The Paper under review: by Henderick C. Lambert, entitled "Cramer-Rao Bounds for Target Tracking Problems Involving Colored Measurement Noise" is extremely well written and takes the reader along for a nice ride so I had to be extra careful because, as an analyst, I was enjoying it so much. Maybe too much (looking only at minutiae) so I stepped back and took a second more critical look. I suggest modifying title of paper to be **"Cramer-Rao Bounds for Tracking Targets Corrupted by Colored Measurement Noise"** as being a little shorter too.

My vote is to publish but also to give the author an opportunity to, perhaps, modify his transcript based on the suggestions, objections, or request for clarification offered below. I only seek improvement to a paper that is already pretty good. My biggest concern is **Issue #6** below, which may be a show stopper.

Idea #1:

I will simplify here to make a strong point and hopefully, by doing so, more directly "cut to the chase":

1. Rigorous and easy to evaluate CRLB procedures exists for lower bounding covariance of estimation error in situations where there is **no process noise present** and consequently associated covariance of process noise is identically the zero matrix as Q=0 (this benign situation, which reaps extremely lucrative tractability of CRLB calculation as a consequence, arises in the mathematical model for targets in a ballistic trajectory but only during the exoatmospheric mid-course phase). This is enough for defense strategies that seek to intercept targets only in this regime (like NMD).
2. Rigorous, but more numerically challenging evaluation methodologies, have recently been developed that claim to extend CRLB methodology to handle cases where there is **non-zero process noise present**. This is a more general case (reflecting atmospheric buffeting associated with reentry drag, or midcourse maneuvering, or with a projectile undergoing late stage maneuvering). However, practical evaluations along this line occur less frequently in the open literature. (Tichavsky, Muravchik, and Nehorai [10] provide a rigorous methodology for performing CRLB evaluations when Q is not identically zero. Those investigators that preceded them were apparently somewhat off the mark in various ways.)

Author H. C. Lambert indicates that he plans to only use the first path above so things should proceed more easily and clearly.

Author Lambert does an excellent job of surveying the CRLB literature and in handling things diplomatically. He covers both options above. I wish that he would clarify the above distinction for the reader so that the reader is aware from the start of the existence of the above dichotomy hinging on the answers "yes" or "no" regarding the presence (or absence) of Q=0 since it is the key to a full understanding.

Idea #2:

Historically, when serial time-correlation arises in either the measurement noise or in the process noise or both, it can be simply handled by merely including the structure of time correlations within the system model by "state augmentation". See details for doing so in:

- Kerr, T. H., "Multichannel Shaping Filter Formulations for Vector Random Process Modeling Using Matrix Spectral Factorization," MIT Lincoln Laboratory Report No. PA-500, Lexington, MA, 27 Mar. 1989 (in an Appendix).
- Gelb, A. (ed.), *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1974 (Sec. 4.5, pp. 133-136).

On page 4, last paragraph, author Lambert states that the nature of correlation in the measurement noise **is assumed** to be prescribed by a known dynamic model. Since how to do so from fundamental measurements collected is a missing link here, the first reference above in Idea #2 demonstrates how to find the corresponding unknown dynamics model when the colored measurement noise correlation function matrix is known or, equivalently, for stationary processes, its power spectral matrix is known, as determined from the statistics of the measurements obtained. Other places that demonstrate how it may be found are:

- Kerr, T. H., "Emulating Random Process Target Statistics (using MSF)," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-30, No. 2, pp. 556-577, Apr. 1994.
- Kerr, T. H., "Comments on 'Precision Free-Inertial Navigation with Gravity Compensation by an Onboard Gradiometer'," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 30, No. 4, pp. 1214-1215, Jul.-Aug. 2007.

While use of state augmentation is a simple routine tactic to reduce the problem to again be one that has only Gausian White Noises for process noise and sensor measurement noise, with known covariance specified (and perhaps now with zero measurement noise covariance unless the measurement noise also had a WGN component too that would remain unchanged), this approach has three drawbacks: (1) dimension of the system model is increased; (2) system has new components of process noise covariance that are not identically zero (only a drawback with regards to the above **Idea #1, Option 2**), (3) effective measurement noise covariance is now the zero-matrix R = 0, which is sometimes a numerical problem for Kalman-like filter tracking formulations if $[H_k P_{k|k-1} H_k^T + R_k]$ is singular and not invertible as a consequence. This aspect also occurs within the CRLB formulation as well. Author Lambert avoids dealing with this more general situation by assuming noise structure of Eq. 2 that will have presence of non-zero GWN measurement noise covariance R even after serial time-correlated noise term is extracted and associated with the system or plant.

Idea #3:

Historically, Arthur Bryson and his student L. J. Henrikson (as referenced by Lambert) in 1968 were able to handle estimation with serial time-correlated noise **without increasing the dimension of the system**. This is the tact that Lambert says he will follow by handling the situation of serially time-correlated measurement noise using the first methodology **(Idea #1 Option 2, above)** by adapting the previous well-know pre-whitening approach by Bryson and Henrikson.

Another place where estimation with serial time-correlated noise was successfully handled **without increasing the dimension of the system** was in:

- Meditch, J. S., *Stochastic Optimal Linear Estimation and Control*, McGraw-Hill, NY, 1969.

Stated Goal of this Paper:

Main topic addressed in this paper, according to the title, is how to handle serially time-correlated measurement noise in CRLB evaluation (which by **Idea #2** is historically associated with presence of non-zero process noise) within an "easy" CRLB evaluation **(Option 1 in Idea #1, above)** where no process noise is present. Lambert states that he will use the result of Idea #3 above. So everything is plausible and we looked forward to the end result.

Objection #1: Single numerical example in Section V, pages 20-23 is for a measurement noise component being a random walk. Random walk is extremely serially time-correlated and its variance increases with time so it is not a stationary process.

The simplest possible non-degenerate numerical example that is serially time-correlated over a finite interval rather than over an infinite interval would seek to illustrate with a first order Markov process as the measurement noise in order to be consistent with what the title of the paper currently says is to be provided at the end of the rainbow.

Issue here centers around what actually constitutes "colored" noise. Colored noise is **not** being Gaussian White Noise (GWN) and the author Lambert agrees (at the bottom of page 1) that he is interested in handling temporally correlated noise (a.k.a. serially time-correlated noise). A Random Walk process is temporally correlated as merely the integral of GWN (with no bounding time constant to drag cross-correlations over successive discrete sampling time steps as a 1st order or higher order stationary Gauss-Markov process would do in the role of the colored noise) but is correlated over all ensuing time steps and its variance increases with time. Using a Random Walk is much worse (harder to handle). (See Issue #6, below.)

Several precedents exist that support my interpretation of what type of example is needed here:

- Stear, E. B., and Stubberud, A. R., "Optimal Filtering for Gauss-Markov Noise," *International Journal of Control*, Vol. 8, No. 2, pp. 129-130, 1968.
- Bryson, A. E. and Johansen, D. E., "Linear Filtering for Time-Varying Systems using Measurements Containing Colored Noise," *IEEE Trans. on Automatic Control*, Vol. AC-10, No. 1, pp. 4-10, Jan. 1965.

Issues and Concerns:

Historically, measurements were represented as being collected in a large super-vector with the earliest or initial one at the top and the remainder ordered sequentially down until the last or most recent is at the bottom.

In Eq. 68, Lambert's convention is opposite to what everyone else has historically used by being exactly opposite. No benefit accrues or loss incurred by his doing so. However, Lambert uses the opposite convention for measurements in Eq. 45. He probably should at least be internally consistent. Please see next complaint below, where this aspect may hurt.

Issue #1: Lambert **takes the difference** of two Fisher forms, which has a risk of incurring something that lacks being positive definite. Usually analysis steps are followed to ensure that positive definiteness is maintained, otherwise problems can ensure. Sums of positive definite matrices are positive definite. Sum of positive definite and positive semi-definite is positive definite, etc. In the paper, there was no reason presented (that I could see) to offer this unpleasant intermediate form in Eq. 48 involving a difference. Perhaps this can be avoided somehow.

Navigation Analysis (more conveniently accessible to readers than somewhat new Ref. [48]):

1. Biezad, D. J., *Integrated Navigation and Guidance Systems*, AIAA Education Series, Reston, VA, 1999.

2. Siouris, G. M., "Navigation: Inertial," *Encyclopedia of Physical Science and Technology*, 2nd Edition, V. 10, pp. 595-647, Academic Press, NY, 1992.

3. Britting, K. R., *Inertial Navigation System Analysis*, Wiley-Interscience, NY, 1971.

4. Huddle, J.R., "Inertial Navigation System Error Model Considerations in Kalman Filter Applications," *Control and Dynamic Systems: Advances in Theory and Applications - Nonlinear and Kalman Filtering* Techniques, Vol. 20, Academic Press, NY, pp. 294-340, Part 2 of 3, 1983.

5. Maybeck, P. S., *Stochastic Models, Estimation, and Control*, Vol. 1, Academic Press, N.Y., 1979 (Chapter 6).

6. Gelb, A. (ed.), *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1974.

Issue # 2: citing on page 5, next to last sentence, and on page 18 first sentence, author Lambert says that his methodology may be extended to include the effect of navigation errors. He references a relatively new report (his [48]) for how to do so as if it is merely an afterthought and how it can be done is too recent to be considered now and addressed thoroughly. The above references listed here above by me have historically said how. The problem is that when INS errors are accounted for, typically the process noise covariance is no longer zero and the easy case Option 1 of Idea #1 above for evaluating CRLB is no longer applicable. Author Lambert would then need to use the less tractable Option 2. It would be nice if he admitted this at the time he brought up handling navigation errors and dismissing it as just a minor extension of what he has provided in his paper since such extensions require use of the other more involved approach of Idea #1 Option 2.

Issue #3: Author Lambert states on page 3 that the following approach utilizing nuisance parameters:

- Miller, R. W, and Chang, C. B., "A Modified Cramer-Rao Bound and Its Application," *IEEE Transactions on Information Theory*, Vol. IT-24, No. 3, pp. 398-400, May 1978.

amounts to a Monte-Carlo sampling technique (like a Metropolis-Hastings or Metropolis-Gibbs sampling and re-sampling associated with attempting to use Particle Filters, as supporting theory that became known much later). However, if one compares how many samples are necessary to participate in the averages for a rigorous interpretation, as discussed in some of the recent papers by Frederick Daum in the last 10 years on Particle Filters for practical applications, one realizes that the amount of numerical computation is daunting by being overwhelming and untenable for most situations. Author Lambert could, perhaps, also make that point in his paper to bring the relevance of the above reference up to date despite the fact that it is an excellent piece of analysis with extremely appealing illustrative examples that were analytically tractable in closed-form. For more general applications, without the benefit of being simple enough for closed-form answers, it would likely be very challenging to average enough data to get useful results.

Issue #4: To prevent readers from incorrectly assuming that the results of Eqs. 40 to 44 are original and new, I suggest referencing oldest occurrences of their use within an estimation context as occurs in:

- Section 1-19 of Liebelt, P. B., *An Introduction to Optimal Estimation*, Addison-Wesley, Reading, MA 1967.

- Appendix 7B: Some Matrix Equalities in Jazwinski, A. H., *Stochastic Processes and Filtering Theory*, Academic Press, N.Y., 1970.

Issue #5: To prevent readers from getting the wrong idea on page 4 about how Tichavsky et al can handle "random dynamics" please avoid this internal short hand code and, instead, say handle "process, plant, or model dynamics with random disturbances (i.e., Q non-zero)". Otherwise, the reader may expect entries in the system matrix may be random and that is not the case. Sorry to "nickel and dime" you but sometimes managers (and other readers) view something and misinterpret it to be exactly what you said rather than "what you meant".

Issue #6: Most serious issue here - On page 23, first sentence at top of page: "components of the measurement vector, y_k, …, are **assumed to have the same statistics** as well as being independent of each other. This assumption **which is needed to invoke CRLB calculation** is at odds with the presence of a random walk, which is non-stationary, with a variance that increases with time. (Please see Gelb Fig. 3.8.5 or Maybeck on this aspect for verification.) Also is contrary to trends depicted in Figs. 1 and 2 (unless we only see a time segment before it is large enough to discern the increase by eye). Please address or finesse. This could be a show-stopper. It is a non-issue if Random Walk component were replaced instead with merely a first order or higher stationary Gauss-Markov process. We realize that it is "necessary to play the ball where it lies" if that is what constitutes the application situation.

May Wish to Reference How to Handle Jamming Considerations:

1. Myers, L., *Improved Radio Jamming Techniques: Electronic Guerilla Warfare*, ISBN 0873645200, Paladin Press, Boulder, CO, 1989.

2. Chapter 8 of Bar-Shalom, Y., Blair, W. D., (Eds.), *Multitarget-Multisensor Tracking Applications ad Advances*, Vol. III, Artech House Inc., Boston, MA, 2000.

Another view of image registration (the application specifically addressed as motivation in this paper) to perhaps be acknowledged as a competitive approach:

- Yetik, I. S., Nehorai, A., "Performance Bounds on Image Registration," *IEEE Trans. on Signal Processing*, Vol. 54, No. 5, pp. 1737-1749, May 2006.

Other significant Cramer-Rao Lower Bound citations (overlooked), perhaps, of interest:

1. Jazwinski, A. H., *Stochastic Processes and Filtering Theory*, Academic Press, N.Y., 1970. (Provides expression for CRLB for Q=0-matrix.)

2. Balakrishnan, A. V., *Kalman Filtering Theory*, Optimization Software, Inc., NY, 1987. (Clarifies difficulties between situation of evaluating CRLB for Q being zero matrix versus not being zero matrix.)

3. Joshi, S. M., Control of Large Flexible Space Structures, Lecture Notes in Control and Information Series, Vol. 131, Springer-Verlag, NY, 1989 (see Section 4.1.1, pp. 165-170).

4. Eldar, Y., "Uniformly Improving the Cramer-Rao Bound and Maximum-Likelihood Estimation," *IEEE Trans. on Signal Processing*, Vol. 54, No. 8, pp. 2943-2956, Aug. 2006.
5. Au-Yueng, C. K., Wong, K. T., "CRB: Sinusoid-Sources' Estimation using Collocated Dipoles/Loops," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 45, No. 1, pp. 94-109, Jan. 2009.
6. Kay, S., Xu, C., "CRLB via the Characteristic Function with Application to the K-Distribution," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 44, No. 3, pp. 1161-1168, July 2008.
7. Smith, S. T., "Statistical Resolution Limits and the Complexified Cramer-Rao Bound," *IEEE Trans. on Signal Processing*, Vol. 53, No. 5, pp. 1597-1609, May 2005.
8. Smith, S. T., "Covariance, Subspace, and Intrinsic Cramer-Rao Bounds," *IEEE Trans. on Signal Processing*, Vol. 53, No. 5, pp. 1610-1630, May 2005.
9. Gini, F., Regianini, R., Mengali, U., "The Modified Cramer-Rao Lower Bound in Vector Parameter Estimation," *IEEE Trans. on Signal Processing*, Vol. 46, No. 1, pp. 52-60, Jan. 1998.