NAVIGATION: *Journal of The Institute of Navigation*
Vol. 28, No. 4, Winter 1981-82
Printed in U. S. A.

Modeling and Evaluating an Empirical INS Difference Monitoring Procedure Used to Sequence SSBN Navaid Fixes

THOMAS H. KERR
Intermetrics, Inc, Cambridge, Mass.
Received June 1981

ABSTRACT

In the past, U.S. Polaris (A-3) and Poseidon (C-3) SSBNs (ballistic missile equipped, nuclear powered submarines) took relatively frequent navigation fixes at specified but clsssified periodic rates to compensate for the degradations of gyro drift-rate. However, a randomization of the fix taking was called for to preclude enemy deciphering of the frequency of fix-taking as the enemy attempts to enhance its surveillance of U.S. SSBNs. Since warm-standby navigation configurations (such as two complete Ships Inertial Navigation Systems, SINS) are usually utilized onboard U.S. SSBNs anyway to reap the benefit of high reliability/availability through online modular repair, the so-called Difference Monitoring procedure consisting of a comparison of the outputs of both these available INSs was instituted in 1976 to randomize the fix taking for C-3 SSBNs.

One contribution of this paper is to use existing results of Helstrom from level-crossing theory for the output of nonlinear operations (e.g., the RSSing of latitude and longitude error to obtain radial position error) to provide an analytically tractable theoretical model for the previous empirical procedure of Difference Monitoring. Using the parameters of RMS level and correlation-time of the underlying INS, this analytic model can be exploited to set the constant decision threshold to achieve a *specified average* interval between position fixes. A second contribution of this paper is to present an expression for the related variance. A comparison between actual SSBN patrol statistics and analytically predicted results for alternative threshold settings is included; however, time and navigation accuracy scales are concealed to prevent divulgence of national security information.

Measures of SSBN detectability to enemy surveillance should apparently also be updated in a manner herein suggested to no longer assume SSBN fix taking at a periodic rate. Adoption of this suggestion enables use of an absolute evaluation technique (based on Pareto-optimality) already developed and utilized as described herein as a third contribution for gauging the "goodness" of either randomized or deterministic SSBN fix strategies over and specified time epoch. Two Electrostaticially Supported Gyro Navigators (ESGNs) have been postulated as a candidate configuration for D-5 Trident II SSBNs, where Difference Monitoring will again be as appropriate as in C-3 Poseidon SSBNs.

1. INTRODUCTION

A description of the empirical procedure of Difference Monitoring, along with the philosophy of operation, and a summary of its experimental verification is provided in the introduction to Section 2. A proposed analytical level-crossing model for Difference Monitoring is offered in Section 2.1, from which expressions are derived in Sections 2.2 and 2.3, respectively, for the *mean* and *variance* in the time between external position fix indications for the INS. Unclassified numerical results are depicted in Section 2.4, where comparisons are made between the expected fix intervals for alternative Difference Monitoring decision theresholds (and summary patrol data is displayed for further justification).

The evolution of an analytic basis for quantifying detectability to enemy surveillance as it relates to the classical sweep rate measure is summarized in Section 3.1. This summary culminates in an expression that has a structural form that is compatible with the relatively recent technique of sensor schedule optimization for Kalman filter applications (as encountered in the integrated INS application for SSBNs). The detailed analytical basis of both properly posing and solving the problem (including solution algorithms and pertinent numerical experience) is provided in Section 3.2. When augmented with the standard techniques of bicriteria optimization theory, the result is a method for quantitatively trading-off the navigation accuracy gained versus the exposure to enemy surveillance availed through the use of alternative navaids for external INS position fixes. Representative quantitative results for the SINS as obtained by the above procedure are depicted in Section 3.3.

2. DIFFERENCE MONITORING FOR RANDOMIZING SSBN NAVIGATION FIXES

Even for the extremely accurate Inertial Navigation Systems (INS) utilized by SSBNs (ballistic missile equipped, nuclear powered submarines denoted as Ships Submersible Ballistic Nuclear) the relatively long (~ months) strategic patrol missions make external position fixes necessary for adequate INS compensation of degradations in navigation accuracy due to gyro drift-rate and other sources. Alternative navigation fixes are available from satellite, Loran, or bathymetry (bottom contour map-matching via sonar), but in each case SSBN utilization presents an increased risk of exposure to enemy surveillance during fix taking, due either to the presence of antennas or to acoustic radiation[1].

Explicitly, Difference Monitoring consists of computing the radial position divergence (by RSSing the latitude and scaled longitude divergence between the outputs* of the two warm-standby INSs) and comparing this test statistic to a fixed decision threshold (denoted herein by D). Since common external effects (e.g., gravity anomalies, velocity reference errors, Schuler oscillations, etc.) mutually cancel in forming INS position divergences, the crossing of the test statistic above the *empirically* specified decision threshold is an indication of essentially growing gryo drift that requires an external position fix to compensate.

To preclude the possibility of both INSs drifting off together with mutually degraded accuracy (for example: as a consequence of uncompensated velocity errors or vertical deflections [and gravity anomalies] experienced in common) yet failing to signal for a navigation fix because their relative divergence and consequently the RSSed test statistic is still small, a maximum allowable time between fixes (MAXTIME) is operationally imposed[2]. Difference Monitoring is therefore

---

* For SINS, these outputs include the standard "corrections," as obtained from the STAistical Reset (STAR) filter, between reset incorporation as actuated through torquing of the SINS[2].

used to both randomize and extend the time between fixes, thus avoiding periodic patterns of fixes and reducing exposure to enemy surveillance.

The specific values (classified) of the two decision parameters, threshold level D and MAXTIME, had been empirically set in the past by considering distributions of various fix interval lengths and time-RMS* accuracy errors for five actual representative SSBN patrols. The thresholding procedure, performed by the responsible organization, consisted of "first selecting arbitrary thresholds, then incrementing them" for each of several iterative passes over the data until finalized thresholds were obtained, where the sampled Circular Error Probable (CEP) from the five particular patrols just equaled the objective CEP spec. Prior to adoption by the SSBN Fleet in March 1976, the selected thresholds were validated† in post-processing tests of five additional operational patrols. Because Difference Monitoring involved no hardware or software changes but merely a change in operational procedures, the usual navigation checkout and shakedown on a surface test ship was circumvented prior to fleet use. This short-cutting may perhaps have opened up the possibility for glitches to occur in its usage as further elaborated upon, with suggested compensation offered by the results of this paper.

2.1. *An Analytical Model for Difference Monitoring*

While the detailed state variable truth model of a SINS (Mark 2 Mod 6 with either G7-B or V-7 gyros) nominally has 34 states[3], reduced-order models of only the most significant states affecting a particular application are not uncommon (e.g., the so-called CON-B STAR filter uses seven states to represent the SINS, while the earlier issue CON-A STAR filter only used six states). In the case of Difference Monitoring, the simplified but adequate reduced-order model suggested for both SINs together (as a consequence of the mutual cancellation of the above enumerated common effects and like terms in forming divergences) is postulated to consist of the following two independent random processes:

$x(t') \triangleq$ divergence in latitude error between the two INSs (taken to be a first-order Gauss-Markov process having correlation time T_1 and variance σ_1^2‡. (2.1-1)

$y(t') \triangleq$ divergence in longitude error times (cosine of latitude as required to obtain units of length compatible with latitude) between the two INSs (taken to be a first-order

* Time averages being assumed to be equal to ensemble averages as a standard ergodicity assumption, appears to be inappropriate in systems with undamped oscillations (such as the Schuler oscillations encountered in navigation system linear error models) since oscillatory linear systems are nonstationary (ex. 45, pp. 167–168 of Ref. 14 and pp. 158–161 of Ref. 39) and consequently nonergodic. Therefore previous threshold setting procedures may perhaps be viewed as being somewhat empirical at best.

† Cautions pertaining to perceived glitches in validation are extended throughout Chapter 3 of Ref. 16.

‡ Since latitude and longitude errors of each individual SINS can be obtained as a linear combination of the computer-frame-to-platform-frame misalignment angles, which in turn represent a single integration of the underlying net uncompensated gyro drift-rate errors, this model *phenomenologically* represents the residual effect persisting after formation of divergences as:

- white noise for the underlying residual gyro drift-rate errors,
- latitude and longitude divergence errors as the serially correlated Coriolis consequence of integrating the above divergence gyro drift-rate noise model,
- use of a factor of $\sqrt{2}$ in the standard deviations of each position difference as if the two identical INSs were independent.

Gauss-Markov process having correlation time T_2 and variance σ_2^2). (2.1-2)

For *conservatism* (achieved by using the worst case) and to facilitate analytic tractability in what follows, the following common correlation times and variances, respectively, are utilized for *both* the $x(t')$ and $y(t')$ processes:

$$T = \max\{T_1, T_2\} \quad (2.1\text{-}3)$$

$$\sigma^2 = \max\{\sigma_1^2, \sigma_2^2 \cos^2 \text{Lat}\} \quad (2.1\text{-}4)$$

thus constituting only a mild assumption since the respective INS latitude and longitude divergences are of a fairly similar character in the physical application. From the above two underlying constituents $x(t')$ and $y(t')$ as modified by the simplifying conditions of Eqs. 2.1-3 and 2.1-4, RSSing yields an expression for the radial position divergence used in Difference Monitoring as

$$r(t') = [x^2(t') + y^2(t')]^{1/2} \quad (2.1\text{-}5)$$

As an additional scaling convenience (and as an obscuring obstacle to cover classified quantifications), let

$$t' = Tt \quad (2.1\text{-}6)$$

so that rather than having to deal with $x(t')$ in Eq. 2.1-1 and its autocorrelation function of

$$R(\tau) = \sigma_1^2 e^{-|\tau|/T_1} \quad (2.1\text{-}7)$$

the conservatively modified (via Eq. 2.1-3 and 2.1-4) $x(t)$ has the conveniently scaled autocorrelation function of

$$R(\tau) = \sigma_1^2 e^{-|\tau|} \quad (2.1\text{-}8)$$

and likewise for $y(t)$. Without any loss of generality, the units of length are also normalized so that:

$$\sigma^2 = 1 \quad (2.1\text{-}9)$$

The Difference Monitoring problem encountered in SSBN fix taking is observed to be of the fundamental form depicted in Fig. 2.1-1. This problem can now be recognized as one classically known as level-crossing for random processes, where requisite details for this particular model have already been worked out analytically (except for a variance that is provided in the derivation of Section 2.2). The $r(t)$ process has a transition probability density function (pdf) of the following form[4, 5, 6]:

$$p(r, t \mid r_0, 0) = \frac{r}{1-\mu^2} \text{exr}\left\{\frac{+\mu^2 r_0^2}{2(1-\mu^2)}\right\} I_0\left[\frac{\mu r_0 r}{1-\mu^2}\right] \quad (2.1\text{-}10)$$

where

$$\mu \triangleq e^{-t} \quad (2.1\text{-}11)$$

and

$I_0[\,\bullet\,]$ is the zero$^{\text{th}}$ order modified Bessel function.

The interpretation of Eq. 2.1-10 is that it represents the pdf for the process $r(t)$ at time t, given that it had the deterministic value r_0 at time = zero.

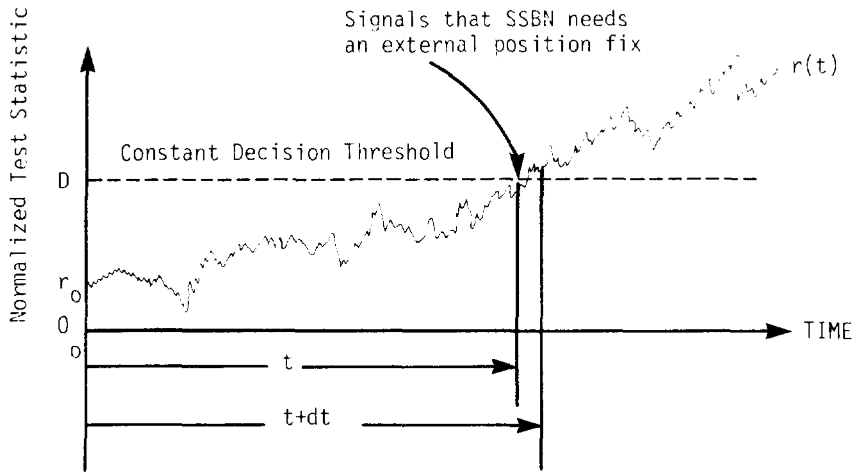


Fig. 2.1-1—Difference Monitoring: test statistic reaching decision threshold signals SSBN to take a navigation fix.

This level-crossing problem portrayed in Fig. 2.1-1 was analytically solved* in Refs. 4, 5, 6 as summarized here (as a necessary precursor to the extension offerred in Section 2.3). Using somewhat corrupted shorthand notation, let

$q_D(t \mid r_0)\, dt \triangleq$ probability that $r(t)$ first crosses level D between the time instants "t" and "$t + dt$", given that the process assumed the deterministic value r_0 at time $= 0$ (2.1.-12)

Consider a particular sample function such that

$$0 \leq r_0 < D < r(t) \qquad (2.1\text{-}13)$$

The sample function of the Markov process $r(t)$ must have crossed the level D at least once within the time interval $(0, t)$, therefore in "sorting" out all the possible realizations [i.e., sample functions] of the aggregate ensemble that can pass through r_0 at time $= 0$ so that they may be categorized as to when they *first* crossed above level D, one obtains the well-known "renewal equation":

$$p(r(t) = r \mid r_0, 0) = \int_0^t q_D(\theta \mid r_0) p(r(t) = r \mid r(\theta) = D)\, d\theta \qquad (2.1\text{-}14)$$

The so-called renewal equation that arises in the context of level-crossing problems is obtained by invoking the Markov property that "what happens to the process after time instant θ (given that $r(\theta) = D$) is indepednent of what occurred previous to instant θ and consequently independent of the fact that the $r(t)$ process may not have crossed level D within the interval $(0, \theta)$."

Following the established solution procedure, notice that when the density of

* While tractably solved level-crossing problems are scarce (apparently only level-crossing solutions for a scalar Wiener/Brownian motion process and an Ornstein-Uhlenbeck [i.e., stationary first-order Markov] process are widely known as reported[11]), both the tractable solution for this nonlinear operation (Eq. 2.1-5) on two random processes as reported[4, 5, 6] (and as acknowledged to be in Ref. 17 as a 1958 precedent) and another neat level-crossing result of Ref. 7 (exploited by this Author for SSBN's in another context[8, 9]) escaped notice in the fairly recent general surveys[10, 11] of available level-crossing results; nor are they discussed in the standard reference on these topics[12].

Eq. 2.1-10 is substituted into Eq. 2.1-14, the result has the following simplified structural form

$$p(r(t-0)=r\,|\,r_0, 0) = \int_0^t q_D(\theta\,|\,r_0)p(r(t-\theta)=r\,|\,r(0)=D)\;d\theta \quad (2.1\text{-}15)$$

Use of the unilateral Laplace transform with respect to t, when applied to Eq. 2.1-15, yields the following scalar algebraic equation

$$P(s, r; r_0) = Q_D(s; r_0)P(s, r; D) \quad (2.1\text{-}16)$$

This equation, in turn, may be rearranged to provide the Laplace transform of the objective pdf as

$$Q_D(s; r_0) = \frac{P(s, r; r_0)}{P(s, r; D)} \quad (2.1\text{-}17)$$

From the definition of the transform of $p(\bullet\,|\,\bullet)$, $p(\bullet, \bullet; \bullet)$ is observed to be a scaled hypergeometric function of the well-known form (p. 251 of Ref. 13):

$$\Phi(\mu, \gamma; x) \triangleq 1 + \frac{\mu}{\gamma}\frac{x}{1!} + \frac{\mu(\mu+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \frac{\mu(\mu+1)(\mu+2)}{\gamma(\gamma+1)(\gamma+2)}\cdot\frac{x^3}{3!} + \ldots \quad \text{for} \quad \gamma > 0 \quad (2.1\text{-}18)$$

Fortunately, upon substituting the hypergeometric function of Eq. 2.1-18 into the expression of $Q_D(s; r_0)$ in Eq. 2.1-17 and properly observing all arguments of the function, the common dependence on r in both the numerator and denominator divides out leaving only

$$Q_D(s; r_0) = \frac{\Phi(s/2, 1; r_0^2/2)}{\Phi(s/2, 1; D^2/2)} \quad (2.1\text{-}19)$$

where the general expression for the hypergeometric distribution simplifies to*

$$\Phi(s/2, 1; x) \triangleq 1 + \frac{(s/2)x}{(1!)^2} + \frac{\frac{s}{2}\left(\frac{s}{2}+1\right)x^2}{(2!)^2} + \frac{\frac{s}{2}\left(\frac{s}{2}+1\right)\left(\frac{s}{2}+2\right)x^3}{(3!)^2} + \ldots \quad (2.1\text{-}20)$$

2.2 *Mean Time to Cross Level D*

To explicitly recover the density $q_D(t; r_0)$ from Eq. 2.1-19, the inverse transform of the ratio of two degenerate confluent hypergeometric functions would have to be performed (a task that *has been impossible to date*). However, for the SSBN Difference Monitoring application, just the mean and variance will suffice. For

* The standard mathematical technique of a ratio test can be used to verify that this series converges absolutely for all finite x.

this more limited objective, it is fruitful to make use of the well-known property that the Laplace transform of the pdf explicitly available in Eq. 2.1-19 is in fact a *moment generating function*:

$$E[\tau_D^n(r_0)] = (-1)^n \frac{\partial^n}{\partial s^n} [Q_D(s; r_0)] \Bigg|_{s=0}$$
$$= (-1)^n \frac{\partial^n}{\partial s^n} \left[\frac{\Phi(s/2, 1; r_0^2/2)}{\Phi(s/2, 1; D^2/2)} \right] \Bigg|_{s=0} \tag{2.2-1}$$

which, when differentiated once, yields the "mean time"* for $r(t)$ to cross level D; when differentiated twice, yields the second moment from which one can obtain the variance.

In Refs. 4, 5, the indicated differentiation of Eq. 2.2-1 was performed once ($n = 1$) to obtain the mean as

$$E[\tau_D(r_0)] = \frac{\partial}{\partial s} \Phi(s/2, 1; D^2/2) \Bigg|_{s=0} - \frac{\partial}{\partial s} \Phi(s/2, 1; r_0^2/2) \Bigg|_{s=0} \tag{2.2-2}$$

where use has been made of the fact that

$$\Phi(0, 1; x) = 1 \quad \text{for all } x \tag{2.2-3}$$

to achieve considerable simplification in the rhs of Eq. 2.2-2. The two terms of Eq. 2.2-2 may be evaluated by differentiating a nine term series† expansion of Eq. 2.1-20 as

$$\begin{aligned}\Phi(\mu, 1; x) = 1 &+ \frac{\mu x}{(1!)} + \frac{(\mu^2 + \mu)x^2}{(2!)^2} \\ &+ \frac{(\mu^3 + 3\mu^2 + 2\mu)x^3}{(3!)^2} + \frac{(\mu^4 + 6\mu^3 + 11\mu^2 + 6\mu)x^4}{(4!)^2} \\ &+ \frac{(\mu^5 + 10\mu^4 + 35\mu^3 + 50\mu^2 + 24\mu)x^5}{(5!)^2} \\ &+ \frac{(\mu^6 + 15\mu^5 + 85\mu^4 + 225\mu^3 + 274\mu^2 + 120)x^6}{(6!)^2} \\ &+ \frac{(\mu^7 + 21\mu^6 + 175\mu^5 + 735\mu^4 + 1624\mu^3 + 1764\mu^2 + 720)x^7}{(7!)^2} \\ &+ \frac{\begin{aligned}(\mu^8 + 28\mu^7 + 322\mu^6 + 1960\mu^5 \\ + 6769\mu^4 + 13{,}132\mu^3 + 13{,}068\mu^2 + 5040\mu)x^8\end{aligned}}{(8!)^2} + \ldots.\end{aligned} \tag{2.2-4}$$

---

* Mean time, expected time, and average time-of-first crossing are all synonyms. Because a first crossing of level D indicates that it is time to take an external navigation fix in the SSBN Difference Monitoring application, the average time to cross level D also has the interpretation of average time between navaid fixes.

† Even though retaining just a few terms of the series suffices for calculating the mean, the explicit enumeration of these nine terms is needed in order to calculate the variance as done in Section 2.3 as a theoretical precedent.

to yield

$$\frac{\partial}{\partial\mu}\Phi(\mu, 1; x) = 0 + \frac{x}{(1!)^2} + \frac{(2\mu + 1)}{(2!)^2}x^2 + \frac{(3\mu^2 + 6\mu + 2)x^3}{(3!)^2}$$
$$+ \frac{(4\mu^3 + 18\mu^2 + 22\mu + 6)x^4}{(4!)^2}$$
$$+ \frac{5\mu^4 + 40\mu^3 + 105\mu^2 + 100\mu + 24)x^5}{(5!)^2}$$
$$+ \frac{(6\mu^5 + 75\mu^4 + 340\mu^3 + 675\mu^2 + 548\mu + 120)x^6}{(6!)^2}$$
$$+ \frac{(7\mu^6 + 126\mu^5 + 875\mu^4 + 2940\mu^3 + 3528\mu + 720)x^7}{(7!)^2}$$
$$+ \frac{(8\mu^7 + 196\mu^6 + 1{,}932\mu^5 + 9{,}800\mu^4 + 27{,}076\mu^3 + 39{,}396\mu^2 + 26{,}136\mu + 5040)x^8}{(8!)^2} + \ldots \quad (2.2\text{-}5)$$

Upon evaluation of Eq. 2.2-5 for $\mu = 0$ as required in Eq. 2.2-2, results in

$$\left.\frac{\partial}{\partial\mu}\Phi(\mu, 1; x)\right|_{\mu=0} = \sum_{n=1}^{\infty}\frac{x^n}{n \bullet n!} \quad (2.2\text{-}6)$$

In Refs. 4, 5, the connection is established between the series of Eq. 2.2-6 and the well-tabulated[15] *exponential integral* $\bar{E}_i$. The expression for the mean-time-to-first-crossing of level D in Eq. 2.2-2 can be evaluated from tables using the convenient equivalence to an exponential integral as

$$E[\tau_D(r_0)] = \tfrac{1}{2}\{\bar{E}_i[D^2/2] - \ln(D^2/2) - \bar{E}_i[r_0^2/2] + \ln(r_0^2/2)\} \quad (2.2\text{-}7)$$

If tables such as Ref. 15 are inaccessible, Eq. 2.2-2 may still be conveniently evaluated approximately using the first few terms of the series expansion of Eq. 2.2-6.

In particular, for the Difference Monitoring application

$$r_0 = 0 \quad (2.2\text{-}8)$$

as a consequence of having just taken an external position fix* and differenced the remainder, so Eq. 2.2-7 degenerates to

$$E[\tau_D(0)] = \tfrac{1}{2}\{\bar{E}_i[D^2/2] - (0.577215\ldots) - \ln(D^2/2)\} \quad (2.2\text{-}9)$$

* Surveying the many significant simplifications that accure for this zero value of r_0 (Eq. 2.2-8) as arises here, note that (1) the modified zeroth order Bessel function has argument zero and consequently a value of unity, (2) the numerator of Eq. 2.1-19 is unity since the value that corresponds to x appearing in Eq. 2.1-20 is zero, (3) the term on the right in Eq. 2.2-2 is zero (as can be seen from Eq. 2.2-6 with $x = 0$). Full generality for a possibly nonzero r_0 was retained herein so that (1) all the benchmarks and milestones of the earlier analysis of Refs. 4, 5, 6 can be conveniently utilized as crosschecks on the correctness of these results, (2) in order to not overlook possible nonzero contributions in the second derivative as the variance is obtained in Section 2.3 by differentiating Eq. 2.2-5 (via Eq. 2.2-1) even though Eq. 2.2-5 is zero for $r_0 = 0$, (3) to provide the first time variance result in its full generality for the possible benefit of others.

As one of the novel contributions of Ref. 16, the above expression was used to calculate the theoretical mean fix interval for SINS Difference Monitoring as depicted in Section 2.4 as a function of normalized threshold setting D.

2.3 Variance in the Time to Cross Level D

Using Eq. 2.2-1, the expression for the variance in the time-to-first-crossing of level D can be obtained by evaluating the following:

$$\mathrm{Var}[\tau_D(r_0)] = E[\tau_D^2(r_0)] - (E[\tau_D(r_0)])^2 \tag{2.3-1a}$$

$$= (-1)^2 \frac{\partial^2}{\partial s^2}\left[\frac{\Phi(s/2,\,1;\,r_0^2/2)}{\Phi(s/2,\,1;\,D^2/2)}\right] - (E[\tau_D(r_0)])^2 \tag{2.3-1b}$$

$$\begin{aligned}= \left(\frac{1}{2}\right)^2 \frac{\partial^2}{\partial \mu^2}\Bigg\{&\Phi(\mu,\,1;\,r_0^2/2) - \frac{\partial^2}{\partial \mu^2}\Phi(\mu,\,1;\,D^2/2)\\ &- 2\frac{\partial}{\partial \mu}\Phi(\mu,\,1;\,r_0^2/2)\\ &+ 2\left[\frac{\partial}{\partial \mu}\Phi(\mu,\,1;\,D^2/2)\right]^2\Bigg\}\Bigg|_{\mu=0} - (E[\tau_D(r_0)])^2\end{aligned} \tag{2.3-1c}$$

The necessary second partial derivative of $\Phi(\mu,\,1;\,x)$ occurring in Eq. 2.3-1c can be conveniently obtained from Eq. 2.2-5 by performing another term by term differentiation to yield:

$$\begin{aligned}\frac{\partial^2}{\partial \mu^2}\Phi(\mu,\,1;\,x) &= \frac{x^2}{2!} + \frac{x^3}{3!} + (0.91\overline{66})\frac{x^4}{4!} + (0.8\overline{333})\frac{x^5}{5!}\\ &+ (0.76\overline{11})\frac{x^6}{6!} + (0.7)\frac{x^7}{7!} + (0.64821)\frac{x^8}{8!} + \ldots\end{aligned} \tag{2.3-2}$$

Using the condition of Eq. 2.2-8 for the Difference Monitoring application in conjunction with Eq. 2.3-2 in Eq. 2.3-1c yields

$$\begin{aligned}\mathrm{Var}[\tau_D(0)] = \left(\frac{1}{2}\right)&\left\{-\frac{\partial^2}{\partial \mu^2}\Phi(\mu,\,1;\,D^2/2) + 2\left[\frac{\partial}{\partial \mu}\Phi(\mu,\,1;\,D^2/2)\right]^2\right\}\Bigg|_{\mu=0}\\ &- \left(\frac{1}{4}\right)\left[\frac{\partial}{\partial \mu}\Phi(\mu,\,1;\,D^2/2)\right]^2\Bigg|_{\mu=0}\end{aligned} \tag{2.3-3a}$$

$$\begin{aligned}= (E[\tau_D(0)])^2\\ - \left(\frac{1}{4}\right)\Bigg\{&\frac{(D^2/2)^2}{2!} + \frac{(D^2/2)^3}{3!} + (0.91\overline{66})\frac{(D^2/2)^4}{4!}\\ &+ (0.8\overline{333})\frac{(D^2/2)^5}{5!} + (0.76\overline{11})\frac{(D^2/2)^6}{6!}\\ &+ 0.7\overline{0}\frac{(D^2/2)^7}{7!} + (0.65821)\frac{(D^2/2)^8}{8!} + \ldots\Bigg\}\end{aligned} \tag{2.3-3b}$$

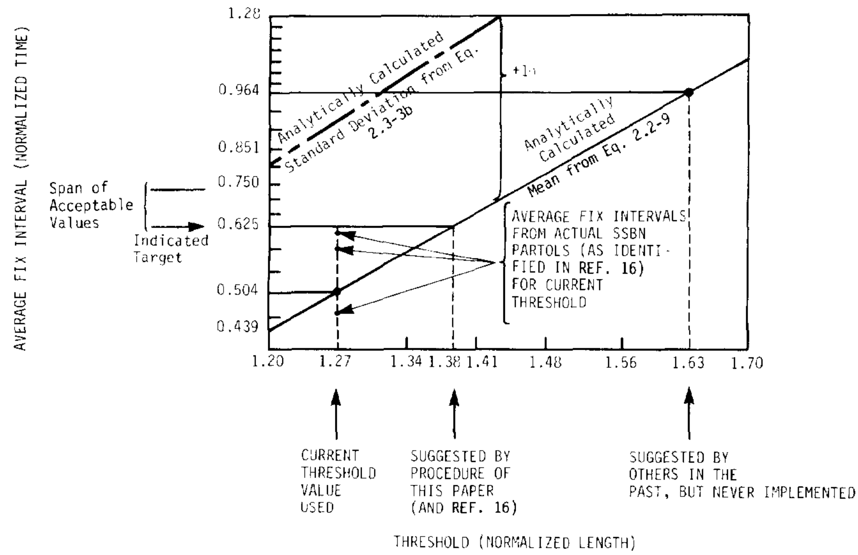


Fig. 2.4-1—Average fix interval vs. threshold level.

The evaluation of the objective variance of the time-of-first-crossing of level D may be conveniently accomplished directly from the series expansion of Eq. 2.3-3*b* by using a programmable hand calculator* (and utilizing Horner's rule to minimize the total number of operations and associated roundoff error incurred) as was done here, with results depicted in Fig. 2.4-1 of Section 2.4 for the SINS as one of the significantly new contributions of this paper.

2.4 *Comparison of Analytically Calculated Theoretical Mean-Time-Between-Fixes to SSBN Fleet Data*

The solid diagonal line in Fig. 2.4-1 represents the evaluation of the theoretical mean-time-between-fixes from the expression derived in Section 2.2 for C-3 SSBN Difference Monitoring. The diagonal line that is alternately dashed and dotted in Fig. 2.4-1 represents the evaluation of the theoretical standard deviation in the time-between-fixes for Difference Monitoring as obtained from the expression derived in Section 2.3. In both cases, the value increases with increasing decision threshold D, which is intuitively reasonable since a higher decision threshold takes longer to reach for the same random process starting at zero. While 0.615 (in normalized units) for time-between-fixes ostensibly was the target spec. objective for the currently used threshold setting, the data obtained using this threshold setting (of 1.27 in normalized units of length) in three separate patrols (sources being utilized are identified in Ref. 16) apparently fall far short of this mark as seen from Fig. 2.4-1. The available patrol data actually fall above and below to bracket the theoretically derived mean of 0.504 as evaluated using the results of Section 2.2 rather than around the target spec. of 0.625 as an average

* A conservative bound on the magnitude of the error incurred in using only the first eight terms of the series in Eq. 2.3-3*b* is available from Lagrange's form of the remainder (or error) as[13];

$$|\,\text{error}\,| < e^{D^2/2}(D^2/2)^9/9! \quad \text{(which is 0.01 for } D^2/2 = 2)$$

corresponding to the largest value of threshold setting considered in the Difference Monitoring Application (where for smaller settings, the error incurred is less).

for times-between-fixes. Additionally, the SSBN patrol data can be seen to be well within 1 standard deviation of the theoretically calculated mean of $D = 1.27$.

Another feature that is also made graphically apparent is the inappropriateness of the decision threshold $D = 1.63$ that had been suggested for use in the past by others,[40] but never actually used by the SSBN fleet in practice. By the analytically tractable, theoretical evaluation technique of Sections 2.2 and 2.3, this threshold of 1.63 is demonstrated to correspond to 0.964, which even exceeds the rather expansive 1σ region of the current threshold setting and is considerably above the span of indicated acceptable values (from 0.625 to 0.750 normalized units of time) having been previously imposed as a prudent range of acceptable variations. The analysis of Section 2.2 indicates that the target spec. of 0.625 (normalized time units) between fixes is just met for $D = 1.38$ (normalized units of length) and so the threshold currently used for C-3 Difference Monitoring should perhaps be raised to this value in order to just meet the target spec.

3. AN EVALUATION TECHNIQUE FOR GAUGING THE ABSOLUTE/ RELATIVE UTILITY OF SSBN FIX STRATEGIES

3.1 *Natural Evolution of the Sweep Rate Measure*

Over the years, various measures of the detectability of objects to enemy surveillance have been developed. One of the earliest measures of detectability that has withstood the test of time is to gauge the "observability*" of the object to a surveillance sensor mounted on the surveillance platform as represented in Fig. 3.1-1 in terms of Sweep Rate[41]. As illustrated in the two smaller diagrams at the bottom of Fig. 3.1-1, alternative sensors have inherent detection patterns such as the "donut" (i.e., torus or annulus) or cookie-cutter (i.e., cylinder) encountered for radar/visual detection of SSBN masts and Forward Looking InfraRed (FLIR), respectively.

Sweep rate (SR) as defined for continuously exposed targets (such as surface ships) is commonly defined (Koopman,[18, 19]) with physical motivation readily apparent from the cross-hatched overview in Fig. 3.1-1 as

$$SR = 2RV \tag{3.1-1}$$

where

$R \triangleq$ detection range of the sensor used,
$V \triangleq$ velocity of the surveillance platform upon which the sensor is mounted.

Detection range is the range at which the target is just detectable with probability 0.50. Everywhere within this range, the probability of detecting the target is greater than 0.50. Also, sweep rate may be routinely converted to probability of detection under an assumed random search[19] for a convenient alternate characterization of detectability. Because most surveillance sensors not only detect the presence of an object orthogonal to the track of a surveillance platform (but also in front and in back), the expression for the area swept has been reasonably modified here to also include the two dotted semicircles depicted in the top view of Fig. 3.1-1 (which can be significant contributors depending on the relative

* "Observables" is Navy military terminology used to denote all physically discernable characteristics that would enable an object's presence to be detected.

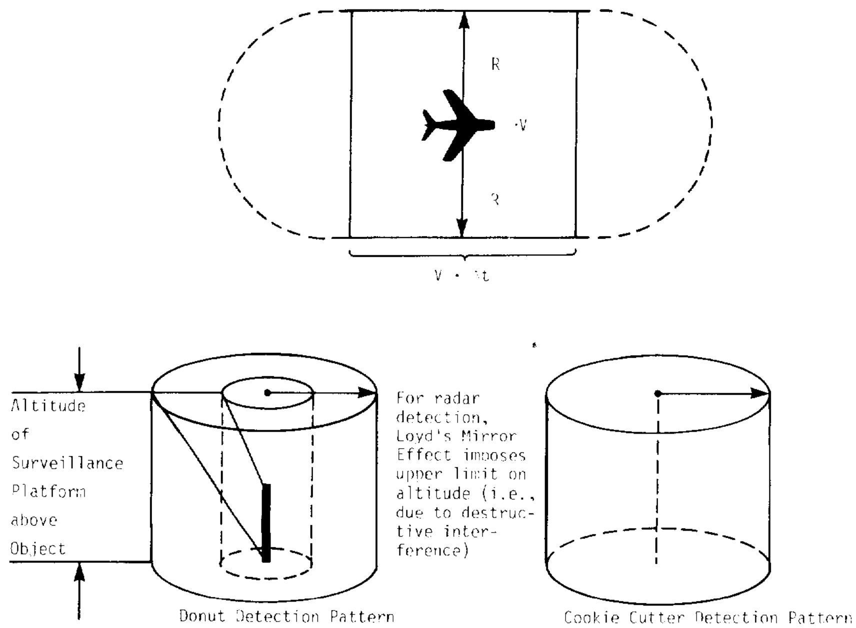


Fig. 3.1-1—Physical basis for quantifying an object's susceptability to enemy surveillance in terms of sweep rate.

magnitudes of R and $V \bullet \Delta t$) as:

$$c \triangleq \text{area swept} = 2RV\Delta t + \pi R^2 \tag{3.1-2}$$

The five C-3 Poseidon and C-4 Trident navigation system position fix alternatives* currently utilized by SSBNs are:

- LORAN-C (Phase-Shift)/NAVSAT Synchronization† and simultaneous NAVSAT fix,
- LORAN-C (Phase-Shift) fix from only *two* stations (or three stations for redundancy),
- LORAN-C (Hyperbolic) fix from three stations,
- Depth Sonar fix (i.e., bathymetric bottom contour map-matching),
- LORAN-C (Phase-Shift)/Depth Sonar synchronization‡,

and a sixth measurement usage option could be considered to be the use of no navaid measurement fix at a candidate fix time.

A new measure of SSBN detectability, *Average Effective Sweep Rate (AESR)*, is recommended in Ref. 16 as being an appropriate generalization which rigorously accommodates *time-varying fix intervals* and *alternating navaid usage*§ as:

---

* Worst case detection threats, assumed with SSBN use of each navaid alternative have been compiled and are periodically reassessed in view of evolving ASW technology (e.g., RF linked sonobuoys, sonar "dippers," Magnetic Anomaly Detection (MAD), infrared wake detectors, etc.) and countermeasures.

† Synchronization (or initialization) pertains to an additional operational consideration in using Phase-Shift LORAN, that the drift of the on-board atomic clock must be compensated for via an occasional NAVSAT or Depth Sonar Bathymetry fix.

‡ Future NAVSTAR Global Positioning System (GPS) availability affords the additional option of LORAN-C (Phase-Shift)/GPS Synchronization and simultaneous GPS fixes for SSBNs taking less time, with much reduced antenna radar cross-sectional area exposure, and with less restrictive constraints on acceptable satellite location translating into available time windows for usage.

§ As routinely encountered since the 1976 introduction of Difference Monitoring into the SSBN Fleet, where any available navaid can be used for the required fixes indicated by Difference Monitoring.

$$\text{AESR} \triangleq \frac{\begin{bmatrix}\text{Surveillance Area Covered During All Navaid}\\ \text{Fix Exposures Over a Specified Time Interval}\end{bmatrix}}{\text{Total Specified Time Interval}} \qquad (3.1\text{-}3a)$$

or

$$\text{AESR} = \frac{\sum_{k=1}^{N} [2R(k)V(k)\Delta t(k) + \pi R^2(k)]}{\sum_{k=1}^{N} \Delta(k)} \qquad (3.1\text{-}3b)$$

where

$R(k) \triangleq$ detection range (of the designated worst case sensor for the navaid used* at the k^{th} time-step) e.g., visual, infrared, *x-band* radar, passive sonar, etc.

$V(k) \triangleq$ velocity of the surveillance craft (either SSN or aircraft, depending on which is mounted with the worst case sensor for the navaid used at the k^{th} time-step)

$\Delta t(k) \triangleq$ time required to take a fix (for the navaid used at the k^{th} time-step)

$N \triangleq$ number of time-steps into which the specified time interval is subdivided (the subdivisions normally occurring at the fix times but can be more frequent for purposes that will be elaborated upon in Section 3.2)

$\Delta(k) \triangleq$ step-size (possibly varying in duration) at the k^{th} time-step

A natural accuracy measure that assesses in summary fashion the effect of SSBN navigation errors (as a consequence of both INSs and navaid fix history) as they would affect a missile that could be called for a launch at any time during the patrol is *Average Uncertainty in Missile Radial Miss Distance*, and is defined as:

$$\sigma_{\text{ARMD}} \triangleq \left[\frac{\sum_{k=1}^{N} \sigma_{DR}^2(k) + \sigma_{CR}^2(k)}{N} + \sigma_{\text{transmission}}^2\right]^{1/2} \qquad (3.1\text{-}4)$$

where

$\sigma_{\text{transmission}} \triangleq$ standard deviation of errors introduced in transmitting navigation information to the SSBN's fire control subsystem through the optical reference unit,

$\sigma_{DR}(k) \triangleq$ standard deviation of down-range missile miss distance at time-step k due to navigation errors,

$\sigma_{CR}(k) \triangleq$ standard deviation of cross-range missile miss distance at time-step k due to navigation errors.

---

* If no navaid fix is taken at the k^{th} time-step, then the appropriate sensor detection range is $R(k) = 0$ and there is no contribution made at time-step $= k$ to the surveillance area covered or to the Average Effective Sweep Rate.

Time-steps k should be taken more frequently than fix/reset occurrences to obtain a representative average.

Besides being natural and convenient, the above two measures AESR and σ_{ARMD} are true generalizations to nonperiodic fix intervals since they agree *identically* with the previous detectability and accuracy measures, respectively, when used to quantitatively evaluate the effects of navaid fix schedules that *are* periodic. Since an SSBN's navigation accuracy degrades with time if fixes from a navigation aid (navaid) of sufficient quality and at a sufficient frequency are not provided, there is an inherent trade-off between *maintaining acceptable navigation accuracy* while seeking to *minimize the risk of SSBN exposure to enemy surveillance from use of navaids.* An additional benefit of using the above AESR and σ_{ARMD} as measures to trade-off the effects on the SSBN of various fix schedules is indicated in Section 3.2. These two measures in particular are demonstrated to allow use of an analytically *tractable* computational procedure that suffices for gauging both the relative and absolute "goodness" or utility of particular patterns of contiguous fix usage (referred to herein as fix strategies or fix schedules) over SSBN patrol segments.

3.2 *Cost Functions, Pareto-Optimality, TPBVPs, and a Min-H Solution Technique*

As detailed for continuous-time[20], the sensor fix usage strategy optimization problem for Kalman filters (as used for keeping track of INS errors on SSBNs) can be represented as finding the vector sequence $\{m^*(k)\}_{k=0}^{N-1}$ which minimizes the following scalar cost function over the time interval from $k = 0$ to $k = N$:

$$J[\{\boldsymbol{m}(k)\}_{k=0}^{N-1}] \triangleq \mu_1 \cdot \mathrm{tr}[\boldsymbol{A}\boldsymbol{P}_N^{(-)}] + \sum_{k=0}^{N-1} \{\mu_1 \cdot \mathrm{tr}[\boldsymbol{B}_k \boldsymbol{P}_k^{(-)}] + \mu_2 \cdot \boldsymbol{c}^T(k)\boldsymbol{m}(k)\} \quad (3.2\text{-}1)$$

where

$\boldsymbol{A}$ is a symmetric positive definite weighting matrix,

$\boldsymbol{B}_k$ is a symmetric positive definite weighting matrix,

$\boldsymbol{c}(\mathrm{k})$ is a vector of the observables cost-per-fix as expressed in terms of area exposed to enemy surveillance during the fix,

μ_1, μ_2 are scalar weightings of fix error and cost of sensor usage having a range of values that is specified as $\mu_2 \triangleq (1 - \mu_1)$ and $0 \leq \mu_1, \leq 1$,

$\boldsymbol{m}(k)$ is a vector that serves to summarize the strategy of sensor usage at time step = k. The sequence $\{\boldsymbol{m}(k)\}_{k=0}^{N-1}$ summarizes the sensor usage strategy over the entire mission time interval under consideration.

The justification for using just a single cost function when the problem is obviously one of trading-off the two considerations of navigation error *vs.* detectability is provided at the end of Section 3.2 where a rigorous link between bicriteria optimization and the so-called method-of-linear-combinations results in Eq. 3.2-1.

In abiding by the above mathematical structure required to properly interface with sensor strategy optimization (as prescribed in Ref. 20), the SSBN observables exposure *vs.* accuracy considerations are modeled in terms of the following combined cost function:

$$J[\{m(k)\}_{k=0}^{N}] = \underbrace{\mu_1 \cdot \sum_{k=0}^{N} \mathrm{tr}[L_k^T L_k P_k^{(-)}]}_{\substack{\text{Reflects Navigation}\\ \text{Error over the}\\ \text{patrol interval}\\ [0,\, N]}} + \underbrace{\mu_2 \cdot \sum_{k=0}^{N-1} c^T(k) m(k)}_{\substack{\text{Reflects cost}\\ \text{of SSBN}\\ \text{"observables"}\\ \text{over the patrol}\\ \text{interval}}} \tag{3.2-2}$$

where

$c_i(k)$ = quantification of area swept during a fix while SSBN is using the i^{th} navaid (Eq. 3.1-2),

L_K = missile impact partials that relate missile miss distance to underlying navigation causes (defined in Chapter 7 of Ref. 3).*

It is noted that the terms under the first summation of Eq. 3.2-2 have the following form

$$\mathrm{tr}[L_k^T L_k P_k^{(-)}] = \mathrm{tr}[L_k P_k^{(-)} L_k^T] = \sigma_{DR}^2 + \sigma_{CR}^2 = \text{variance of radial miss distance at time} = k \tag{3.2-3}$$

where

σ_{DR}—down-range miss distance along the great circle joining the launch point to the target,

σ_{CR}—component of miss distance orthogonal to the down range error (i.e., cross-range error).

Notice that by virtue of Eq. 3.2-3, the first term on the right of the cost function of Eq. 3.2-2 is μ_1 times $(\sigma_{\text{ARMD}}^2 \cdot N - \sigma_{\text{transmission}}^2)$ while the second term is μ_2 times the numerator of Eq. 3.1-3*b*.

Both the C-3 Poseidon, C-4 Trident, as well as all of several proposed Trident II configurations have SSBN navigation system error models of the following state variable form:

$$x(k+1) = \Phi(k+1, k)x(k) + w(k) \tag{3.2-4}$$

with the following external position fix measurement structure

$$z_i(k) = H_i x(k) + v_i(k) \tag{3.2-5}$$

(i = 1, . . . , 6 corresponding to the six SSBN sensor fix options listed following Eq. 3.1-2)

---

* The symmetric matrices $L_k^T L_k$ and $L_N^T L_N$, respectively, play the role of B_k and A in the cost function of Eq. 3.2-1.

where

Φ is the invertible discrete-time transition matrix,
$x(k)$ is the state vector at time k
$w(k)$ is the zero mean Gaussian white corrupting process noise having covariance level Q,
$z_i(k)$ is the measurement at time k as obtained from the i^{th} external position fix/reset sensor,
H_i and $v_i(k)$ are the observation matrix and zero mean Gaussian measurement noise for the i^{th} external position fix/reset sensors, respectively, where the corrupting measurement noise has covariance level R_i.

The use of either *no-measurement or else just one external position fix/reset measurement* at each time step k from one of the six alternative SSBN navaid usage options is modeled as

$$z(k) = \sum_{i=1}^{6} m_i(k) z_i(k) = \sum_{i=1}^{6} m_i(k)[H_i x(k) + v_i(k)] \tag{3.2-6}$$

where

$$m_i(k) = 0 \quad \text{or} \quad 1 \tag{3.2-7}$$

(i.e., a one signifies that the i^{th} navigation aid device *is* being used, a zero signifies that the i^{th} device *is not* being used). As in Ref. 20, a convenient constraint is imposed that for each k

$$\sum_{i=1}^{6} m_i(k) = 1 \tag{3.2-8}$$

which has the physical interpretation that, at each time step k, no more than one of the six external position fix navaid usage options is being used.

By paralleling in discrete-time the continuous-time methodology of Ref. 20, an appropriate scalar Hamiltonian is formed from the cost function of Eq. 3.2-1 as

$$\begin{aligned} H(P_k^{(-)}, \Lambda_{k+1}, m(k), k) &\triangleq \mu_1 \cdot \mathrm{tr}[B_k P_k^{(-)}] + \mu_2 \cdot c^T m(k) \\ &+ \mathrm{tr}\Bigg[\Bigg(\Phi(k+1,k)\Bigg\{P_k^{(-)} - P_k^{(-)}\Bigg(\sum_{i=1}^{M} m_i(k) H_i^T[H_i P_k^{(-)} H_i^T + R_i]^{-1} H_i\Bigg) \\ &\cdot P_k^{(-)}\Bigg\}\Phi^T(k+1,k) + Q\Bigg)\Lambda_{k+1}^T\Bigg] \end{aligned} \tag{3.2-9}$$

which also incorporates the appropriate dynamical constraint of

$$\begin{aligned} P_{k+1}^{(-)} = \Phi(k+1,k)\Bigg\{P_k^{(-)} - P_k^{(-)}\Bigg(\sum_{i=1}^{M} m_i(k) H_i^T[H_i P_k^{(-)} H_i^T + R_i]^{-1} H_i\Bigg) \\ \cdot P_k^{(-)}\Bigg\}\Phi^T(k+1,k) + Q \end{aligned} \tag{3.2-10a}$$

$$\begin{aligned} = \Phi(k+1,k)\Bigg\{\Bigg(\sum_{i=1}^{M} m_i(k) H_i^T R_i^{-1} H_i\Bigg) \\ + (P_k^{(-)})^{-1}\Bigg\}^{-1} \Phi^T(k+1,k) + Q \end{aligned} \tag{3.2-10b}$$

(with initial condition P_0) associated with the SSBNs's use of a Kalman filter to track the navigation error states of the INS.

Upon taking the appropriate gradients that define an optimum to minimize the cost function of Eq. 3.2-1, the result is*

$$\Lambda_k^* = \left.\frac{\partial H(P_k, \Lambda_{k+1}, m(k), k)}{\partial P_k}\right|_* \tag{3.2-11a}$$

$$\begin{aligned} &= \mu_1 \cdot B_k^T + \Phi^T \Lambda_{k+1}^* \Phi - \Phi^T \Lambda_{k+1}^* \Phi_k^* \\ &\quad \cdot \left(\sum_{i=1}^{M} m_i^*(k) H_i^T [H_i P_k^* H_i^T + R_i]^{-1} H_i \right) \\ &\quad - \left(\sum_{i=1}^{M} m_i^*(k) H_i^T [H_i P_k^* H_i^T + R_i]^{-1} H_i \right) P_k^* \Phi^T \Lambda_{k+1}^* \Phi \\ &\quad + \sum_{i=1}^{M} m_i^*(k) H_i^T [H_i P_k^* H_i^T + R_i]^{-1} H_i P_k^* \Phi^T \Lambda_{k+1}^* \Phi P_k^* H_i^T \\ &\quad \cdot [H_i P_k^* H_i^T + R_i]^{-1} H_i \end{aligned} \tag{3.2-11b}$$

which upon completing the square simplifies as

$$\begin{aligned} \Lambda_k^* &= \mu_1 \cdot B_k^T + \left[I - P_k^* \left(\sum_{i=1}^{M} m_i^*(k) H_i^T (H_i P_k^* H_i^T + R_i)^{-1} H_i \right) \right]^T \\ &\quad \cdot \Phi^T \Lambda_{k+1}^* \Phi \left[I - P_k^* \left(\sum_{i=1}^{M} m_i^*(k) H_i^T (H_i P_k^* H_i^T + R_i)^{-1} H_i \right) \right] \end{aligned} \tag{3.2-11c}$$

with final condition

$$\Lambda_N^* = \frac{\partial}{\partial P_N} \mathrm{tr}[\mu_1 \cdot A P_N]|_* = \mu_1 \cdot A^T \tag{3.2-12}$$

(recognized to be a matrix Lyapunov equation). Upon taking gradients† of the Hamiltonian in Eq. 3.2-9 with respect to Λ_{k+1}, just returns the dynamical constraint of Eq. 3.2-10 as the Riccati equation

$$\begin{aligned} P_{k+1}^* &= \Phi(k+1, k) \left\{ P_k^* - P_k^* \left(\sum_{i=1}^{M} m_i^*(k) H_i^T [H_i P_k^* H_i^T + R_i]^{-1} H_i \right) P_k^* \right\} \\ &\quad \cdot \Phi^T(k+1, k) + Q \end{aligned} \tag{3.2-13}$$

with initial condition

$$P_0^* = P_0 \tag{3.2-14}$$

Eqs. 3.2-11c and 3.2-13 to be solved backward and forward, respectively, over a specified time interval, constitute the fundamental Two Point Boundary Value Problem (TPBVP) that is underlying the optimization or minimization of the cost function of Eq. 3.2-2; however, the equations are in fact well-posed (i.e., are directly solvable without encountering any instabilities in either the forward or backward time directions[36]) and are only finite dimensional.

---

* The asterisk appearing in the evaluation of the Hamiltonian derivatives in the canonical equations and as a superscript in P^*, Λ^*, m^* is standard usage and denotes evaluation at the optimum.

† In this application, it is permissible without ill effects to ignore the slight complications that arise in taking matrix gradients with respect to symmetric matrices[21].

A further necessary condition on the optimum fix strategy m^* is that it minimize the Hamiltonian as

$$H(P_k^*, \Lambda_{k+1}^*, m_k^*, k) \leq H(P_k^*, \Lambda_{k+1}^*, m_k, k) \qquad (3.2\text{-}15)$$

which simplifies to yield the following condition

$$\sum_{i=1}^{M} m_i^*(k) s_i^*(k) \leq \sum_{i=1}^{M} m_i(k) s_i^*(k) \qquad (3.2\text{-}16)$$

where for convenience

$$s_j^*(k) \triangleq \mu_2 \cdot c_j - \mathrm{tr}[(H_j^T[H_j P_k^* H_j^T + R_j]^{-1} H_j) P_k^* \Phi^T \Lambda_{k+1}^* \Phi P_k^*] \qquad (3.2\text{-}17)$$

This simplified representation in Eq. 3.2-16 of the essence of the statement of the minimum principle for the SSBN problem, in conjunction with the constraint of Eq. 3.2-8 and in conjunction with the physical constraint of Eq. 3.2-7 that $m_i(k)$ is either one or zero (i.e., sensor off or in use) leads to the following necessary condition for satisfying the inequality of Eq. 3.2-16;

$$m_i^*(k) = \begin{cases} 1 & \text{if} \quad s_i^*(k) \leq s_j^*(k) \quad \text{for} \quad j = 1, 2, \ldots, M \quad \text{and} \quad s_i^*(k) \leq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (3.2\text{-}18)$$

In practical terms, when the policy of Eq. 3.2-18 is adherred to, the result is that the minimum principle of Eq. 3.2-15 is satisfied as a necessary condition for minimizing the cost function J of Eq. 3.2-1.

Returning to consider the appropriateness of the simple scalar cost function of Eq. 3.2-2 for the SSBN application, the two conflicting objectives of "reducing navigation error through navaid utilization" and "reducing the exposure to enemy surveillance in using navaids" are not so diametrically opposed that they are incompatible. If there were originally just a single scalar criterion, two distinct competing fix usage strategies could be compared unambigously to demonstrate superiority of one strategy over the other. However, in applications such as the SSBN navaid fix utilization problem which inherently involves two criteria:

J_{AE}—a measure of navigation accuracy error
J_{COST}—cost of SSBN exposure to enemy surveillance in fix taking

there is *no unambigous optimum* since the *plane cannot be "ordered"*. However, when provided with two (or more) criteria there is a set of optimal strategies classified as being *Pareto-optimal*[26, 25] for which:

- For fixed J_{AE}, J_{COST} is minimized,
- For fixed J_{COST}, J_{AE} is minimized.

The pareto-optimal set is depicted in Fig. 3.2-1 and can be calculated using the method-of-linear combinations. The "method-of-linear combinations"[22, 23, 24, 26] requires that the associated scalar cost function of Eq. 3.2-2 of the form

$$J\left[\{m(k)\}_{k=0}^{N-1}\right] = \mu \cdot J_{\mathrm{AE}} + (1 - \mu) \cdot J_{\mathrm{COST}} \qquad (3.2\text{-}19)$$

be minimized for each fixed μ within the range

$$0 \leq \mu \leq 1 \qquad (3.2\text{-}20)$$

Bi-criteria optimization theory (pp. 24–28 of Ref. 26, Refs. 22, 23) guarantees that

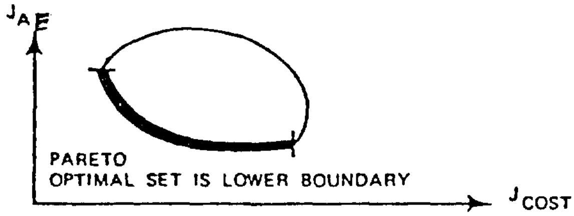


Fig. 3.2-1—Pareto-optimal set provides a lower bound for performance trade-offs.

for the structure* provided by the SSBN navaid fix utilization problem, minimization of Eq. 3.2-2 for all values of μ in the range of Eq. 3.2-20 provides all of the Pareto-optimal solutions.

The switching policy of Eq. 3.2-18 for specifying $m^*(k)$ is crucial as it forms the computational basis of the min-H solution technique†[20, 30, 33]. The basic structure of the discrete-time-formulation presented here (use being available through Intermetrics, Inc. as a 1000 line validated PL/1 program) is identical to the min-H flow chart of Fig. 5 in Ref. 20 for continuous-time sensor schedule optimization. However, the current updated implementation includes an additional outside loop for obtaining solutions that minimize the cost function J of Eq. 3.2-1 for a span of fixed weightings μ (handled as a stacked case run) and in addition incorporate a reduced-order filter formulation (unlike Ref. 16). The use of varying weightings in this manner serves as the requisite link with bi-criteria optimization theory to allow computational delineation of the Pareto-optimal lower boundary for the SSBN application as pictorially illustrated in Section 3.3.

Major technical differences between the optimization approach of Ref. 20 and that which is reported here (as used in Ref. 16) are:

- Cost functions used here include effect of accuracy considerations over the entire time interval, rather than just a single terminal time accuracy constraint (as discussed in detail in Section 3, Ref. 38),
- Current formulation is posed in discrete-time to more closely match the SSBN application of discrete navaid events,
- Continuous-time formulation of Ref. 20 required piece-wise continuous flow of measurements at all times, while explicit modification herein allows possibility of *no* measurements at some times (being the prevalent case for the SSBN application),
- Discrete-time formulation (having only a finite number of candidate fix times over an interval) circumvents a continuous-time "chattering" problem acknowledged in Ref. 20, where successive iterations of the min-H algorithm‡ produced slightly different instant-of-switchings within the time continuum,

---

* Structure does *not* require strict "convexity" of the two cost function components, merely directional convexity suffices[22, 23]. While the right hand cost-of-exposure component of Eq. 3.2-2 is linear in m and therefore convex in a degenerate sense but convex none-the-less, directional convexity of the left hand accuracy error component of Eq. 3.2-2 is established by using the convexity of the matrix inverse for symmetric positive definite matrices[28] in conjunction with the alternative form of Eq. 3.1-10*b* and the fact that the trace operation preserves convexity.

† While the min-H algorithm represents an approximate technique for many non-linear optimization applications, use of the min-H algorithm is *exact* for the linear problem structure of sensor schedule optimization[27].

‡ The rate of convergence of the min-H algorithm is fast[20, 16, 31] and has been studied in detail in Ref. 27 where convergence (in-the-sense-of-orthogonal-search-algorithms) is guaranteed for this sensor scheduling optimization problem.

- Application to C-3 Poseidon in Ref. 16 untilized a 46 state model (as contrasted to the 3 state model in the application of Ref. 20)* consisting of:

—34 state linearized SINS navigation error model,
—10 state Loran fix/synchronization model[29],
—2 state Depth Sonar bias effects[3],

which was computationally accommodated due to faster computation speeds and more efficient handling of sparse matrices now available despite the current computer burden that goes as the cube of the state size (46^3).

Despite the prevalence of independent analytical support for different aspects of the procedure, it still has not been reduced to a crank-and-grind technique. Considerable human "mothering" of the computational procedure is required. An example is in the required provision of "informed" initial guesses for reasonably close fix strategies to start off the iterative optimization, since the underlying two cost functions are not strictly convex the possibility exists for encountering some local extrema or limit-cycling if adequate precautions are not taken to steer clear of these.

3.3 *Assessing Limits of C-3 SSBN Performance*

Fig. 3.3-1 depicts the Pareto-optimum lower boundary obtained using the procedures of Section 3.2 for C-3 Poseidon SSBNs[16] under *nominal* conditions (as reflected in the values of c_i used in Eq. 3.2-2). Worst or best case environmental conditions as effects on detection range (e.g., as a consequence of electronic countermeasures ECM), velocity of surveillance platform, or time required to take a fix (e.g., sea state related effects can alter fix taking expediency) for a particular navaid can be absorbed merely as minor changes in the values of the c_i† (defined in Eq. 3.1-2) as used in Eq. 3.2-2. While a lower boundary slightly different from the one portrayed in Fig. 3.3-1 results for best and worst case conditions, it is still easily calculated as a parametric study using the same techniques of Section 3.2.

Also depicted in Fig. 3.3-1, as numbered from one to six, are the relative evaluations of various representative C-3 SSBN fix strategies (for a four day epoch) as explicitly identified in Ref. 16 to include several characteristic of Difference Monitoring (i.e., Nos. 3 and 6)‡. Extremes in detectability are obtained for different sensor mixes and differing fix intervals. These alternate C-3 navaid fix strategies can be compared to each other as a *relative* gauge of utility or to the lower boundary where proximity is an *absolute* gauge of "goodness."

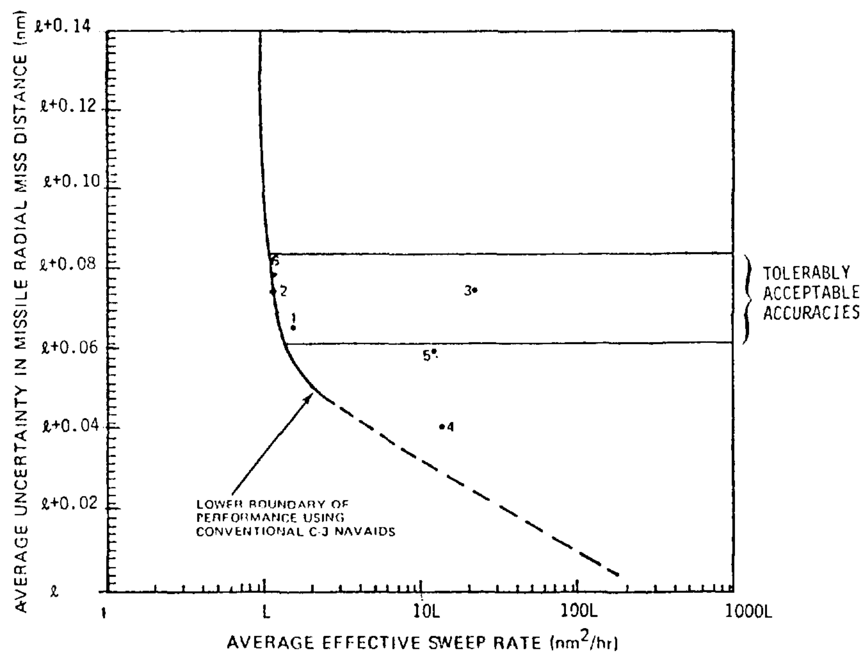
While the methodology remains the same, complete evaluations such as the one depicted in Fig. 3.3-1 have yet to be performed for the Electrostatically Supported Gyro (ESG) navigation technology of Trident I and II SSBNs for SINS/ESGM, SINS/ESGN, or ESGN/ESGN candidate navigation configura-

---

* The application of Ref. 20 was to a Lincoln Lab study of an aircraft (modeled using 3 state variables) with alternative radar measurements meeting a required terminal-time accuracy constraint (for targeting purposes).

† Methodology of Section 3.2 also accommodates time-varying cost $c(k)$[16, 20]. This enables a rigorous contrivance for the SSBN application of making the cost of using a depth Sonar fix essentially infinite between designated map-matching areas (P-points) to avoid otherwise inappropriate fix taking.

‡ No fix schedules are explicitly depicted here since any order of alternative fix usage and time intervals between fixes would be classified.



(ℓ and L are used here for declassifying data)

Fig. 3.3-1—Relative comparisons between selected fix schedules in an accuracy vs. observables trade-off.

tions (including options of velocity measuring sonar and/or gradiometers), despite the fact that specific alternative navaid usage schedules have been postulated[34] for C-4 Trident SSBNs. Other considerations that may help alleviate the impacted computer burden of the three CP-890/UYK navigation computers and further facilitate use of the analysis technique of Section 3.2 is to utilize recently developed results from decentralized estimation (for more detail, please see conclusions on p. 326 of Ref. 35 and Ref. 37). Since accuracy in targeting is a significant function of the navigation accuracy of own-ship position but effectiveness requires maintaining covertness, similar trade-off analyses should be performed for full consideration of TOMAHAWK and SUBROC launches as well as for other cruise-missile launching submarines.

ACKNOWLEDGMENTS

This work is an outgrowth of previous work supported by the Department of the Navy, Strategic Systems Project Office, under TASC Contract N00030-79-C-0042. Thanks are due to Laura Callahan for the painstaking preparation of this document and to John Heymann for the technical illustrations.

REFERENCES

1. Clautice, W. G., "Submarine Navigation," *Navigation, Journal of the Institute of Navigation*, Vol. 25, No. 1, pp. 34–39, Spring, 1978.
2. "Difference Monitoring (U)," Navigation Information Bulletin (SUPPLEMENT) SSPO, Department of the Navy, NIB-S-18, September 1977 (CONFIDENTIAL).
3. "Improved Accuracy Program Navigation Error Analysis Notebook (U)," Eighth Issue,

SSM Pub. No. GJ-2236-2454B, November 1978 (CONFIDENTIAL).

4. Helstrom, C. W., "Note on a Markov Envelope Process," *IRE Transactions on Information Theory,* Vol. IT-5, pp. 139–140, September 1959.
5. Helstrom, C. W. "The Accuracy of Probability Distributions Computed by the Sampling Approximation," Westinghouse Research Report No. 8-1259-R2, May 1956.
6. Balakrishnan, A. V., ed., *Communication Theory,* McGraw-Hill, New York, 1968.
7. Gallager, R. G., and Helstrom, C. W., "A Bound on the Probability that a Gaussian Process Exceeds a Given Function," *IEEE Transactions on Information Theory,* Vol. IT-15, No. 1, January 1969.
8. Kerr, T. H., "False Alarm and Correct Detection Probabilities Over a Time Interval for Restricted Classes of Failure Detection Algorithms," *IEEE Transactions on Information Theory,* Vol. IT-28, 1982.
9. Kerr, T. H., "Improving ESGM Failure Detection in the SINS/ESGM System (U)," The Analytic Sciences Corporation, Report No. TR-678-3-1, 31 October 1976 (CONFIDENTIAL).
10. Wong, E., "Recent Progress in Stochastic Processes: A Survey," *IEEE Transactions on Information Theory,* Vol. IT-19, No. 3, pp. 262–275, May 1973.
11. Blake, I. F. and Lindsey, W. C., "Level-Crossing Problems for Random Processes," *IEEE Transactions on Information Theory,* Vol. IT-19, No. 3, pp. 295–315, May 1973.
12. Cramer, H. and Ledbetter, M. R., *Stationary and Related Stochastic Processes: Sample Function Properties and Their Applications,* John Wiley and Sons, Inc., New York, 1967.
13. Patel, J. K., Kapadia, G. H., and Owen, D. B., *Handbook of Statistical Distributions,* Marcel Dekker Inc., New York, 1976.
14. Maybeck, P. S., *Stochastic Models, Estimation, and Control,* Vol. 1, Academic Press, New York, 1979.
15. Jahnke, E. and Emde F., *Tables of Higher Functions,* McGraw-Hill Book Company, New York, 1960.
16. Kerr, T. H., "Improving C-3 SSBN Navaid Utilization," TASC Technical Information Memorandum TIM-1390-3-1, August 1979 (SECRET).
17. Rice, S. O., "Distribution of Fades in Radio Transmission: Gaussian Noise Model," *Bell System Technical Journal,* Vol. 37, pp. 581–635, May 1958.
18. Koopman, B. O., "Search and Screening," Vol. 2B, Division 6, Operations Evaluation Group, Office of the Chief of Naval Operations, Navy Department, Washington, D.C., 1946 (new expanded edition by Pergamon Press, New York, 1980).
19. Koopman, B. O., "The Theory of Search: II Target Detection," *Operations Research,* Vol. 4, No. 5, pp. 503–531, October 1956.
20. Athans, M., "On the Determination of Optimal Costly Measurement Strategies for Linear Stochastic Systems," *Automatica,* Vol. 8, pp. 397–412, 1972.
21. Walsh, P. M., "On Symmetric Matrices and the Matrix Minimum Principle," *IEEE Transactions on Automatic Control,* Vol. AC-22, No. 6, pp. 995–996, December 1977.
22. Polak, E. and Payne, A. N., "On Multicriteria Optimization" in *Directions in Large-Scale Systems* (Ed. by Y. C. Ho and S. K. Mitter), Plenum Press, New York, 1976.
23. Lin, J. G., "Three Methods for Determining Pareto-Optimal Solutions of Multiple-Objective Problems," in *Directions in Large-Scale Systems,* Ed. by Y. C. Ho and S. K. Mitter, Plenum Press, New York, 1976.
24. Zadeh, L. A., "Optimality and Non-Scalar-Valued Performance Criteria," *IEEE Transactions on Automatic Control,* Vol. AC-8, January, 1963.
25. Pareto, V., *Cours de'Economie Politique,* Lausanne, Rouge, 1896.
26. Ho, Y. C., "The First International Conference on the Theory and Applications of Differential Games," Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass., Final Report to Air Force Office of Scientific Research, Grant AFOSR-69-1768, January 1970.
27. Platzman, L. K., "On the Interative Design of Optimal and Suboptimal Controllers,"

M.S. Thesis, Electronic Systems Laboratory, Report ESL-R-498 Massachusetts Institute of Technology, Cambridge, Mass., May 1973.

28. Kerr, T. H., "Three Important Matrix Inequalities Currently Impacting Control and Estimation Applications," *IEEE Transactions on Automatic Control,* Vol. AC-23, No. 6, pp. 1110–1111, December 1978.
29. Griffith, E. W., Extension of the Phase-Shift Loran-C Synchronization Interval (U)," The Analytic Sciences Corporation, Technical Information Memorandum TIM-678-2-1, 31 October 1976 (CONFIDENTIAL).
30. Kelley, H. J., "Method of Gradients" in *Optimization Techniques* (Ed. by G. Leitman), Academic Press, New York, 1962.
31. Gottlieb, R. G., "Rapid Convergence to Optimum Solutions Using a Min-H Strategy," *AIAA Journal,* Vol. 5, No. 2, pp. 322–329, February 1967.
32. Bryson, A. E., and Ho, Y. C., *Applied Optimal Control,* Ginn and Company, Waltham, Massachusetts, 1969.
33. Balakrishnan, A. V., and Neustadt, L. W. (Eds.), *Computing Methods in Optimization Problems,* Academic Press, New York, pp. 211–239, 1964.
34. "Trident Navigation Operating Policy/Scenario (Normal Operations) (U)," Pub. No. GJ-19-1710F, SSM, Great Neck, New York, 23 January 1976 (SECRET).
35. Kerr, T. H., and Chin, L., "A Stable Decentralized Filtering Implementation for JTIDS RelNav," *Proceedings of Position, Location, and Navigation Symposium* (PLANS), Atlantic City, New Jersey, December 8–11, 1980.
36. Scott, M. R., *Invariant Imbedding and Its Applications to Ordinary Differential Equations: An Introduction,* Addison-Wesley, New York, 1973.
37. Kerr, T. H., and Chin, L., "The Theory and Techniques of Discrete-Time Decentralized Filters," in *Advances in the Theory and Technology of Applications of Nonlinear Filters and Kalman Filters,* edited by C. T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph Noordhoff International Publishing, Leiden, 1981.
38. Kerr, T. H., "Rectifying Several Pervasive Errors Appearing in Estimation and Control Theory," submitted to *IEEE Transactions on Automatic Control,* 1981.
39. Soong, T. T., *Random Differential Equations in Science and Engineering,* Academic Press, New York, 1973.
40. "FBM Navigation Subsystem Progress Letter, 16 September 1978-15 October 1978," Sperry Systems Management Pub. No. GA-2264-0017-225, October 1978.
41. Morse, P. M. and Kimball, G. E., "How to Hunt a Submarine," in *The World of Mathematics,* Vol. IV, Part XX: Mathematics in Warfare, (Ed. by J. R. Newman), Fifth Printing, Simon and Schuster, New York, 1965.