presentation for Reentry Systems Program Review at Lincoln Laboratory, Lexington, MA, 10 Jan. 1989 (unclassified).

[33] Kerr, T. H., "Assessing and Improving the Status of Existing Angle-Only Tracking (AOT) Results," *Proceedings of the International Conference on Signal Processing Applications & Technology (ICSPAT)*, Boston, MA, 24-26 Oct. 1995, pp. 1574-1587.

[34] Powell, T. D., "Automated Tuning of an Extended Kalman Filter Using the Downhill Simplex Algorithm," *AIAA Jour. of Guid., Control, and Dyn.*, Vol. 25, No. 5, Sep./Oct. 2002, pp. 901-909. (cf. [70].)

[35] Kerr, T. H., "Streamlining Measurement Iteration for EKF Target Tracking," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 27, No. 2, Mar. 1991, pp. 408-420 (correction in Nov. 1991). http://www.tekassociates.biz/KerrStreamliningMeasurementIter.pdf

[36] Gura, I. A., "Extension of Linear Estimation Techniques to Nonlinear Problems," *The Journal of Astronautical Sciences*, Vol. XV, No. 4, Jul./Aug. 1968, pp. 194-205.

[37] Liang, D. F., and Christensen, G. S., "Exact and Approximate State Estimation for Nonlinear Dynamic Systems," *Automatica*, Vol. 11, 1975, pp. 603-612.

[38] Liang, D. F., "Exact and Approximate Nonlinear Estimation Techniques," in Advances in the Techniques and Technology in the Application of Nonlinear Filters and Kalman Filters, edited by C. T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph No. 256, Noordhoff International Publishing, Lieden, Chap. 2, 1981.

[39] Widnall, W. S., "Enlarging the Region of Convergence of Kalman Filter Employing Range Measurements," *AIAA Journal*, Vol. 11, No. 3, Mar. 1973, pp. 283-287.

[40] Baheti, R. S., O'Halloron, D. R., Itzkowitz, H. R., "Mapping Extended Kalman Filters onto Linear Arrays," *IEEE Trans. on Autom. Contr.*, Vol. 35, No. 12, Dec. 1990, pp. 1310-1319.

[41] Chui, C. K., Chen, G., Chui, H. C., "Modified EKF and Real-Time Parallel Algorithms for System Parameter Identification," *IEEE Trans. on Auto. Cont.*, Vol. 35, No. 1, Jan. 1990, pp. 100-104.

[42] Kerr, T. H., "An Analytic Example of a Schweppe Likelihood Ratio Detector," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 25, No. 4, Jul. 1989, pp. 545-558. (A Kalman filter is at its core.)

[43] Kerr, T. H., "Real-Time Failure Detection: A Static Nonlinear Optimization Problem that Yields a Two Ellipsoid Overlap Test," *Journal of Optimiz. Theory and Applic.*, Vol. 22, No. 4, Aug. 1977, pp. 509-535.

[44] Kerr, T. H., "Modeling and Evaluating an Empirical INS Difference Monitoring Procedure Used to Sequence SSBN Navaid Fixes," *Proceedings of the Annual Meeting of the Institute of Navigation*, U.S. Naval Academy, Annapolis, Md., 9-11 Jun. 1981. (reprinted in *Navigation: Journal of the Institute of Navigation*, Vol. 28, No. 4, Winter 1981-82, pp. 263-285.)

[45] Kerr, T. H., "Computational Techniques for the Matrix Pseudoinverse in Minimum Variance Reduced-Order Filtering and Control," in Control and Dynamic Systems-Advances in Theory and Applications, Vol. XXVIII: *Advances in Algorithms and computational Techniques for Dynamic Control Systems*, Part 1 of 3, C. T. Leondes (Ed.), Academic Press, N.Y., 1988, pp. 57-107. (Offers several corrections to [139].)

[46] Kerr, T. H., "Multichannel AR Modeling for the Active Decoy (U)," MIT Lincoln Laboratory Report No. PA-499, Lexington, MA, Mar. 1987 (SECRET).

[47] Kerr, T. H., "Multichannel Shaping Filter Formulations for Vector Random Process Modeling Using Matrix Spectral Factorization," MIT Lincoln Laboratory Report No. PA-500, Lexington, MA, 27 Mar. 1989.

[48] Kerr, T. H., "Fallacies in Computational Testing of Matrix Positive Definiteness/Semidefiniteness," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 26, No. 2, Mar. 1990, pp. 415-421.

[49] Kerr, T. H., "Emulating Random Process Target Statistics (using MSF)," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 30, No. 2, Apr. 1994, pp. 556-577.

[50] Kerr, T. H., "Rationale for Monte-Carlo Simulator Design to Support Multichannel Spectral Estimation and/or Kalman Filter Performance Testing and Software Validation/Verification Using Closed-Form Test Cases," MIT Lincoln Laboratory Report No. PA-512, Lexington, MA, 22 Dec. 1989.

[51] Kerr, T. H., "Numerical Approximations and Other Structural Issues in Practical Implementations of Kalman Filtering," chap. in Approximate Kalman Filtering, edited by Guanrong Chen, 1993, pp. 193-220.

[52] Kerr, T. H., "Extending Decentralized Kalman Filtering (KF) to 2D for Real-Time Multisensor Image Fusion and/or Restoration: Optimality of Some Decentralized KF Architectures," *Proceedings of the Inter-*

national Conference on Signal Processing Applications & Technology (ICSPAT96), Boston, MA, 7-10 Oct. 1996, pp. 155-170.

[53] Kerr, T. H., *UEWR Design Notebook-Section 2.3: Track Analysis*, TeK Associates, Lexington, MA, (for XonTech, MA), XonTech Report No. D744-10300, 29 Mar. 1999. [CRLB Evaluations are now good using correct input data supplied by XonTech and General Dynamics from TDSAT [69]. Parameter settings for random noises was zeroed in these runs to yield the mean response but at the expected realistic nonuniform time steps. The MatLab Interpolation function was invoked by Dan Polito (GD) to provide these values at the periodic time steps used as input to the simulation of CRLB in 1999 for each mission scenario.]

[54] Kerr, T. H., "Considerations in whether to use Marquardt Nonlinear Least Squares vs. Lambert Algorithm for NMD Cue Track Initiation (TI) calculations," TeK Associates Technical Report No. 2000-101, Lexington, MA, (for Raytheon, Sudbury), 27 Sep. 2000. for noise http://www.tekassociates.biz/TeKMemoLambertNMDcueSRS29Sept2KAM.pdf

[55] Kerr, T. H., "Statistical Analysis of a Two Ellipsoid Overlap Test for Real-Time Failure Detection," *IEEE Trans. on Autom. Contr.*, Vol. 25, No. 4, Aug. 1980, pp. 762-773.

[56] Kerr, T. H., "Sensor Scheduling in Kalman Filters: Evaluating a Procedure for Varying Submarine Navaids," *Proceedings of 57th Annual Meeting of the Institute of Navigation*, Albuquerque, NM, 9-13 Jun. 2001, pp. 310-324.

[57] Kerr, T. H., "Vulnerability of Recent GPS Adaptive Antenna Processing (and all STAP/SLC) to Statistically Non-Stationary Jammer Threats," *Proceedings of SPIE, Session 4473: Tracking Small Targets*, San Diego, CA, 29 Jul.-3 Aug. 2001, pp. 62-73.

[58] Kerr, T. H., "Exact Methodology for Testing Linear System Software Using Idempotent Matrices and Other Closed-Form Analytic Results," *Proceedings of SPIE, Session 4473: Tracking Small Targets*, San Diego, CA, 29 Jul.-3 Aug. 2001, pp. 142-168.

[59] Wishner, R. P., Larson, R. E., and Athans, M., "Status of Radar Tracking Algorithms," *Proc. of* 1^{st} *Symposium on Nonlinear Estimation Theory & Its Applications*, San Diego, CA, 1970, pp. 32-54.

[60] Athans, M., Wishner, R. P., and Bertolini, A., "Suboptimal State Estimation for Continuous-Time Nonlinear Systems from Discrete Noisy Measurements," *IEEE Trans. on Autom. Contr.*, Vol. 13, No. 5, Oct. 1968, pp. 504-514.

[61] Wishner, R. P., Tabaczynski, J. A., and Athans, M., "A Comparison of Three Non-Linear Filters," *Automatica*, Vol. 5, 1969, pp. 487-496.

[62] Mehra, R. K., "A Comparison of Several Nonlinear Filters for Reentry Vehicle Tracking," *IEEE Trans. on Autom. Contr.*, Vol. 16, No. 4, Aug. 1971, pp. 307-319.

[63] Mendel, J. M., "Computational Requirements for a Discrete Kalman Filter," *IEEE Trans. on Autom. Contr.*, Vol. 16, No. 6, Dec. 1971, pp. 748-758.

[64] Stratonovich, R. L., *Topics in the Theory of Random Noise*, Vol. I, Translated by Richard Silverman, Gordon and Breach, Second Printing, New York, 1967.

[65] Athans, M., "An Example for Understanding Non-Linear Prediction Algorithms," MIT Electronics Systems Lab., Technical Memorandum ESL-TM-552, Cambridge, MA, Jun. 1974.

[66] Kerr, T. H., "Three Important Matrix Inequalities Currently Impacting Control and Estimation Applications," *IEEE Trans. on Autom. Contr.*, Vol. 23, No. 6, Dec. 1978, pp. 1110-1111.

[67] Kalandros, M., Pao, L. Y., "Covariance Control for Multisensor Systems," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 38, No. 4, Oct. 2002, pp. 1138-1157.

[68] Kerr, T. H., "Comments on 'Federated Square Root Filter for Decentralized Parallel Processes'," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 27, No. 6, Nov. 1991.

[69] Chang, C. B., Tabaczynski, J. A., "Application of State Estimation to Target Tracking," *IEEE Trans. on Autom. Contr.*, Vol. 29, No. 2, Feb. 1984, pp. 98-109.

[70] Boers, Y., Driessen, H., Lacle, N., "Automatic Track Filter Tuning by Randomized Algorithms," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 38, No. 4, Oct. 2002, pp. 1444-1449. (cf. [34].)

[71] Norgaand, M., Paulsen, N. K., Ravn, O., "New Developments in State Estimation for Nonlinear Systems," *Automatica*, Vol. 36, No. 11, Nov. 2000, pp. 1627-1638.

[72] Schei, T. S., "A Finite Difference Method for Linearization in Nonlinear Estimation Algorithms," *Automatica*, Vol. 33, No. 11, Nov. 1997, pp. 2051-2058.

[73] Netto, A., Gimeno, N. L., and Mendes, M. J., "On the Optimal and Suboptimal Nonlinear Problem for Discrete-Time Systems," *IEEE Trans. on Autom. Contr.*, Vol. 23, No. 6, Dec. 1978, pp. 1062-1067.

[74] C. Y. Chong, S. Mori, "Convex Combination and Covariance Intersection Algorithms in Distributed Fusion," *Proc. of 4th Intern. Conf. on Information Fusion*, Montreal, CA, Aug. 2001.

[75] Kerr, T. H., "Decentralized Filtering and Redundancy Management for Multisensor Navigation," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 23, No. 1, Jan. 1987, pp. 83-119.

[76] Kerr, T. H., and Chin, L., "The Theory and Techniques of Discrete-Time Decentralized Filters," in *Advances in the Techniques and Technology in the Application of Nonlinear Filters and Kalman Filters*, edited by C.T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph No. 256, Noordhoff International Publishing, Lieden, 1981, pp. 3-1 to 3-39.

[77] Schweppe, F. C., *Uncertain Dynamic Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[78] Levy, L. J., "Sub-optimality of Cascaded and Federated Filters," *Proc. of* 53^{rd} *Annual ION Meeting: Navigation Technology in the 3rd Millennium*, Cambridge, MA, Jun. 1996, pp. 399-407.

[79] Mutambra, A. G. O., *Decentralized Estimation and Control Systems*, CRC Press, NY, 1998.

[80] Alfano, S., Greer, M. L., "Determining if Two Solid Ellipsoids Intersect," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 1, Jan.-Feb. 2003, pp. 106-110.

[81] Lee, J.-W., Khargonekar, P.P., "A Convex Optimization-Based Nonlinear Filtering Algorithm with applications to Real-Time sensing for patterned wafers," *IEEE Trans. on Autom. Contr.*, Vol. 48, No. 2, Feb. 2003, pp. 224-235.

[82] Hsieh, C.-S., "General Two-Stage Extended Kalman Filters," *IEEE Trans. on Autom. Contr.*, Vol. 48, No. 2, Feb. 2003, pp. 289-293.

[83] Li, X.-R., "Multiple Model Estimation with Variable Structure-Part II: Model Set Adaptation," *IEEE Trans. on Autom. Contr.*, Vol. 45, No. 11, Nov. 2000, pp. 2047-2060.

[84] Ninness, B., "Strong Law of Large Numbers Under Weak Assumptions with Applications," *IEEE Trans. on Autom. Contr.*, Vol. 45, No. 11, Nov. 2000, pp. 2117-2122.

[85] Blackman, S. S., "Multiple Hypothesis Tracking for Multiple Target Tracking," *Systems Magazine Tutorials of IEEE Aerospace and Electron. Sys.*, Vol. 19, No. 1, Jan. 2004, pp. 5-18.

[86] Mahler, R. P. S., "Statistics 101 for Multisensor, Multitarget Data Fusion," *Systems Magazine Tutorials of IEEE Aerospace and Electronic Systems*, Vol. 19, No. 1, Jan. 2004, pp. 53-64.

[87] Haykin, S. (Ed.), *IEEE Proceedings: Special Issue on Sequential State Estimation*, Vol. 92, No. 3, Mar. 2004.

[88] Doyle, J. C., "Guaranteed Margins for LQG Regulators," *IEEE Trans. on Autom. Contr.*, Vol. 23, No. 4, Aug. 1978, pp. 756-757.

[89] Astrom, K. J., Hagguend, T., "Automatic Tuning of Simple Regulators with Specification on Phase and Amplitude Margins," *Automatica*, Vol. 20, No. 5, 1984, pp. 645-651.

[90] Lewis, F. L., *Applied Optimal Control and Estimation*, Prentice-Hall and Texas Instruments Digital Signal Processing Series, 1992.

[91] Gran, R., "An Approach to the Separation of Control and Estimation in Nonlinear Systems," *Proceedings of First Symposium on Nonlinear Estimation Theory & Its Applications*, pp. 215-220, San Diego, CA, 21-25 Sep. 1970.

[92] Wonham, W. H., "The Separation Theorem of Stochastic Control," (abstract only), *Proceedings of First Symposium on Nonlinear Estimation Theory & Its Applications*, San Diego, CA, 21-25 Sep. 1970, p. 8.

[93] Lainiotis, D. G., Upadhyay, T. N., Deshpande, J. G., "A Nonlinear Separation Theorem," *Proceedings of Second Symposium on Nonlinear Estimation Theory & Its Applications*, San Diego, CA, 13-15 Sep. 1971, pp. 184-187.

[94] Davis, M. H. A. "The Separation Principle in Stochastic Control Via Girsanov Solutions," *Proceedings of Fifth Symposium on Nonlinear Estimation Theory & Its Applications*, San Diego, CA, 23-25 Sep. 1974, pp. 62-68.

[69] After booster rockets are jetisoned, the RV target in an inverse square gravity field is properly described by Newton's 2^{nd} law: $\frac{d(mv)}{dt} = -\frac{GmM}{r^2}$ and since the mass m is subsequently constant and divides out so simplification is [365, p. 56]: $\frac{dv}{dt} = -\frac{GM}{r^2} \Rightarrow \frac{dv}{dr}\frac{dr}{dt} = -\frac{GM}{r^2} \Rightarrow v\frac{dv}{dr} = -\frac{GM}{r^2} \Rightarrow vdv = -\frac{GMdr}{r^2} \Rightarrow \frac{v^2}{2} = \frac{GM}{r} \Rightarrow v = \frac{\sqrt{2GM}}{\sqrt{r}}$. Proceeding further, $r^{1/2}dr = \sqrt{2GM}dt \Rightarrow r^{3/2} = \frac{3\sqrt{2GM}t}{2}$. However, TDSAT was faulted by someone (HSS) who incorrectly thought, from his prior terrestrial-only experience, that the effect of gravity should be $-\frac{gt^2}{2}$. Unfortunately, this discrepancy due to human error was incorrectly blamed on TDSAT. TDSAT correctly used an inverse square Gravity model internally.

[95] Atassi, A. N., Khalil, H. K., "A Separation Principle for the Control of a Class of Nonlinear Systems," *IEEE Trans. on Autom. Contr.*, Vol. 46, No. 5, May 2001, pp. 742-746.

[96] Costa, O. L. V., Tuesta, E. F., "Finite Horizon Quadratic Optimal Control and A Separation Principle for Markovian Jump Linear Systems," *IEEE Trans. on Autom. Contr.*, Vol. 48, No. 10, Oct. 2003, pp. 1836-1842.

[97] Witenhausen, H. S., "A counterexample in stochastic optimal control," *SIAM J. of Control*, Vol. 6, No. 1, 1968, pp. 131-147.

[98] Smith, S.C., Seiler, "Estimation with Lousy Measurements: Jump Estimators for Jump Systems," *IEEE Trans. on Autom. Contr.*, Vol. 48, No. 12, Dec. 2003, pp. 2163-2171.

[99] Leiva, H., Siegmund, S., 'A Necessary Algebraic Condition for Controllability and Observability of Linear Time-Varying Systems," *IEEE Trans. on Autom. Contr.*, Vol. 48, No. 12, Dec. 2003, pp. 2229-2232.

[100] Asif, A., "Fast Implementations of the Kalman-Bucy Filter for Satellite Data Assimilation," *IEEE Signal Processing Letters*, Vol. 11, No. 2, Feb. 2004, pp. 235-238.

[101] Kirubarajan, T. Bar-Shalom, Y., "Kalman Filter Versus IMM Estimator: When do we need the latter?," *IEEE Trans. on Aerospace and Electron. Sys.*, Vol. 39, No. 4, Oct. 2003, pp. 1452-1457.

[102] Mahler, R. P. S., "Multitarget Bayes Filtering via First Order Multitarget Moments," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 39, No. 4, Oct. 2003, pp. 1152-1178.

[103] Monim, A. , "Submarine Floating Antenna Model for Loran-C Signal Processing," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 39, No. 4, Oct. 2003, pp. 1304-1315.

[104] Hassibi, B., Sayed, A. H., and Kailath, T., "Linear Estimation in Krein Spaces-Part I: Theory," *IEEE Trans. on Autom. Contr.*, Vol. 41, NO. 1, Jan. 1996, pp. 18-33.

[105] Hsieh, C.-S., "General Two-Stage Extended Kalman Filters," *IEEE Trans. on Autom. Contr.*, Vol. 48, No. 2, Feb. 2003, pp. 289-293.

[106] Kerr, T. H., "Critique of Some Neural Network Architectures and Claims for Control and Estimation," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 34, No. 2, Apr. 1998, pp. 406-419.

[107] Zadunaisky, P. E., "Small Pertubations on Artificial Satellites," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 39, No. 4, Oct. 2003, pp. 1270-1276.

[108] Krishnamurthy, V., Dey, S., "Reduced Spatio-Temporal Complexity for MMPP and Image Based Tracking Filters for Maneuvering Targets," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 39, No. 4, Oct. 2003, pp. 1277-1291.

[109] Li, X.-R., Jilkov, V.P., "Survey of Maneuvering Target Tracking. Part I: Dynamic Models," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 39, No. 4, Oct. 2003, pp. 1333-1364.

[110] Bruni, C., DiPillo, G., Koch, G., "Bilinear Systems: An appealing class of 'almost linear' systems," *IEEE Trans. on Autom. Contr.*, Vol. 18, No. 4, Aug. 1972, pp. 334-348.

[111] Abutaleb, A. S., "A Tracking Filter for Maneuvering Sources," *IEEE Trans. on Autom. Contr.*, Vol. 34, No. 4, Apr. 1989, pp. 471-475.

[112] Chaffee, J., Kovach, K., Robel, G., "Integrity and the Myth of Optimal Filtering," *Proceedings of the Institute of Navigation*, Winter Technical Meeting, Jan. 1997.

[113] Rogers, S. R., "Instability of a Decoupled Tracking Filter," *IEEE Trans. on Autom. Contr.*, Vol. 34, No. 4, Apr. 1989, pp. 469-471.

[114] Cox, C. M., Chao, B. F., "Detection of a Large-Scale Mass Redistribution in the Terrestrial System Since 1998," *Science*, Vol. 297, 2 Aug. 2002, pp. 831-833 (also see [217]).

[115] Anaes, H. B., and Kailath, T., "Initial-Condition Robustness of Linear Least Squares Filtering Algorithms," *IEEE Trans. on Autom. Contr.*, Vol. 19, No. 4, Aug. 1974, pp. 393-397.

[116] Fitzgerald, R. J., "Divergence of the Kalman Filter," *IEEE Trans. on Autom. Contr.*, Vol. 16, No. 6, Dec. 1971, pp. 736-747.

[117] Zames, G., "Input-Output Feedback Stability and Robustness: 1959-1985," *IEEE Control System Magazine*, Vol. 16, No. 3, Jun. 1996, pp. 61-66.

[118] Kell, L. H., Bhattacharyya, S. P., "Robust, Fragile, or Optimal," *IEEE Trans. on Autom. Contr.*, Vol. 42, No. 8, Aug. 1997, pp. 1098-1105 (also see comments on pp. 1265-1267 and authors' polite reply to bellowing on p. 1268 in Vol. 43, No. 9, pp. 1265-1267, Sep. 1998 issue).

[119] Rosenbrock, H. H., McMorran, P. D., "Good, Bad, or Optimal," *IEEE Trans on Autom. Contr.*, Vol. 16, No. 6, Dec. 1971, pp. 552-553.

[120] Tam, L.-F., Wong, W.-S., and Yau, S.S.-T., "On a Necessary and Sufficient Condition for Finite Dimensionality of Estimation Algebras," *SIAM Journal on Control and Optimization*, Vol. 28, No. 1, Jan. 1990, pp. 173-185.

[121] Boutayeb, M., Rafaralahy, H., Darouach, M., "Convergence Analysis of the Extended Kalman Filter Used as an Observer for Nonlinear Discrete-Time Systems," *IEEE Trans. on Autom. Contr.*, Vol. 42, No. 4, Apr. 1997, pp. 581-586.

[122] Bell, B. M., Cathey, F. W., "The Iterated Kalman Filter Update as a Gauss-Newton Method," *IEEE Trans. on Autom. Contr.*, Vol. 38, No. 2, Feb. 1993, pp. 294-298.

[123] Lee, J. T., Lay, F., Ho, Y.-C., "The Witsenhausen Counterexample: A Hierarchical Search Approach for Nonconvex Optimization Problems," *IEEE Trans. on Autom. Contr.*, Vol. 46, No. 3, Mar. 2001, pp. 382-397.

[124] Arnold, B. C., Castillo, E., Sarabia, J. M., *Conditional Specification of Statistical Models*, Springer Series on Statistics, Spinger-Verlag, NY, 1999.

[125] *Recent Advances in Stochastic Calculus*, Edited by J. S. Baras and V. Mirelli, Progress in Automation and Information Systems, Springer-Verlag, NY, 1990.

[126] Hu, G.-Q., Yau, S. S.-T., "Finite-Dimensional Filters with Nonlinear Drift XV: New Direct Method for Construction of Universal Finite-Dimensional Filter," *IEEE Trans. on Autom. Contr.*, Vol. 47, No. 3, Mar. 2002, pp. 50-57.

[127] Pulford, G. W., La Scala, B. F., "MAP Estimation of Target Maneuver Sequence with the Expectation-Maximization Algorithm," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 38, No. 2, Apr. 2002, pp. 367-377.

[128] Shumway, R. H., Stoffer, D. S., "An Approach to Time Series Smoothing and Forecasting Using the EM Algorithm," *Journal of Time Series Analysis*, Vol. 3, No. 4, 1982, pp. 253-264.

[129] Tse, E., "Parallel Computation of the Conditional Mean State Estimate for Nonlinear Systems," *Proceedings of Symposium on Nonlinear Estimation and its Applications*, San Diego, CA, Sep. 1971, pp. 385-394.

[130] Tse, E., Larson, R. E., and Senne, K., "A Survey of Parallel Processing Algorithms for Nonlinear Estimation," *Proceedings of the Symposium on Nonlinear Estimation and its Applications*, San Diego, CA, 23-25 Sep. 1974, pp. 257-269.

[131] Baheti, R. S., O'Halloron, D. R., Itzkowitz, H. R., "Mapping Extended Kalman Filters onto Linear Arrays," *IEEE Trans. on Autom. Contr.*, Vol. 35, No. 12, Dec. 1990, pp. 1310-1319.

[132] Holtzman, J. M., *Nonlinear System Theory: A Functional Analysis Approach*, Prentice-Hall, 1970.

[133] Olson, D. K., "Converting Earth-Centered, Earth-Fixed Coordinates to Geodetic Coordinates," *IEEE Trans. on Aero. and Electr. Sys.*, Vol. 32, No. 1, Jan. 1996, pp. 473-475.

[134] Wu, Y., Wong, P., Hu, X., "Algorithm of Earth-Centered, Earth-Fixed Coordinates to Geodetic Coordinates," *IEEE Trans. on Aero. and Electr. Sys.*, Vol. 39, No. 4, Oct. 2003, pp. 1457-1461.

[135] Carr, E. M., Jones, H. M., Carey, M. O., *MAXLIK: A Computer Program to Determine Satellite Orbits from Radar Metric Data (U)*, MIT Lincoln Laboratory Report No. PSI-126, Lexington, MA, Dec. 1981 (limited distribution).

[136] Zames, G., "Feedback and Optimal Sensitivity: model reference transformations, multiplicative semi-norms, and approximate inverses," *IEEE Trans. on Auto. Contr.*, Vol. 26, Apr. 1981, pp. 744-752.

[137] Kozin, F., "On the Probability Densities of the Output of Some Random Systems," *Journal of Applied Mechanics*, Vol. 28, 1961, pp. 161-165.

[138] Song, T. T., *Random Differential Equations in Systems and Engineering*, Academic Press, NY, 1973.

[139] Gelb, A. (Ed.), *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1974.

[140] Kerr, T. H., "False Alarm and Correct Detection Probabilities Over a Time Interval for Restricted Classes of Failure Detection Algorithms," *IEEE Trans. on Inform. Theory*, Vol. 28, No. 4, Jul. 1982, pp. 619-631.

[141] Kerr, T. H., "Examining the Controversy Over the Acceptability of SPRT and GLR Techniques and Other Loose Ends in Failure Detection," *Proceedings of the American Control Conference*, San Francisco, CA, 22-24 Jun. 1983, pp. 966-977.

[142] Kerr, T. H., "A Critique of Several Failure Detection Approaches for Navigation Systems," *IEEE Trans. on Autom. Contr.*, Vol. 34, No. 7, Jul. 1989, pp. 791-792.

[143] Kerr, T. H., "On Duality Between Failure Detection and Radar/Optical Maneuver Detection," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 25, No. 4, Jul. 1989, pp. 581-583.

[144] Sayed, A. H., and Kailath, T., "A State-Space Approach to Adaptive RLS Filtering," *IEEE Signal Processing Magazine*, Vol. 11, No. 3, Jul. 1994, pp. 18-60. (Please see its sequel.)

[145] Scharf, L., Chong, E., McWhorter, L. T., Zoltowski, M., Goldstein, J. S., "Algebraic Equivalence of the Multistage Wiener Filter and the Conjugate Gradient Weiner Filter," *Proceedings of the 7th Annual*

Adaptive Sensor Array Processing Workshop, Lincoln Laboratory of MIT, Lexington, MA, 11-13 Mar. 2003.

[146] Payne, A. N., "Observability Problem for Bearings-Only Tracking," *International Journal of Control*, Vol. 49, No. 3, 1989, pp. 761-768.

[147] Zhou, Y., Sun, Z., "Observability Analysis of Single Passive Observer," *Proceedings of the 1995 IEEE National Aerospace and Electronics Conference*, NY, 1995, pp. 215-219.

[148] Jauffret, C., Pillon, D., "Observability in Passive Target Motion Analysis," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 32, No. 4, Oct. 1996, pp. 1290-1300.

[149] Rao, S. K., "Comments on 'Observability in Passive Target Motion Analysis'," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 34, No. 2, Apr. 1998, p. 700.

[150] Meurer, G. W., "The TRADEX MTT Multitarget Tracker," *The Lincoln Laboratory Journal*, Vol. 5, No. 3, 1992, pp. 317-349.

[151] Lawton, J. A., Jesionowski, R. J., Zarchan, P., "Comparison of Four Filtering Options for Radar Tracking Problems," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 4, Jul.-Aug. 1998, pp. 618-623.

[152] Ramachandra, K. V., *Kalman Filtering Techniques for Radar Tracking*, Marcel Dekker, Inc., NY, 2000.

[153] Bridgewater, A. W., "Analysis of Second and Third Order Steady-State Tracking Filters," in *AGARD Conference Proceedings No. 252: Strategies for Automatic Track Initiation*, Ed. By Dr. S. J. Rabinowitz, Monterey, CA, 16-17 Oct. 1978, pp. 9-1 to 9-11.

[154] Brookner, E., *Tracking and Kalman Filtering Made Easy*, John Wiley & Sons, Inc., NY, 1998 (especially Chapt. 3 for practical issues associated with radar target tracking, especially for Early Earning radars).

[155] Press, W. H., Teukolsky, S. A., et al, *Numerical Recipes in Fortran 90: The art of parallel scientific computing*, $2^{\rm nd}$ Edition, Vol. 2 of *Fortran Numerical Recipes*, Cambridge University Press, NY, 1996 (1999 reprint with corrections).

[156] Kreucher, C., Hero, A., Kastella, K., "Multiple Model Particle Filtering for Multi-Target Tracking," *Proceedings of the 12th Annual Adaptive Sensor Array Processing Workshop*, Lincoln Laboratory of MIT, Lexington, MA, 16-18 Mar. 2004.

[157] Lohmiller, W. and Slotine, J.-J. E., "On Contraction Analysis for Nonlinear Systems," *Automatica*, Vol. 34, No. 6, 1998.

[158] Slotine, J.-J. E., "Modular Stability Tools for Distributed Computation and Control," *Int. Jour. on Adaptive Control and Signal Processing*, Vol. 17, No. 6, 2003.

[159] *Special Issue on Stochastic Control methods applied to Financial Engineering*, *IEEE Trans.on Autom. Contr.*, Vol. 49, No. 3, Mar. 2004.

[160] *Selected Papers of Frank Kozin: Stochastic Analysis and Engineering Applications*, Ed. by Y. Sunahara, MITA Press, Tokyo, Japan, 1994 (entirely in English).

[161] Hassibi, B., Sayed, A. H., Kailath, T., "H^{∞} Optimality of the LMS Algorithm," *IEEE Trans. on Sig. Proc.*, Vol. 44, No. 2, Feb. 1996, pp. 267-281.

[162] Hassibi, B., Sayed, Kailath, T., "Linear Estimation in Krein Spaces– Part I: Theory," *IEEE Trans. on Autom. Contr.*, Vol. 41, No. 1, Jan. 1996, pp. 18-33.

[163] Hassibi, B., Sayed, Kailath, T., "Linear Estimation in Krein Spaces– Part II: Applications," *IEEE Trans. on Autom. Contr.*, Vol. 41, No. 1, Jan. 1996, pp. 34-50.

[164] Brunke, S., Cambell, M. E., "Square Root Sigma Point Filtering for Real-Time Nonlinear Estimation," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 27, No. 2, Mar.-Apr. 2004, pp. 314-17. (2005, p. 383).

[165] Klinger, A., "Information and Bias in Sequential Estimation," *IEEE Trans. on Autom. Contr.*, Vol. 10, No. 1, Feb. 1968, pp. 102-103.

[166] Kalatchin, P., Chebotko, I., et al, *The Revolutionary Guide to Bitmapped Graphics*, Wrox Press Ltd., Birmingham, UK, 1994.

[167] Kerr, T. H., "Comments on 'An Algorithm for Real-Time Failure Detection in Kalman Filters'," *IEEE Trans. on Autom. Contr.*, Vol. 43, No. 5, May 1998, pp. 682-683.

[168] Golub, G. H., Van Loan, C. F., *Matrix Computations*, $3^{\rm rd}$ Edition, Johns Hopkins University Press, Baltimore, MD, 1996.

[169] Chan, K., "A Simple Mathematical Approach for Determining Intersection of Quadratic Surfaces," *Proceedings of American Astronautical Society*, Part III, AAS Paper 01-358, Jul.-Aug. 2001, pp. 785-801.

[170] Leva, J. L., "An Alternative Closed-Form Solution to the GPS Pseudorange Equations," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 32, No. 4, Oct. 1996, pp. 1430-1439.

[171] Galvin, W. P., "Matrices with 'Custom-Built' Eigenspaces," *SIAM Mathematical Monthly*, May 1984, pp. 308-309.

[172] Yau, S.-T., Yau, S. S.-T., "Nonlinear Filtering and Time Varying Schrodinger Equation 1," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 40, No. 1, Jan. 2004, pp. 284-292.

[173] Mahata, K., Soderstrom, T., "Large Sample Properties of Separable Nonlinear Least Squares Estimators," *IEEE Trans. on Sig. Proc.*, Vol. 52, No. 6, Jun. 2004, pp. 1650-1658.

[174] Bar-Shalom, Y., Chen, H., Mallick, M., "One Step Solution for the Multistep Out-of-Sequence-Measurement Problem," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 40, No. 1, Jan. 2004, pp. 27-37.

[175] Mendel, J. M., *Lessons in Digital Estimation*, Prentice-Hall, Englewood, Cliffs, NJ, 1987.

[176] Mendel, J. M., *Lessons in Estimation Theory for Signal Processing, Communications, and Control*, Prentice-Hall, Upper Saddle River, NJ, 1995.

[177] Ristic, B., Arulampalam, S., Gordon, N., *Beyond the Kalman Filter: Particle Filters for Tracking Applications*, Artech House, Boston, MA, 2004.

[178] Kulhavy, R., *Recursive Nonlinear Estimation: A Geometric Approach*, Lecture Notes in Control and Information Sciences, Springer, NY, 1996.

[179] Eldar, Y. C., "Minimum Variance in Biased Estimation: Bounds and Asymptotically Optimal Estimators," *IEEE Trans. on Sig. Proc.*, Vol. 52, No. 7, Jul. 2004, pp. 1915-1930.

[180] Bernado, J. M., Smith, A. F. M., *Bayesian Theory*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, NY, 1993.

[181] Snieder, R., *A guided Tour of Mathematical Methods for the Physical Sciences*, (Especially Chap. 20 on Potential Theory for gravity of non-spherical earth) Cambridge University Press, Cambridge, UK, 2001.

[182] Hanzon, B., "A differential-geometric approach to approximate nonlinear filtering," in C. T. J. Dodson (Ed.), *Geometrization of Statistical Theory*, ULDM Publications, Lancaster, UK, 1987, pp. 219-224.

[183] Hanzon, B. Hut, R., "New Results on the projection filter," *Proceedings of the* $1^{\rm st}$ *European Control Conference*, Grenoble, France, 1991, pp. 623-628.

[184] Brigo, D., "On the nice behavior of the Gaussian Projection Filter with small observation noise," *Proceedings of the* $3^{\rm rd}$ *European Control Conference*, Rome, Italy, Vol. 3, 1995, pp. 1682-1687.

[185] Brigo D., Hazon, B., and Gland, F. Le, "A differential-geometric approach to nonlinear filtering: the projection filter," *Proceedings of the* 34^{th} *IEEE Conference on Decision and Control*, New Orleans, LA, Vol. 4, 1995, pp. 4006-4011.

[186] Gorlatch, S., Lengauer, C. (Eds.), *Constructive Methods for Parallel Programming, Advances in Computation: Theory and Practice*, Vol. 10, Nova Science Publishers Inc. NY, 2000.

[187] Tadonki, C., Philippe, B., "Parallel Multiplication of Vector by a Kronecker Product of Matrices," Chap. 5 in *Parallel Numerical Linear Algebra*, J. Dongarra, Erricos John Kontoghiorghes (ds.), Nova Science Publishers, Huntington, NY, 2001.

[188] Voinov, V. G., Nikulin, M. S., *Unbiased Estimators and Their Applications*, Vol. 2: Multivariate Case, (especially App. 2: On Evaluating Some Multivariable Integrals), Kluwer Academic publishers, Boston, MA, 1996.

[189] Young, P., *Recursive Estimation and Time Series Analysis: An Introduction*, (especially App. 2: Gauss's derivation of recursive least squares & App. 3: Instantaneous Cost function associated with recursive least squares) Springer-Verlag, NY, 1984.

[190] Ibragimov, N. H. (Ed.), *CRC Handbook of Lie Group Group Analysis of Differential Equations*, Vol. 1: Symmetries, Exact Solutions, and Conservation Laws, CRC Press, Inc., Boca Raton, FL, 2000.

[191] Mohler, R. R., *Nonlinear Systems: Vol. II: Applications to Bilinear Control*, Prentice-Hall, Englewood Cliffs, NJ, 1991.

[192] Chikte, S. D., "Bilinear Systems with Nilpotent Lie Algebras: Least Squares Filtering," *IEEE Trans. on Autom. Contr.*, Vol. 25, No. 6, Dec. 1979, pp. 948-953.

[193] Speyer, J. L., "Computation and Transmission Requirements for a Decentralized Linear-Quadratic-Gaussian Contol Problem," *IEEE Trans. on Autom. Contr.*, Vol. 24, No. 2, Apr. 1979, pp. 266-269.

[194] Saridis, G. N., *Self-Organizing Control of Stochastic Systems*, Control and Systems Theory Series, Vol. 4, Marcel Dekker, NY, 1977.

[195] Abou-Kandil, H., Freiling, G., Ionescu, V., Jank, G., *Matrix Riccati Equations in Control*, Birkhauser, Basel, Switzerland, 2003.

[196] Milman, M. H., Scheid, Jr., R. E., "A Note on Finite Dimensional Estimators for Infinite Dimensional Systems," *IEEE Trans. on Autom. Contr.*, Vol. 30, No. 12, Dec. 1985, pp. 1214-1217.

[197] Elliott, R. J., Krishnamurthy, V., Poor, H. V., "Exact Filters for Certain Moments and Stochastic Integrals of the State of Systems with Beněs

Nonlinearity," *IEEE Trans. on Autom. Contr.*, Vol. 44, No. 10, Oct. 1999, pp. 1929-1933.

[198] Nikoukhah, R., Campbell, S. L., Delebecque, F., "Kalman Filtering for General Discrete-Time Linear Systems," *IEEE Trans. on Autom. Contr.*, Vol. 44, No. 10, Oct. 1999, pp. 1829-1839.

[199] Nikoukhah, R., Taylor, D., Willsky, A. S., Levy, B. C., "Graph Structure and Recursive Estimation of Noisy Linear Relations," *Journal of Mathematical Systems, Estimation, and Control*, Vol. 5, No. 4, 1995, pp. 1-37.

[200] Nikoukhah, R., Willsky, A. S., Levy, B. C., "Kalman Filtering and Riccati Equations for Descriptor Systems," *IEEE Trans. on Autom. Contr.*, Vol. 37, 1992, pp. 1325-1342.

[201] Chung, D., Park, C. G., Lee J. G., "Robustness of Controllability and Observability of Continuous Linear Time-Varying Systems with Parameter Perturbations," *IEEE Trans. on Autom. Contr.*, Vol. 44, No. 10, Oct. 1999, pp. 1919-1923.

[202] Aihara, S. I., Bagchi, A., "On the Mortensen Equation for Maximum Likelihood State Estimation," *IEEE Trans. on Autom. Contr.*, Vol. 44, No. 10, Oct. 1999, pp. 1955-1961.

[203] Lasserre, J. B., "Sample-Path Average Optimality for Markov Control Processes," *IEEE Trans. on Autom. Contr.*, Vol. 44, No. 10, Oct. 1999, pp. 1966-1971.

[204] Aliev, F. A., Ozbek, L., "Evaluation of Convergence Rate in the Central Limit Theorem for the Kalman Filter," *IEEE Trans. on Autom. Contr.*, Vol. 44, No. 10, Oct. 1999, pp. 1905-1909.

[205] Spall, J. C., Wall, K. D., "Asymptotic Distribution Theory for the Kalman Filter State Estimator," *Communications on Statistical Theory and Methods*, Vol. 13, 1984, pp. 1981-2003.

[206] Spall, J. C., "Validation of State-Space Models from a Single Realization of Non-Gaussian Measurements," *IEEE Trans. on Autom. Contr.*, Vol. 30, 1985, pp. 1212-1214.

[207] Ibrahim, S., Rajagopalan, A. N., "Image Estimation in Film-Grain Noise," *IEEE Signal Processing Letters*, Vol. 12, No. 3, Mar. 2005, pp. 238-241.

[208] Chang, C., Ansari, R., "Kernel Particle Filtering for Visual Tracking," *IEEE Signal Processing Letters*, Vol. 12, No. 3, Mar. 2005, pp. 242-246.

[209] Luenberger, D. G., *Introduction to Dynamic Systems: Theory, Models, and Applications*, John Wiley & Sons, NY, 1979.

[210] Luenberger, D. G., "Dynamic Equations in Descriptor Form," *IEEE Trans. on Autom. Contr.*, Vol. 22, No. 3, Jun. 1977, pp. 312-321.

[211] Shampine, L. F., Reichett, M. W., "The MatLab ODE Suite," *SIAM Journal on Scientific Computing*, Vol. 18, 1997, pp. 1-22.

[212] Gear, C. W., Watanabe, D. S., "Stability and Convergence of Variable Order Multi-step Methods," SIAM Journal of Numerical Analysis, Vol. 11, 1974, pp. 1044-1058. (Also see Gear, C. W., *Automatic Multirate Methods for Ordinary Differential Equations*, Rept. No. UIUCDCS-T-80-1000, Jan. 1980.)

[213] Gini, F., Regggianini, R., Mengali, U., "The Modified Cramer-Rao Lower Bound in Vector Parameter Estimation," *IEEE Trans. on Sig. Proc.*, Vol. 46, No. 1, Jan. 1998, pp. 52-60.

[214] Prasad, M. K., "Stack Filter Design Using Selection Probabilities," *IEEE Trans. on Sig. Proc.*, Vol. 53, No. 3, Mar. 2005, pp. 1025-1037.

[215] Brcich, R. F., Iskander, D. R., Zoubir, A. M., "The Stability Test for Symmetric Alpha-Stable Distributions," *IEEE Trans. on Sig. Proc.*, Vol. 53, No. 3, Mar. 2005, pp. 977-986.

[216] Zheng, Z. W., Zhu, Y.-S., "New Least-Squares Registration Algorithm for Data Fusion," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 40, No. 4, Oct. 2004, pp. 1410-1416.

[217] de Morais, T. N., Oliveira, A. B. V., Walter, F., "Global Behavior of the Equatorial Anomaly Since 1999 and Effects on GPS," *IEEE AES Systems Magazine*, Vol. 20, No. 3, Mar. 2005, pp. 15-23. (cf., [114])

[218] Minvielle, P., "Decades of Improvements in Re-entry Ballistic Vehicle Tracking," *IEEE A&E Systems Magazine*, Vol. 20, No. 8, part 1 of 2, Aug. 2005, pp. CF-1 to CF-14.

[219] Wu, Y., Hu, X., Hu, D., Wu, M., "Comments on 'Gaussian Particle Filtering'," *IEEE Trans. on Sig. Proc.*, Vol. 53, No. 8, Aug. 2005, pp. 3350-3351.

[220] Evans, R., Krishnamurthy, V., Nair, G., Sciacca, L., "Networked Sensor Management and Data Rate Control for Tracking Maneuvering Targets," *IEEE Trans. on Sig. Proc.*, Vol. 53, No. 6, Jun. 2005, pp. 1979-1991.

[221] Farina, A., *Antenna-Based Signal Processing Techniques for Radar Systems,* Artech House," Boston, MA, 1992.

[222] Daum, F. E., "Nonlinear Filters: Beyond the Kalman Filter," *IEEE Aerospace and Electronic Systems Magazine*, Tutorials II, Vol. 20, No. 8, Part 2 of 2, Aug. 2005, pp. 57-69.

[223] Bar-Shalom, Y., Challa, S., Blom, H. A. P., "IMM Estimator Versus Optimal Estimator for Hybrid Systems," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 41, No. 3, Jul. 2005, pp. 986-991.

[224] Cai, J., Sinha, A., Kirubarajan, T., "EM-ML Algorithm for Track Initiation using Possibly Noninfomative Data," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 41, No. 3, Jul. 2005, pp. 1030-1048.

[225] Bruno, M. G. S., Pavlov, A., "Improved Sequential Monte-Carlo Filtering for Ballistic Target Tracking," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 41, No. 3, Jul. 2005, pp. 1103-1109.

[226] Coraluppi, S., Carthel, C., "Distributed Tracking in Multistatic Sonar," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 41, No. 3, Jul. 2005, pp. 1138-1147.

[227] Yan, Z., Duan, G., "Time Domain Solution to Descriptor Variable Systems," *IEEE Trans. on Autom. Contr.*, Vol. 50, No. 11, Nov. 2005, pp. 1796-1798.

[228] Bucy, R. S., Joseph, P. D., *Filtering for Stochastic Processes with Applications to Guidance*, 2nd Edition, Chelsea, NY, 1984 (1st Edition, Interscience, NY, 1968).

[229] Bucy, R. S., "Nonlinear Filtering," *IEEE Trans. on Autom. Contr.*, Vol. 10, No. 2, Apr. 1965, p. 198.

[230] Bucy, R. S., "Information and Filtering," *Information Sciences*, Vol. 18, 1979, pp. 179-187.

[231] Bucy, R. S., "Distortion Rate Theory and Filtering," IEEE Trans. on Information Theory, Vol. 28, 1982, pp. 336-339.

[232] Bucy, R. S. (with assistance of B. G. Williams), *Lectures on Discrete Time Filtering*, Springer-Verlag, NY, 1994.

[233] Bucy, R. S., Moura, J. M. F., *NATO Advanced Study Institute on Nonlinear Stochastic Problems*, NATO ASI Series, Series C: Mathematical and Physical Sciences, No. 104, Kluwer, Boston, MA, 1982.

[234] Blondel, V. D., Megretski, A. (eds.), *Unsolved Problems in Mathematical Systems and Control Theory*, Princeton University Press, Princeton, NJ, 2004.

[235] Sawitski, G., "Finite-Dimensional Filter Systems in Discrete Time," *Stochastics*, Vol. 5, 1981, pp. 107-114.

[236] Damm, T., *Rational Matrix Equations in Stochastic Control*, Springer-Verlag, NY, 2004.

[237] Kushner, H. J., Dupuis, P. G., *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer-Verlag, NY, 1992.

[238] Costa, O. L. V., Fragoso, M. D., Marques, R. P., *Discrete-Time Jump Linear Systems, Series on Probability and Its Applications*, Gani, J., Heyde, C. C., Jagers, P., Kurtz, T. G. (Eds.), Springer-Verlag, NY, 2005.

[239] Murata, K., *Matrices and Matroids for Systems Analysis*, Series on Algorithms and Combinatorics 20, Springer-Verlag, NY, 2000.

[240] Stone, L. D., Barlow, C. A., Corwin, T. L., *Bayesian Multiple Target Tracking*, Artech, Boston, MA, 1999.

[241] Blahut, R. E., Miller Jr., W., Wilcox, C. H., *Radar and Sonar*, Part 1, The IMA Volumes in Mathematics and Its Applications, Vol. 32, Springer-Verlag, NY, 1991.

[242] Chestnut, H., "Bridging the Gap in Control-Status 1965," *IEEE Trans. on Autom. Contr.*, Vol. 10, Apr. 1965, pp. 125-126 (evidently still an problem in 2016). [G.E. vice president Harold Chestnut discussed his concerns regarding this issue with me at General Electric R&D Center in 1971.]

[243] Rapoport, I, Oshman, Y., "Fault-Tolerant Particle Filtering by Using Interactive Multiple Model-Based Rao-Blackwellization," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 6, Nov.-Dec. 2005, pp. 1171-1177.

[244] Xin, J., Sano, A., "Efficient Subspace-Based Algorithm for Adaptive Bearing Estimation and Tracking," *IEEE Trans. on Sig. Proc.*, Vol. 53, No. 12, Dec. 2005, pp. 4485-4505.

[245] Kerr, T. H., "Comments on 'Determining if Two Solid Ellipsoids Intersect'," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 1, Jan.-Feb. 2005, pp. 189-190.

[246] Kerr, T. H., "Integral Evaluation Enabling Performance Trade-offs for Two Confidence Region-Based Failure Detection," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 3, May-Jun. 2006, pp. 757-762.

[247] Zhang, L., Lam, J., Zhang, Q., "Lyapunov and Riccati Equations of Discrete-Time Descriptor Systems," *IEEE Trans. on Autom. Contr.*, Vol. 44, No. 11, Nov. 1999, pp. 2134-2139.

[248] Stykel, T., "On Some Norms for Descriptor Systems," *IEEE Trans. on Autom. Contr.*, Vol. 51, No. 5, May 2006, pp. 842-847.

[249] Fridman, E., "Descriptor Discretized Lyapunov Functional Method: Analysis and Design," *IEEE Trans. on Autom. Contr.*, Vol. 51, No. 5, May 2006, pp. 890-897.

[250] McEneaney, W. M., *Max-Plus Methods for Nonlinear Control and Estimation*, Birkhauser, Boston, 2006.

[251] Lu, P., "Nonlinear Predictive Controllers for Continuous Systems," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 3, May-Jun. 1994, pp. 553-560.

[252] Crassidis, J. L., Markley, F. L., "Predictive Filtering for Nonlinear Systems," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 3, May-Jun. 1997, pp. 566-572.

[253] Crassidis, J. L., Markley, F. L., "Predictive Filtering for Attitude Estimation Without Rate Sensors," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 3, May-Jun. 1997, pp. 522-527.

[254] Li, J., Zhang, H.-Y., "Stochastic Stability Analysis of Predictive Filters," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 3, May-Jun. 2006, pp. 738-741.

[255] Konrad, R., Srefan, G., Engin, Y., "Stochastic Stability of the Discrete Time Extended Kalman Filter," *IEEE Trans. on Autom. Contr.*, Vol. 44, No. 4, Apr. 1999, pp. 714-728.

[256] Isaac, A., Zhang, X., Willet, P., Bar-Shalom, Y., "A Particle Filter for Tracking Two Closely Spaced Objects Using Monopulse Radar Channel Signals," *IEEE Signal Processing Letters*, Vol. 13, No. 6, Jun. 2006, pp. 357-360.

[257] Roberts, W. J. J., Ephraim, Y., Dieguez, E., "On Ryden's EM Algorithm for Estimating MMPPs," *IEEE Signal Processing Letters*, Vol. 13, No. 6, Jun. 2006, pp. 373-376.

[258] Koening, D., "Observer Design for Unknown Input Nonlinear Descriptor Systems via Convex Optimization," *IEEE Trans. on Autom. Contr.*, Vol. 51, No. 6, Jun. 2006, pp. 1047-1052.

[259] Kailath, T., "A View of Three Decades of Linear Filtering," *IEEE Trans. on Information theory*, Vol. 20, No. 2, Mar. 1974, pp. 146-181.

[260] Wong, E., "Recent Progress in Stochastic Processes-A Survey," *IEEE Trans. on Information theory*, Vol. 19, No. 3, May 1973, pp. 262-275.

[261] Duttweiler, D., Kailath, T., "RKHS Approach to Detection and Estimation Problems-IV: Non-Gaussian Detection," *IEEE Trans. on Information theory*, Vol. 19, No. 1, Jan. 1973, pp. 19-28.

[262] Kailath, T., Poor, H. V., "Detection of Stochastic Processes," *IEEE Trans. on Information theory*, Vol. 44, No. 6, Oct. 1998, pp. 2230-2231.

[263] Brammer, R., "A Note on the Use of Chandrasekhar Equations for the Calculation of the Kalman Gain," *IEEE Trans. on Information theory*, Vol. 21, No. 3, May 1975, pp. 334-336.

[264] Kailath, T., "Author's reply to 'A Note on the Use of Chandrasekhar Equations for the Calculation of the Kalman Gain'," *IEEE Trans. on Information theory*, Vol. 21, No. 3, May 1975, pp. 336-337.

[265] Bensoussan, A., *Stochastic Control by Functional Analysis Methods*, Vol. II, North-Holland Publishing, NY, 1982.

[266] Kerr, T. H. "Further Critical Perspectives on Certain Aspects of GPS Development and Use," *Proceedings of 57th Annual Meeting of the Institute of Navigation*, pp. 592-608, Albuquerque, NM, 9-13 Jun. 2001.

[267] Loria, A., Lamnabhi-Lagarrigue, F., Panteley, E. (Eds.), *Advanced Topics in Control Systems: Lecture Notes from FAP 2005*, Lecture Notes in Control and Information Sciences, M. Thoma and M. Morari (Series Eds.), Springer, NY, 2006.

[268] Triggs, B. Sdika, M., "Boundary Conditions for Young-van Vliet Recursive Filtering," *IEEE Trans. on Sig. Proc.*, Vol. 54, No. 6, Part 1 of Two Parts, pp. 2365-2367, Jun. 2006.

[269] Heidergott, B.,. Olsder, G. I., van derWoude, J., *MaxPlus at Work: Modeling and Analysis of Synchronized Systems*, Princeton University Press, Princeton, NJ, 2006.

[270] El-Sheimy, N., Shin, E.-H., Niu, X., "Kalman Filter Face-Off: Extended vs. Unscented Kalman Filters for Integrated GPS and MEMS Inertial," *Inside GNSS: Engineering Solutions for the Global Navigation Satellite System Community*, Vol. 1, No. 2, Mar. 2006, pp. 48-54.

[271] Yaesh, I., Shaked, U., "Discrete-Time Min-Max Tracking," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 42, No. 2, Apr. 2006, pp. 540-547.

[272] Crassidis, J. L., "Sigma-Point Kalman Filtering for Integrated GPS and Inertial Navigation," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 42, No. 2, Apr. 2006, pp. 750-756.

[273] Rabbat, M. G., Nowak, R. D., "Decentralized Source Localization and Tracking," *Proceedings of International Conference on Acustics, Speech, and Signal Processing*, Vol. 3, Montreal, QC, Canada, May 2004, pp. 921-924.

[274] Ribeiro, A., Giannakis, G. B., "Bandwidth-Constrained Distributed Estimation Using Wireless Sensor Networks-Part I: Gaussian Case," *IEEE Trans. on Sig. Proc.*, Vol. 54, No. 3, Mar. 2005, pp. 1131-1143.

[275] Marano, S., Matta, V., Willett, P., Tong, L., "Support-Based and ML Approaches to DOA Estimation in a Dumb Sensor Network," *IEEE Trans. on Sig. Proc.*, Vol. 54, No. 4, Apr. 2006, pp. 1563-1567.

[276] He, T., Ben-David, S., Tong, L., "Nonparametric Change Detection and Estimation in Large-Scale Sensor Networks," *IEEE Trans. on Sig. Proc.*, Vol. 54, No. 4, Apr. 2006, pp. 1204-1217.

[277] Gao, Z., Ho, D. W. C., "State/Noise Estimator for Descriptor Systems with Application to Sensor Fault Diagnosis," *IEEE Trans. on Sig. Proc.*, Vol. 54, No. 4, Apr. 2006, pp. 1316-1326.

[278] Zhan, R., Wan, J., "Neural Network-Aided Adaptive Unscented Kalman Filter for Nonlinear State Estimation," *IEEE Sig. Proc. Letters*, Vol. 13, No. 7, Jul. 2006, pp. 445-448.

[279] "Don't tie up your app with threads: Know how and when multi-threading works," *Visual Studio Developer*, Vol. 13, No. 5, Pinnacle Publishing, May 2006, pp. 1-5.

[280] Ma, W.-K., Vo, B.-N., Singh, S. S., Baddeley, A., "Tracking an Unknown Time-Varying Number of Speakers Using TDOA Measurements: A Random Finite Set Approach," *IEEE Trans. on Sig. Proc.*, Vol. 54, No. 9, Sep. 2006, pp. 3291-3304.

[281] Ishihara, J. Y., Terra, M. H., Campos, J. C. T., "Robust Kalman Filter for Descriptor Systems," *IEEE Trans. on Autom. Contr.*, Vol. 51, No. 8, Aug. 2006, pp. 1354-1358.

[282] Cho, A., "A New Way to Beat the Limits on Shrinking Transistors," *Science*, Vol. 313, Issue 5774, 5 May 2006, p. 672.

[283] Krebs, V., "Nonlinear Filtering Theory," in *Advances in the Techniques and Technology in the Application of Nonlinear Filters and Kalman Filters*, edited by C.T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph No. 256, Chap. 1, Noordhoff International Publishing, Lieden, 1981.

[284] Salazar, M. R., "State Estimation of Ballistic Trajectories with Angle Only Measurements," in *Advances in the Techniques and Technology in the Application of Nonlinear Filters and Kalman Filters*, edited by C.T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph No. 256, Chap. 18, Noordhoff International Publishing, Lieden, 1981.

[285] Wakker, K. F., Ambrosius, B. A. C., "Kalman Filter Satellite Orbit Improvement Using Laser Ranging from a Single Station," in *Advances in the Techniques and Technology in the Application of Nonlinear Filters and Kalman Filters*, edited by C.T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph No. 256, Chap. 17, Noordhoff International Publishing, Lieden, 1981.

[286] Liang, D. F., "Exact and Approximate Nonlinear Estimation Techniques," in *Advances in the Techniques and Technology in the Application of Nonlinear Filters and Kalman Filters*, edited by C.T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph No. 256, Chap. 2, Noordhoff International Publishing, Lieden, 1981.

[287] Liang, D. F., "Comparison of Nonlinear Filters for Systems with Non-Negligible Nonlinearities," in *Advances in the Techniques and Technology in the Application of Nonlinear Filters and Kalman Filters*, edited by C.T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph No. 256, Chap. 16, Noordhoff International Publishing, Lieden, 1981.

[288] Zhao, Z., Li, X.-R., Jolkov, V. P., "Best Linear Unbiased Filtering with Nonlinear Measurements," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 40, No. 4, Oct. 2004, pp. 1324-1330.

[289] Jin, Z., Gupta, V., "State Estimation Over Packet Dropping Networks using Multiple Description Codes," *Automatica*, Vol. 42, No. 9, Sept. 2006, pp. 1441-1452.

[290] Larson, R. E., Dressle, R. M., Ratner, R. S., "Application of the Extended Kalman Filter to Ballistic Trajectory Estimation," Final Report 5188-103 for Stanford Research Institute, Monlo Park, CA, Jan. 1967.

[291] Gruber, M., "An Approach to Target Tracking," Technical Note 1967-8, DDC 654272, MIT Lincoln Laboratory, Lexington, MA, Feb. 1967.

[292] Mehra, R. K., "A Comparison of Two Nonlinear Filters for Ballistic Trajectory Estimation," *Proc. of* $3^{\rm rd}$ *Symposium on Nonlinear Estimation Theory & Its Applications*, San Diego, CA, Sept. 1972, pp. 277-280.

[293] Athans, M., Whiting, R. H., Gruber, M., "A Suboptimal Estimation Algorithm with Probabilistic Editing for False Measurements with Application to Target Tracking with Wake Phenomena," *Proc. of* $3^{\rm rd}$ *Symposium on Nonlinear Estimation Theory & Its Applications*, San Diego, CA, Sept. 1972, pp. 17-33.

[294] Chang, C.-B., Whiting, R. H., Athans, M., "On the State and Parameter Estimation for Manuvering Re-entry Vehicles," *Proc. of* $3^{\rm rd}$ *Symposium on Nonlinear Estimation Theory & Its Applications*, San Diego, CA, Sept. 1972, pp. 63-78.

[295] Fitzgerald, R. J., "Range-Doppler Coupling and Other Aspects of Reentry Vehicle Tracking," *Proc. of* 4^{th} *Symposium on Nonlinear*

Estimation Theory & Its Applications, San Diego, CA, Sept. 1973, pp. 57-63.

[296] Brodzik, A. K., "On the Fourier Transform of Finite Chirps," *IEEE Signal Processing Letters,* Vol. 13, No. 9, Sep. 2006, pp. 541-544.

[297] Brown, C. D., *Spacecraft Mission Design,* AIAA Education Series, J. S. Przemieniecki (Ed.), AIAA, Wash. DC, 1992.

[298] Regan, F. J., Anandakrishnan, S. M., *Dynamics of Atmospheric Reentry,* AIAA Education Series, J. S. Przemieniecki (Ed.), AIAA, Wash. DC, 1993.

[299] Hong, S., Bolić, M., Djurić, P. M., "An Efficient Fixed-Point Implementation of Residual Resampling Scheme for High- Speed Particle Filters," *IEEE Signal Processing Letters*, Vol. 11, No. 5, May 2004, pp. 482-485.

[300] Rudd, J. G., Marsh, R. A., Roecker, J. A., "Surveillance and Tracking of Ballistic Missile Launches," *IBM Journal of Research and Develoment*, Vol. 38, No. 2, Mar. 1994, pp. 195-216.

[301] Levine, N., "A New Technique for Increasing the Flexibility Of Recursive Least Squares Data Smoothing," *The Bell System Technical Journal*, Vol. 40, No. 3, May 1961, pp. 821-840.

[302] *Computing in Science and Engineering: Special Issue on Monte Carlo Methods*, A Publication of the IEEE Computer Society, Vol. 8, No. 2, Mar./Apr. 2006, pp. 7-65.

[303] Rao, S. K., "Comments on 'Discrete-Time Observability and Estimability for Bearings-Only Target Motion Analysis," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 34, No. 4, Oct. 1998, pp. 1361-1367.

[304] Vetterling, W., Teukolsky, S., Press, W., and Flannery, B., *Numerical Recipies-Example Book (FORTRAN)*, Cambridge University Press, Cambridge, UK, 1986.

[305] Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, 1^{st} ed., McGraw-Hill Book Company, NY, 1965.

[306] Gray, J. E., Murray, W., "A Derivation of an Analytic Expression for the Tracking Index for the $\alpha - \beta - \gamma$ Filter," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 29, No. 3, Jul. 1993, pp. 1064-1065 (also see http://en.wikipedia.org/wiki/Alpha_beta_filter).

[307] Kerr, T. H., "Drawbacks of Residual-Based Event Detectors like GLR or IMM Filters in Practical Situations," *IEEE Trans. on Sig. Proc.*, Submitted in 2006 (under review). http://www.tekassociates.biz/savemeplease.pdf

[308] Horowitz, L. L., Senne, K. D., "Performance Advantage of Complex LMS for Controlling Narrow-Band Adaptive Arrays, *IEEE Trans. on Circuits and Systems*, Vol. 28, No. 6, Jun. 1981, pp. 562-576.

[309] Brookner, E., *Radar Technology*, Artech, Norwood, MA, 1977.

[310] Dionne, D., Michalska, H., Oshman, Y., Shinar, J., "Novel Adaptive Generalized Likelihood Ratio Detector with Application to Maneuvering Target Tracking, *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 2, Mar./Apr. 2006, pp. 465-474.

[311] Schmidt, G. C., "Designing Nonlinear Filters Based on Daum's Theory, *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 2, Mar./Apr. 1993, pp. 371-376.

[312] Duan, Z., Han, C., Li, X.-R., "Comments on 'Unbiased Converted Measurements for Tracking'" *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 40, No. 4, Oct. 2004, pp. 1374-1376.

[313] Friedland, B., "Separated-Bias Estimation and Some Applications," in *Advances in the Techniques and Technology in the Application of Nonlinear Filters and Kalman Filters*, edited by C.T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph No. 256, Noordhoff International Publishing, Lieden, 1981, pp. 15-1 to 15-13.

[314] Jesionowski, R., Zarchan, P., "Comparison of Filtering Options for Ballistic Coefficient Estimation," *Proceedings of* $7^{\rm th}$ *Annual AIAA/BMDO Technology Readiness Conference and Exhibit*, Session 10: Surveillance Technology Demonstations, paper No. 10-03, Fort Carlson, Colorado Springs, CO, 3-6 Aug. 1998 (Unclassified).

[315] Reed, I. S., Mallet, J. D., Brennan, L. E., "Rapid Convergence Rate in Adaptive Arrays," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 10, No. 6, pp. 853-868, Nov. 1974.

[316] Galatsanos, N. P., Chin, R. T., "Restoration of Color Images by Multichannel Kalman filtering," *IEEE Trans. on Sig. Proc.*, Vol. 39, No. 10, pp. 2237-2252, Oct. 1991.

[317] Buzzi, S., Lops, M., Venturino, L., Ferri, M., "Track-Before-Detect Procedures in Multi-Target Environments," *IEEE Trans. Aerosp. Electron. Syst.*, Vol. 44, No. 3, pp. 1135-1150, Jul. 2007.

[318] Lambert, H. C., Daum, F. E., Weatherwax, J. L., "A Split-Step Solution of the Fokker-Planck Equation for the Conditional Density," *Proceedings of the Asilomar Conf.*, DOI: 10.1109/ACSSC.2006.355119, pp. 2014-2018, 2006.

[319] Daum, F. , Krichman, M., "Non-Particle Filters," *Proceedings of MIT Lincoln Laboratory ASAP Workkshop*, Session 1t: Tracking, paper no. 23, Lexington, MA, 6-7 Jun. 2006.

[320] Daum, F. , Krichman, M., "Meshfree Adjoint Methods for Nonlinear Filtering," *Proceedings of IEEE Aerospace Conf.*, BigSky, Montana, pp. ?-?, Mar 2006.

[321] Oh, M.-S., "Monte-Carlo Integration via Importance Sampling: dimensionality effects and an adaptive algorithm," *Contemporary Mathematics*, Vol. 115, pp. 165-187, 1991.

[322] Giles, M. B., Suli, E., "Adjoint Methods for PDEs," *Acta Numerica*, Cambridge University Press, pp. 145-236, 2002.

[323] Traub, J., and Werschultz, A., "Complexity and Information," *Proceedings of IEEE Aerospace Conf.*, Cambridge University Press, 1998.

[324] Buzzi, S., Lops, M., Venturino, L., Ferri, M., "Track-before-Detect Procedures in a Multi-Target Environment," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 44, No. 3, pp. 1135-1150, Jul. 2008.

[325] Woffenden, D, C., Geller, D. K., "Observability Criteria for Angles-Only Navigation," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 45, No. 3, pp. 1194-1208, Jul. 2009.

[326] Buzzi, S., Lops, M., Venturino, L., Ferri, M., "Something that uses Covariance Intersection," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. ?, No. 3, pp. 1135-1150, Jul. 2008.

[327] Carlton, A. G., Follin, J. W., "Recent Development in Fixed and Adaptive Filtering," *NATO AGARDograph*, No. 21, 1956. (This date is why R. Bucy had originally claimed precedence for James Follin at JHU/APL over Kalman).

[328] Battin, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics*, 2^{nd} Edition AIAA Education Series, Boston, MA, 1999.

[329] http://ccar.colorado.edu/asen5050/ASEN5050/Lectures_files/lecture11.pdf

[330] http://www.tekassociates.biz/TeKMemoLambertNMDcueSRS29Sept2KAM.pdf

[331] La Scala, B. F., Bitmead, R. R., James, M. R., "Conditions for stability of the Extended Kalman Filter and their application to the frequency tracking problem," *Math. Control, Signals Syst*. (MCSS), vol. 8, No. 1, pp. 1-26, Mar. 1995.

[332] Reif, K., Gunther, S., Yaz, E., Unbehauen, R., "Stochastic stability of the continuous-time extended Kalman filter," *Proc. Inst. Elect. Eng.*, Vol. 147, p. 45, 2000.

[333] Crassidis, J. L., Markley, F. L., Cheng, Y., "Survey of Nonlinear Attitude Estimation Methods," *Journal of Guidance, Control, and Dynamics,* Vol. 30, No. 1, pp. 12-28, Jan. 2007.

[334] Majji, M., Junkins, J. L., Turner, J. D., "J^{th} Moment Extended Kalman Filtering for Estimation of Nonlinear Dynamic Systems," *AIAA Guidance, Navigation, and Control Conference and Exhibit*, Honolulu, HI, Paper No. AIAA 2008-7386, pp. 1-18, 18-21 Aug. 2008.

[335] Scorse, W. T., Crassidis, A. L., "Robust Longitudinal and transverse Rate Gyro Bias Estimation for Precise Pitch and Roll Attitude Estimation in Highly Dynamic Operating Environments Utilizing a Two Dimensional Accelerometer Array," *AIAA Atmospheric Flight Mechanics Conference*, Paper No. AIAA 2011-6447, Portland, OR, pp. 1-28, 8-11 Aug. 2011.

[336] Jensen, Kenneth J., "Generalized Nonlinear Complementary Attitude Filter," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 34, No. 5, pp. 1588-1593, Sept.-Oct. 2011.

[337] Salcudean, S., "A globally convergent angular velocity observer for rigid body motion," *IEEE Trans. on Autom. Control*, Vol. 36, No. 12, pp.1493-1497, Dec. 1991.

[338] Choukroun, D., Weiss, H., Bar-Itzhack, I. Y., Oshman, "Kalman Filtering for Matrix Estimation," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 42, No. 1, pp. 147-159, Jan. 2006.

[339] Choukroun, D., Weiss, H., Bar-Itzhack, I. Y., Oshman, "Direction Cosine Matrix Estimation from Vector Observations Using a Matrix Kalman Filter," *AIAA Guidance, Navigation, and Control Conference and Exhibit*, pp. 1-11, Aug. 2003.

[340] Choukroun, D., "A Novel Quaternion Kalman Filter using GPS Measurements," *Proceedings of ION GPS*, Portland, OR, pp. 1117-1128, 24-27 Sep. 2002.

[341] Choukroun, D., Weiss, H., Bar-Itzhack, I. Y., Oshman, "Kalman Filtering for Matrix Estimation," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 42, No. 1, pp. 147-159, Jan. 2006.

[342] Choukroun, D., Bar-Itzhack, I. Y., Oshman, "Novel Quaternion Kalman Filter," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 42, No. 1, pp. 174-190, Jan. 2006.

[343] Choukroun, D., Weiss, H., Bar-Itzhack, I. Y., Oshman, "Direction Cosine Matrix Estimation From Vector Observations Using A Matrix Kalman Filter," *Proceedings of AIAA Guidance*, Navigation, and

Control Conference and Exhibit, Austin, TX, pp. 1-11, 11-14 Aug. 2003.

[344] Choukroun, D., "Ito Stochastic Modeling for Attitude Quarternion Filtering," *Proceedings of Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, P. R. China, pp. 733-738, 16-18 Dec. 2009.

[345] Cheng, Y., Landis Markley, F., Crassidis, J. L. Oshman, Y., "Averaging Quaternions," *Advances in the Astronautical Sciences series*, Vol. 127, American Astronautical Society, AAS paper No. 07-213, 2007.

[346] Landis Markley, F., "Attitude Filtering on SO(3)," *Advances in the Astronautical Sciences series*, Vol. 122, American Astronautical Society, AAS paper No. 06-460, 2006.

[347] Cheng, Y., Crassidis, J. L., and Landis Markley, F., "Attitude Estimation for Large Field-of-View Sensors," *Advances in the Astronautical Sciences series*, Vol. 122, American Astronautical Society, AAS paper No. 06-462, 2006.

[348] Landis Markley, F., "Attitude Estimation or Quaternion Estimation?," *Advances in the Astronautical Sciences series*, Vol. 115, American Astronautical Society, AAS paper No. 03-264, 2003.

[349] Reynolds, R., Landis Markley, F., Crassidis, J. L., "Asymptotically Optimal Attitude and Rate Bias Estimation with Guaranteed Convergence," *Advances in the Astronautical Sciences series*, Vol. 132, American Astronautical Society, AAS paper No. 08-286, 2008.

[350] Daum, F. E., Huang, J., "Seven dubious methods to mitigate stiffness in particle flow with non-zero diffusion for nonlinear filters, Bayesian decisions and transport," *Proceedings of SPIE Conference 9092: Signal and Data Processing of Small Targets 2014*, Session 2: Tracking Small Targets I, Paper 9092-11, Baltimore, Maryland, 7, 8 May 2014.

[351] Daum, F. E., Huang, J., *Particle Flow for Nonlinear Filters*, presentation at local Boston IEEE AES meeting, 21 Jan. 2014.

[352] Battin, R. H., Vaughan, R. M., "An elegant Lambert algorithm," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 7, pp. 662-670, 1984.

[353] Loechler, L. A., *An Elegant Lambert Algorithm for Multiple Revolution Orbits,*" M.S. Thesis, Department of Aero. & Astro., MIT, Cambridge, MA, 1988.

[354] Bar-Shalom, Y., Li, X.-R., Kirubarajan T., *Estimation with Applications to Tracking and Navigation: algorithms and software for information extraction*, Wiley, 2001.

[355] Mortensen, R. E., *Optimal Control of Continuous-Time Stochastic Systems*, Ph.D. Thesis (engineering), Univ. of California, Berkeley, CA, 1966.

[356] Sofir, I., "Improved Method for Calculating Exact Geodetic Latitude and Attitude-Revisited," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 2, ff. 369, 2000.

[357] Knuth, D. E., *The Art of Computer Programming*, Vol. 2: Seminumerical Algorithms, 2nd Edition, Addison-Wesley, Reading, MA, 1981.

[358] Soysal, G., Efe, M., "Data fusion in a multistatic radar network using covariance intersection and particle filtering," Proc. of the 14^{th} International Conference on Information Fusion (FUSION), Chicago, Il, 5-8 Jul. 2011.

[359] Brown, R. G., Hwang, P. Y. C., *Introduction to Random Signals and Applied Kalman Filtering,* 2^{nd} Edition, John Wiley and Sons, 1992.

[360] Magill, D. T., "Optimal Adaptive Estimation of Sampled Stochastic Processes," *IEEE Trans. on Automatic control*, Vol. 10, No. 4, pp. 434-439, 1965.

[361] Patel, J. K., Kapadia, C. H., Owen, D. B., *Handbook of Statistical Distributions,* Marcel Dekker, N.Y., 1976.

[362] Gibbes, B. P., *Advanced Kalman Filtering, Least-Squares and Modeling: a practical handbook,* John Wiley % Sons,Hoboken, NJ, 2011. http://onlinelibrary.wiley.com/doi/10.1002/9780470890042.fmatter/pdf

[363] Khaleghi, B., Khamis, A., Karray, F. O., Razavi, S. N., "Multisensor data fusion: A review of the state-of-the-art," *Information Fusion*, Vol. 14, No. 1, Jan. 2013, pp. 2844

[364] Khaleghi, B., Khamis, A., Karray, F. O., Razavi, S. N., "Corrigendum to 'Multisensor data fusion: A review of the state-of-the-art'," *Information Fusion*, Vol. 14, No. 10, Oct. 2013, pp. 562.

[365] Ayers, F., *Theory and Problems of differential Equations,*, Schaum's Outline Series, Schaum Publishing Co., NY, 1952.

[366]
- https://www.siam.org/pdf/news/744.pdf
- http://www.sciencedirect.com/science/article/pii/S0167819198000106
- https://en.wikipedia.org/wiki/Linear_congruential_generator
- http://link.springer.com/article/10.1023%2FA%3A1022333627834
- http://maths-people.anu.edu.au/˜brent/pd/rpb132tr.pdf

[367] Grunberg, D. B., Athans, M., *Guaranteed Properties of The Extended Kalman Filter*, MIT Laboratory for Information and Decision Systems, LIDS-P-1724, MIT, Cambridge, MA, Dec. 1987

[368] L'Ecuyer, P., "Software for Uniform Random Number Generation: Distinguishing the Good and the Bad," *Proceedings of the 2001 Winter Simulation Conference entitled: A Simulation Odyssey*, Ed. by B.A.Peters, J.S.Smith, D.J.Mederios, M.W.Rohrer, Vol. 1, pp. 95-105, Arlington, VA, 9-12 Dec. 2001.

[369] Callegari, S., Rovatti, R., Setti, G., "Embeddable ADC-Based True Random Number Generator for Cryptographic Applications Exploiting Nonlinear Signal Processing and Chaos," *IEEE Trans. on Signal Processing*, Vol. 53, No. 2, pp. 793-805, Feb. 2005.

[370] Brcich, R. F., Iskander, D. R., Zoubir, A. M., "The Stability Test for Symmetric Alpha-Stable Distributions," *IEEE Trans. on Sig. Proc.*, Vol. 53, No. 3, Mar. 2005, pp. 977-986.

[371] Abramowitz, M., and Stegun, I. A., *Handbook of Mathematical Tables*, National Bureau of Standards, AMS Series 55, 1966.

[372] Kerr, T. H., *Applying Stochastic Integral Equations to Solve a Particular Stochastic Modeling Problem,* Ph.D. thesis, Department of Electrical Engineering, Univ. of Iowa, Iowa City, IA, 1971.

[373] Kay, S., "Efficient Generation of Colored Noise," *Proc. of IEEE*, Vol. 69, No. 4, pp. 480-481, Apr. 1981.

[374] Bach, E., "Efficient Prediction of Marsaglia-Zaman Random Number Generator," *IEEE Trans. on Information Theory,* Vol. 44, No. 3, pp. 1253-1257, May 1998.

[375] Morgan, D. R., "Analysis of Digital Random Numbers Generated from Serial Samples of Correlated Gaussian Noise," *IEEE Trans. on Infor. Theory*, Vol. 27, No. 2, Mar. 1981.

[376] Atkinson, A. C., "Analysis Tests of Pseudo-random Numbers," *Applied Statistics*, Vol. 29, No. 2, pp. 154-171, 1980.

[377] Sanwate, D. V., and Pursley, M. B., "Analysis Crosscorelation Properties of Pseudo-random and Related Sequences," *Proc. of the IEEE*, Vol. 68, No. 5, pp. 593-619, May 1980.

[378] Daum, F., Huang, J., "Exact particle flow for nonlinear filters: Seventeen dubious solutions to a first order linear underdetermined PDE, *44th Asilomar Conf. on Signals, Systems and Computers,* Pacific Grove, CA, 7-10 Nov. 2010, pp. 64 - 71.

[379] Michael Griebel, Marc Alexander Schweitzer (eds.) *Meshfree Methods for Partial Differential Equations IV,* Springer, 2008.

[380] Daum, F., Huang, J., "Nonlinear filters with particle flow induced by log-homotopy," *SPIE Proc., Vol. 7336: Signal Processing, Sensor Fusion, and Target Recognition XVIII,* Ivan Kadar (Ed.), 11 May 2009.

[381] Strauss, W.A., *Partial Differential Equations: an introduction,* 2nd Ed., John Wiley & Sons, Ltd, 2008.

[382] Athans, M. and Tse, E., "A Direct Derivation of the Optimal Linear Filter using the Maximum Principle," *IEEE Trans. on AC*, Vol. 12, pp. 690-698, 1967.

[383] Balakrishnan, A. V., *Stochastic Differential Systems I - Filtering and Control: a function space approach*, Springer-Verlag, NY, 1973.

[384] Jekeli, C., "Precision Free-Inertial Navigation with Gravity Compensation by an Onboard Gradiometer," *SAIAA Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 3, May-June 2006, pp. 704-713.

[385] Kerr, T. H., "Comment on 'Precision Free-Inertial Navigation with Gravity Compensation by an Onboard Gradiometer'," *SAIAA Journal of Guidance, Control, and Dynamics*, Vol. 30, No. 4, Jul.-Aug. 2007, pp. 1214-1215.

[386] Richert, T., El-Sheimy, N., "Low-Noise Linear Combination of Triple-Frequency Carrier Phase Measurements," *SNavigation: Journal of the Institute of Navigation*, Vol. 53, No. 1, Spring 2006, pp. 61-67.

[387] Kerr, T. H., "Comment on 'Low-Noise Linear Combination of Triple-Frequency Carrier Phase Measurements'," *SNavigation: Journal of the Institute of Navigation*, Vol. 57, No. 2, Summer 2010, pp. 161162.

[388] Arnaud Doucet, Nando DeFritas, Neil Gordon (Eds.), *Sequential Monte Carlo Methods in Practice*, Princeton University Press, Princeton, NJ, 2001.

[389] Cheng, Y., Singh, T., "Efficient Particle Filtering for Road Constrained Target tracking," *IEEE Trans. on AES,*" Vol. 43, No. 4, Oct. 2007, pp. 1454-1469.

[390] Ockendon, J., Howison, S., Locey, A., and Movchan, A., *Applied Partial Differential Equations*, Oxford University Press, Oxford, NY, 2005.

[391] Kerr, T. H., *ADA70 Steady-State Initial-Value Convergence Techniques*, General Electric Report, Technical Information Series No. 72 CRD095, Schenectady, NY, 1972.

[392] G. Janashia, E. Lagvilava, and L. Ephremidze, "A New Method of Matrix Spectral Factorization," *IEEE Trans. Inform. Theory,* Vol. 57, No. 4, 2011, pp. 2318-2326.

[393] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian

tracking," *IEEE Trans. Signal Processing,* Vol. 50, pp. 174188, Feb. 2002.

[394] R.van der Merwe and E. Wan, "Gaussian mixture sigma-point particle filters for sequential probabilistic inference in dynamic state-space models" *Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing (ICASSP),* Hong Kong, 2003, pp. 701704.

[395] R. Fonod, M. Idan, and J. L. Spyer, "Approximate estimators for linear systems with additive Cauchy Noises," *AIAA Journal of Guidance, Control, and Dynamics,* Vol. 40, No. 11, pp. 28202827, Nov.. 20017.

PLACE PHOTO HERE

Thomas H. Kerr III received the BSEE (magna cum laude) in electronics from Howard Univ. in 1967 and MSEE & Ph.D. degrees in EE from the Univ. of Iowa in 1969 & 1971, respectively, both in Modern Control and Estimation.

His experience since 1971 as an R&D algorithm engineer and software developer has encompassed various Kalman filter theoretical evolutionary and revolutionary developments for DoD applications. As a Controls Engineer at General Electric's Corporate Research & Development Center (Schenectady, NY; 1971-'73), he worked on simulations in, analysis for, and improvements to G.E.'s Automated Dynamic Analyzer (ADA). At TASC (1973-'79), he developed real-time failure detection algorithms for Poseidon and Trident SSBN submarine SINS/ESGM navigation and evaluated/improved algorithms and strategies for SSBN navaid fix-taking. At Intermetrics, Inc. (Cambridge, MA; 1979-'86), he was involved in applying evolving decentralized Kalman filters for Navy JTIDS RelNav refinement and for Air Force ICNIA decentralized filter development, along with performing IV&V for sonobuoy and DT&E(OR) for GPS in the SSN-701 attack sub. He also worked on security aspects (h/w & s/w) on the improvement program for WWMCCS. At Lincoln Laboratory (Lexington, MA; 1986-'92), he performed radar target tracking of reentry vehicles (RV's), and performed GPS navigation analysis for airborne terrain mapping in support of a Neural Network-based ATR application (by others). He is currently CEO/Principal Engineer at TeK Associates, an engineering consulting, R&D, and s/w development company he founded in 1992. As a subcontractor for MITRE, XonTech, and Raytheon (2 contracts), he consulted on UEWR/NMD. Under a TeK Associates subcontract from Arete, he performed accuracy trade studies on various alternative INS using GPS and/or differential GPS position fixes at a high rate for airborne INS/GPS for the Navy ACOSS Littoral surveillance program. As a contractor at Goodrich ISR, he worked on Kalman filters for U-2 surveillance camera pointing accuracy improvements. He also taught *Optimal Control* in the Graduate ECE Department of Northeastern Univ. in the evenings (1990-95).

His software programming experience is in Assembly Language, PL/1, Fortran, HTML, but now usually uses Visual Basic and MatLab/Simulink almost exclusively (as well as modules of his own TK-MIP) for Monte-Carlo simulation in the above applications to set parameters and substantiate conclusions.

He received the M. Barry Carlton Award for Outstanding Paper to appear in *IEEE Trans. on Aerospace and Electronic Systems* for 1987 has been chairman of the local Boston IEEE Control Systems Section twice (1990-92; 2002-04), is an IEEE Life Senior member and an AIAA GNC Assoc. Fellow, and has been a member of the Institute of Navigation (since 1981), ISA, and MSDN (level 2) and a life Member of the NDIA. He served for decades as a reviewer for IEEE Trans. on AC, IT, AES, SMC, Computer Science and for AIAA Journal of Guidance, Control, and Dynamics. He served as Vice-Chairman of "Stochastic Control Session" at 1975 Conference on Decision and Control (CDC) and as Co-chairman of "Sensors, Components, and Algorithms for Navigation" session at the Institute of Navigation (ION) Annual Summer Conference (1999) in Cambridge, MA. Academic affiliations: TBΠ, ΠME, ΣΠΣ, EKN, and ΣΞ. He has completed 12 marathons and bikes with the Charles River Wheelman (since '77 & served as a ride leader for 5 yeats).