Mon. 25, 1971

Hi Joe,

Here is the new method of approach (numerical rather than graphical) of specifying convergence acceleration factors that I ~~th~~ stumbled upon over the weekend that I mentioned to you this morning. It makes use of a result by Gershgorin (1931).

Presented here is a proof of the method (result applied to ~~acceleration~~ convergence factor selection not a proof of Gershgorin's result), discussion of how applied, and a worked numerical example which has been worked previously in the report by two other distinct techniques.

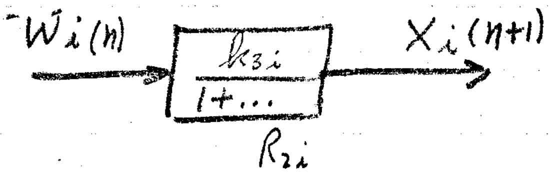
Respectfully,

Tom Kerr

An Application of a Result by Gershgorin (1931)

Another conservative approach that can be of use in assuring that the eigenvalues of the characteristic equation corresponding to the difference equations which describe the iterations all lie within the unit circle. (Based on a 1931 result by Gershgorin, Bull. Acad. Sciences de l'U.R.S.S., Classe mathém., 7-e série, Leningrad, 1931, p. 749.)
Can also be found in Kreyszig, <u>Adv. Eng. Math.</u>, John Wiley and Sons, New York, 1962, p. 456.

For a general linear system, let the output of each type 1 box be a state variable:



The equation which describes the iterations internal to ADA is

$$X_i(n+1) = k_{3i} W_i(n) + R_{2i}\left[k_{3i} W_i(n) - X_i(n)\right],$$

a difference equation.
Since the system is linear, the input $W_i(n)$

must be of the following form:

$$W_i(n) = \sum_{j=1}^{n} a_{ij} X_j(n) + U_i(0),$$ where n is the number of type 1 boxes in the simulation.

For a specific type 1 box, the difference equation now becomes

$$X_i(n+1) = k_{3i}\left[\sum_{j=1}^{n} a_{ij}\, X_j(n) + U_i(0)\right]$$
$$+ R_{2i}\left[k_{3i} \sum a_{ij}\, X_j(n) + k_{3i}\, U_i(0) - X_i(n)\right]$$

or

$$X_i(n+1) = \sum_{j=1}^{n}\left[k_{3i}\,(1+R_{2i})\, X_j(n) - R_{2i} X_i(n)\, \delta_{ij}\right]$$

$$+ k_{3i}(1+R_{2i})\, U_i(0),$$

for $i = 1, 2, \ldots n$.

Let $\underline{X}(n) = (X_1(n), X_2(n), \ldots, X_n(n))^T$; the characteristic equation corresponding to the above system of difference equations is:

$$\underline{X}(n+1) = A\,\underline{X}(n), \text{ where } A = \begin{bmatrix} \cdots a_1 \cdots \\ \cdots a_2 \cdots \\ \vdots \\ \cdots a_n \cdots \end{bmatrix}.$$

E.g., $a_1 = [(1+R_{21})k_{31}a_{11} - R_{21}, (1+R_{21})k_{31}a_{12}, \ldots, (1+R_{21})k_{31}a_{1n}]$.

Now applying Gershgorin's result, we have that

$$|(1+R_{2i})k_{3i}a_{ii} - R_{2i} - \lambda| \le |1+R_{2i}| \sum_{\substack{j=1 \\ j\ne i}}^{n} |k_{3i}||a_{ij}|$$

$i = 1, 2, \ldots, n$.

This means that for each eigenvalue, the left-hand side of the above inequality says that we have a bound for the eigenvalue which is a circle having a center at $(1+R_{2i})k_{3i}a_{ii} - R_{2i}$ and radius given by the right-hand side of the inequality,

$$|1+R_{2i}| \sum_{\substack{j=1 \\ j\ne i}}^{n} |k_{3i}||a_{ij}|.$$

If the center of the bounding circle is within the unit circle and since $\sum_{\substack{j=1 \\ j\ne i}}^{n} |k_{3i}||a_{ij}|$ is a finite sum of finite elements, hence is finite, then we can adjust R_{2i} so that the bounding circle is within the unit circle. If we do this for each i $(i = 1, \ldots, n)$, we are assured that all of the eigenvalues lie within the unit circle.

As an example of how this technique is applied, let us elaborate on this procedure for $i = 1$.

Gershgorin's result is

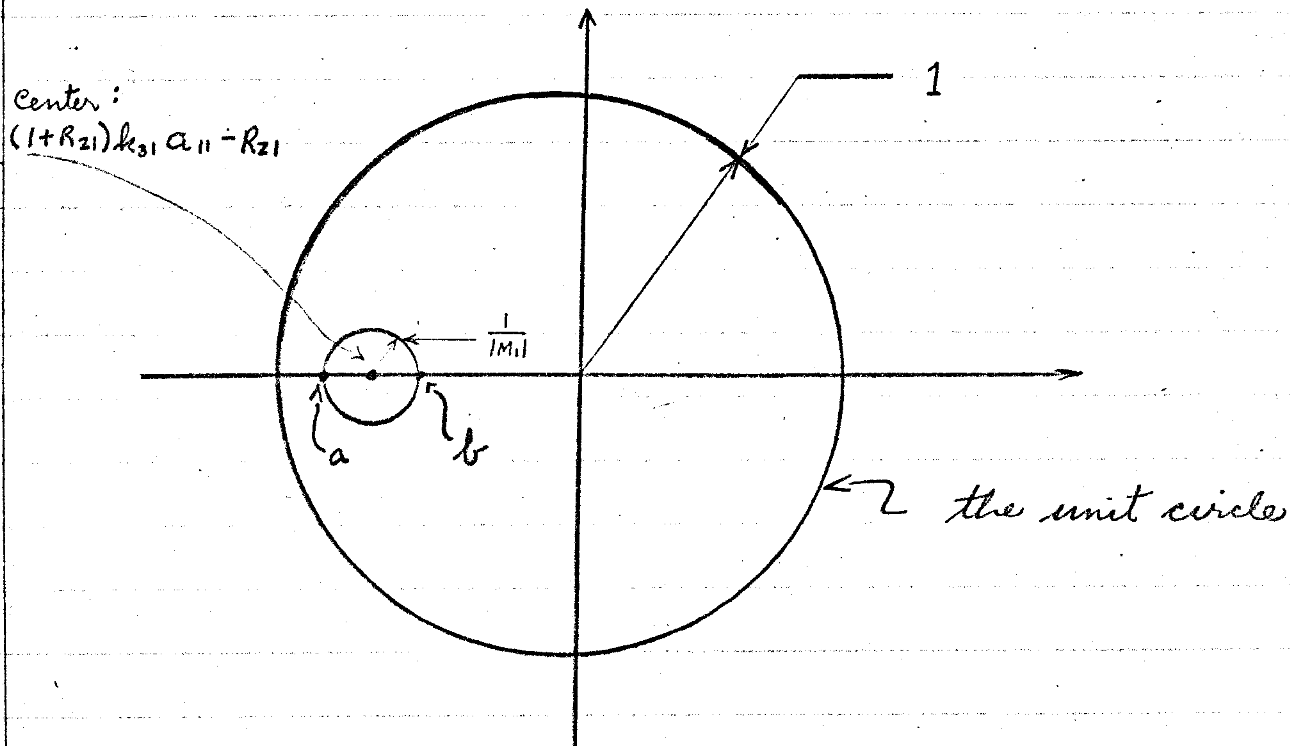
$$\left| (1+R_{21}) k_{31} a_{11} - R_{21} - \lambda \right| \leq |1+R_{21}| \sum_{j=2}^{n} |k_{31}| |a_{1j}|.$$

Since the sum on the right-hand side is a finite sum of finite values equal to, say, N_1 (i.e., $\sum_{j=2}^{n} |k_{31}| |a_{1j}| = N_1 < \infty$).

Let $R_{21} = -1 + \frac{1}{M_1 N_1}$, then we have that

$$|1+R_{21}| \sum_{j=2}^{n} |k_{31}| |a_{1j}| \leq \frac{1}{|M_1| N_1} N_1 = \frac{1}{|M_1|}.$$

By adjusting M_1 we can make $\frac{1}{|M_1|}$ as small as is required to make the bounding circle which has its center within the unit circle lie entirely within the unit circle. This concept may be graphically portrayed in the following diagram:



By making $R_{21} = -1 + \frac{1}{M_1 N_1}$, the center of the circle is at

$$(1+R_{21})k_{31}a_{11} - R_{21} = \frac{k_{31}a_{11}}{M_1 N_1} + 1 - \frac{1}{M_1 N_1}$$

$$= 1 + \frac{(k_{31}a_{11} - 1)}{M_1 N_1}$$, where M_1 can be either positive or negative. We want the center of the eigenvalue bounding circle to lie within the unit circle so we require M_1 be chosen so that

$$-1 < 1 + \frac{k_{31}a_{11} - 1}{M_1 N_1} < 1$$

or $$-2 < \frac{k_{31}a_{11} - 1}{M_1 N_1} < 0 ,$$

which is always possible as long as $k_{31}\, a_{11} \neq 1$.

If the points $a = 1 + \frac{k_{31}a_{11} - 1}{M_1 N_1} - \frac{1}{|M_1|}$ and $b = 1 + \frac{k_{31}a_{11} - 1}{M_1 N_1} + \frac{1}{|M_1|}$ of the eigenvalue bounding circle are within the unit circle, then the whole eigenvalue bounding circle is within the unit circle (and so is the eigenvalue). Since it is obvious that $a < b$, all that must be done is to find (or adjust) M_1 to make $-1 < a$ and $b < 1$

or, equivalently,

$$-2 < \frac{k_{31}a_{11} - 1}{M_1 N_1} - \frac{1}{|M_1|}$$

and $$\frac{k_{31}a_{11} - 1}{M_1 N_1} + \frac{1}{|M_1|} < 0 \quad , \text{ with } k_{31}a_{11} \neq 1.$$

For the inequality $\frac{k_{31}a_{11}-1}{M_1 N_1} + \frac{1}{|M_1|} < 0$

If $M_1 > 0$

$$\frac{k_{31}a_{11}-1}{N_1} + \frac{M_1}{|M_1|} < 0$$

$$\frac{k_{31}a_{11}-1}{N_1} + 1 < 0$$

$$\frac{k_{31}a_{11}-1}{N_1} < -1$$

If $M_1 < 0$

$$\frac{k_{31}a_{11}-1}{N_1} + \frac{M_1}{|M_1|} > 0$$

$$\frac{k_{31}a_{11}-1}{N_1} - 1 > 0$$

$$\frac{k_{31}a_{11}-1}{N_1} > 1$$

So for the other inequality, $-2 < \frac{k_{31}a_{11}-1}{M_1 N_1} - \frac{1}{|M_1|}$

If $\frac{k_{31}a_{11}-1}{N_1} < -1$, then $M_1 > 0$ and pick $M_1 > -\frac{1}{2}\left(\frac{k_{31}a_{11}-1}{N_1} - 1\right)$.

If $\frac{k_{31}a_{11}-1}{N_1} > 1$, then $M_1 < 0$ and pick $M_1 < -\frac{1}{2}\left(\frac{k_{31}a_{11}-1}{N_1} + 1\right)$.

If $-1 < \frac{k_{31}a_{11}-1}{N_1} < 1$, this whole method cannot be applied.

In order to apply this conservative method to a simulation involving n ADA type 1 boxes, it must not be that

$$-1 < \frac{k_{3i} a_{ii} - 1}{N_i} < 1$$

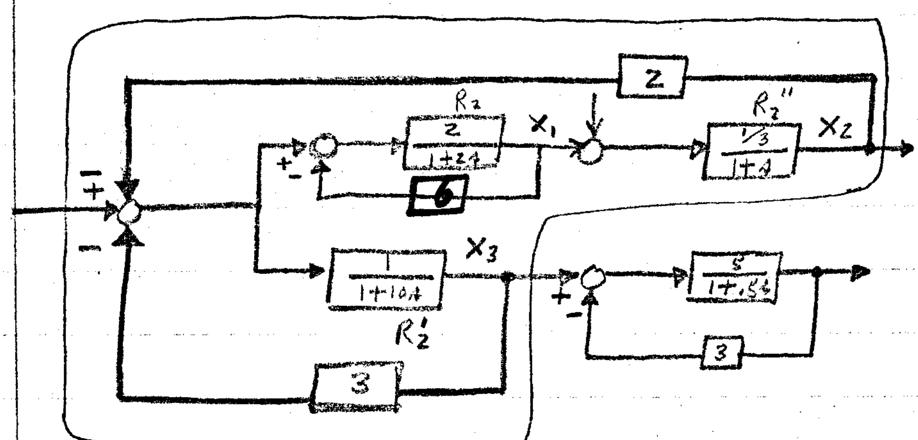
for any i, $i = 1, 2, \ldots, n$.

For a general linear problem which does not satisfy $-1 < \frac{k_{3i} a_{ii} - 1}{N_i} < 1$

for any $i = 1, \ldots, n$, the method of forcing the bounding circle to be entirely within the unit circle is applied for each i, $i = 1, \ldots, n$. This bounds all n of the eigenvalues within the unit circle. This causes the system of difference equations to converge, hence the steady state initial condition iterations of ADA converge.

Let us apply this new method to a numerical example which is similar to an example which has been previously worked by two other techniques.

Example 7c:



$$X_1(n+1) = 2\left[-2X_2(n) - 3X_3(n) - 6X_1(n)\right] +$$

$$R_{21}\left\{2\left[-2X_2(n) - 3X_3(n) - 6X_1(n)\right] - X_1(n)\right\}$$

$$X_2(n+1) = \tfrac{1}{3}\left[X_1(n)\right] + R_{22}\left\{\tfrac{1}{3}\left[X_1(n)\right] - X_2(n)\right\}$$

$$X_3(n+1) = 1\left[-2X_2(n) - 3X_3(n)\right]$$

$$+ R_{23}\left\{\left[-2X_2(n) - 3X_3(n)\right] - X_3(n)\right\}$$

First $\quad N_1 \triangleq \sum_{j=2}^{3} |k_{31}||a_{1j}| = 2\left(|-2|+|-3|\right) = 10,$

$$\frac{k_{31}a_{11}-1}{N_1} = \frac{2(-6)-1}{10} = -\frac{13}{10},$$

$$N_2 \triangleq \sum_{\substack{j=1 \\ j\neq 2}}^{3} |k_{32}||a_{2j}| = \tfrac{1}{3}\left(|1|+0\right) = \tfrac{1}{3},$$

$$\frac{k_{32}a_{22}-1}{N_2} = \frac{\frac{1}{3}(0)-1}{\frac{1}{3}} = -3,$$

$$N_3 \triangleq \sum_{j=1}^{2} |k_{33}||a_{3j}| = 1\left(0+|-2|\right) = 2,$$

and $\quad \dfrac{k_{33}a_{33}-1}{N_3} = \dfrac{1(-3)-1}{2} = -2.$

Since $\quad \dfrac{k_{31}a_{11}-1}{N_1} = -\dfrac{13}{10} < -1,$

$$\frac{k_{32}a_{22}-1}{N_2} = -3 < -1,$$

and $\quad \dfrac{k_{33}a_{33}-1}{N_3} = -2 < -1,$ this

conservative method can be applied. Where this method can be applied the work is simplified considerably since convergence factors can be specified one at a time

For x_1, we have that

$$|(1+R_{21})k_{31}a_{11} - R_{21} - \lambda| \le |1+R_{21}| \sum_{j=2}^{3} |k_{31}||a_{1j}|,$$

$$N_1 \triangleq \sum_{j=2}^{3} |k_{31}||a_{1j}| = 2\sum_{j=2}^{3} |a_{1j}| = 2\left(|-2|+|-3|\right) = 10,$$

and
$$\frac{k_{31}a_{11}-1}{N_1} = \frac{2(-6)-1}{10} = \frac{-13}{10}.$$

Since $-\frac{13}{10} < -1$, choose M_1

such that $M_1 > -\frac{1}{2}\left(\frac{k_{31}a_{11}-1}{N_1} - 1\right) = -\frac{1}{2}\left(-\frac{13}{10} - 1\right) = \frac{23}{20}.$

An acceptable value is $M_1 = 2$, therefore

$$R_2 = R_{21} = -1 + \frac{1}{2\cdot 10} = -1 + 0.05 = -0.95.$$

For x_2, we have that

$$\left|(1+R_{22})k_{32}a_{22}-R_{22}-\lambda\right| \le |1+R_{22}| \sum_{\substack{j=1\\ j\neq 2}}^{3} |k_{32}||a_{2j}|$$

$$N_2 \triangleq \sum_{\substack{j=1\\ j\neq 2}}^{3} |k_{32}||a_{2j}| = \frac{1}{3}(|1|+0) = \frac{1}{3},$$

and $\dfrac{k_{32}a_{22}-1}{N_2} = \dfrac{\frac{1}{3}(0)-1}{\frac{1}{3}} = -3.$

Since $-3 < -1$, choose M_2 such that

$$M_2 > -\frac{1}{2}\left(\frac{k_{32}a_{22}-1}{N_2} - 1\right) = -\frac{1}{2}(-3-1) = 2.$$

An acceptable value is $M_2 = 4$, therefore

$$R_2'' = R_{22} = -1 + \frac{1}{\frac{1}{3}\cdot 4} = -1 + \frac{3}{4} = -0.25.$$

For x_3, we have that

$$|(1+R_{23})k_{33}a_{33}-R_{23}-\lambda| \le |1+R_{23}|\sum_{j=1}^{2}|k_{33}||a_{3j}|,$$

$$N_3 \triangleq \sum_{j=1}^{2}|k_{33}||a_{3j}| = 1(0+|-2|) = 2,$$

and $$\frac{k_{33}a_{33}-1}{N_3} = \frac{1(-3)-1}{2} = -2.$$

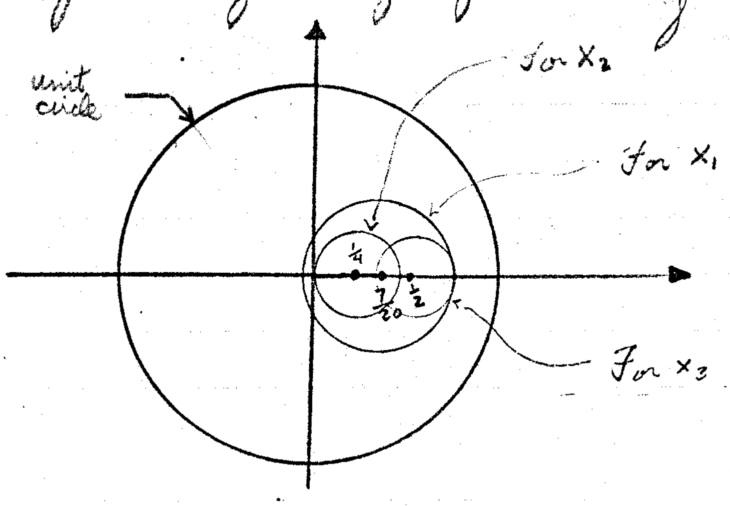
Since $-2<-1$, choose M_1

such that $M_3 > -\frac{1}{2}\left(\frac{k_{33}a_{33}-1}{N_3}-1\right) = -\frac{1}{2}(-2-1) = \frac{3}{2}$.

An acceptable value is $M_3 = 4$, therefore

$$R_2' = R_{23} = -1+\frac{1}{4\cdot 2} = -1+.125 = -.875.$$

Notice that using this approach, the convergence acceleration factors are picked one at a time. The results of selecting convergence acceleration factors by this method may be portrayed graphically as



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$LIST ØUT
10/26/71    13:25


 ADA70  10/26/71  13.416


  CASE    1      STEADY-STATE INITIAL VALUES  ITERATIØN     35

 ALGEBRAIC FUNCTIØNS

  NAME       VALUE        NAME       VALUE       NAME       VALUE
  E1       -8.1220E-02   E2       -6.2477E-03   E3       1.9875E 00
  E4       -5.0767E-03


 CØNSTANT, STEP, AND TABULAR FUNCTIØNS

  NAME       VALUE        NAME       VALUE       NAME       VALUE
  U1        1.0000E 00   U2        2.0000E 00


 DIFFERENTIAL EQUATIØNS

  NAME        Y           DY1 ...
  X        -2.5383E-02
  Z        -8.1220E-02
  W         6.6244E-01
  Y        -1.2495E-02


  CASE    1      FINAL VALUES AT TIME  1.0000E 00

 ALGEBRAIC FUNCTIØNS

  NAME       VALUE        NAME       VALUE       NAME       VALUE
  E1       -8.1214E-02   E2       -6.2475E-03   E3       1.9875E 00
  E4       -5.1022E-03


 CØNSTANT, STEP, AND TABULAR FUNCTIØNS

  NAME       VALUE        NAME       VALUE       NAME       VALUE
  U1        1.0000E 00   U2        2.0000E 00


 DIFFERENTIAL EQUATIØNS

  NAME        Y           DY1 ...
  X        -2.5372E-02
  Z        -8.1219E-02
  W         6.6244E-01
  Y        -1.2494E-02


  CASE   1   GENERAL APPRØACH TØ FINDING CØNV. ACCEL FACTØRS FØR ARB. SY
***********TEM

    TIME        Y          Z          W          X          U1
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