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Due 2 wks from March 5, 1969

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Homework Assignments

(i) Given a system described by the following stochastic differential equation:

$$dx(t) = f(x,t)\,dt + G(x,t)\,d\xi(t) \qquad (\mathrm{I})$$
$$x(t_0) = x_0,$$

where

$f(x,t)$ = n-vector, $G(x,t)$ = $n \times m$ matrix, $\xi(t)$ = m-vector Wiener process with $\langle d\xi(t)\rangle = 0$ and $\langle d\xi(t)\, d\xi^T(t)\rangle = Q(t)\,dt$, $x_0 = N[\hat{x}_0, \Gamma_0]$.

(a) Show that the Fokker-Planck equation corresponding to eq. (I) is given by:

$$\frac{\partial p(x,t|x_0,t_0)}{\partial t} = -\left(\frac{\partial}{\partial x}\right)^T\left[f(x,t)\,p(x,t|x_0,t_0)\right] + \frac{1}{2}\,\mathrm{tr}\left\{\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)^T\left[G(x,t)\,Q(t)\,G^T(x,t)\,p(x,t|x_0,t_0)\right]\right\}.$$

(b) Find the differential equations for the mean $\hat{x}(t) = \langle x(t)\rangle$ and the covariance $\Gamma(t) = \langle [x - \hat{x}(t)][x - \hat{x}(t)]^T\rangle$ of the $x(t)$ process.

(c) If the system (I) is linear with additive noise, i.e.,

$$dx(t) = A(t)\,x(t)\,dt + G(t)\,d\xi(t) \qquad (\mathrm{II})$$
$$x(t_0) = x_0,$$

find the equations for $\hat{x}(t)$ and $\Gamma(t)$ from part (b).

(d) Find the differential equation for the characteristic function $c(t)$ corresponding to eq. (II).

(2) Let a dynamical system be described by the following scalar stochastic differential equation:

$$dx(t) = -\frac{x(t)}{1+x^2(t)}\,dt + g(t)\,d\xi(t)$$

$$x(t_0) = x_0,$$

where $\langle d\xi(t)\rangle = 0$ and $\langle [d\xi(t)]^2\rangle = q(t)\,dt$.

(a) Find the Fokker-Planck equation.

(b) Find the differential equations for the mean $\hat{x}(t)$ and the covariance $\gamma(t)$ of $x(t)$.

(c) Give an expression for the steady-state solution of the Fokker-Planck equation obtained in part (a), i.e., set $\frac{\partial p(x,t|x_0,t_0)}{\partial t} = 0$ and solve the Fokker-Planck equation (only an expression is needed).

(3) Consider the Lienard's equation:

$$dx_1(t) = x_2(t)\, dt$$

$$dx_2(t) = -\left[2x_1(t) + 3x_1^2(t) + x_2(t)\right]dt + g(t)\, d\xi(t)$$

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$ where $\langle d\xi(t)\rangle = 0$ and $\langle [d\xi(t)]^2\rangle = q(t)\, dt$.

(a) Find the Fokker-Planck equation.

(b) Find the differential equations for the mean

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$$

and the covariance

$$\Gamma(t) = \begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{21}(t) & \Gamma_{22}(t) \end{bmatrix} = \begin{bmatrix} \langle [x_1 - \hat{x}_1(t)]^2 \rangle & \langle [x_1 - \hat{x}_1(t)][x_2 - \hat{x}_2(t)] \rangle \\ \langle [x_2 - \hat{x}_2(t)][x_1 - \hat{x}_1(t)] \rangle & \langle [x_2 - \hat{x}_2(t)]^2 \rangle \end{bmatrix}.$$

Note that $\Gamma(t)$ is symmetric so that only equations for $\Gamma_{11}(t)$, $\Gamma_{12}(t)$ and $\Gamma_{22}(t)$ are needed.

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1.) $d\underline{x}(t) = \underline{f}(\underline{x},t)dt + \underline{G}(\underline{x},t)$
$\underline{x}(t_0) = \underline{x}_0$

where:

$\underline{f}(\underline{x},t)$ is an n-vector
$\underline{G}(\underline{x},t)$ is an $n \times m$ matrix
$\underline{\xi}(t)$ is an m-vector Wiener process
with $\langle d\underline{\xi}(t)\rangle = \underline{0}$, $\langle d\underline{\xi}(t)\, d\underline{\xi}^T(t)\rangle = \underline{Q}(t)\, dt$
and $p[\underline{x}_0, t_0] = N[\hat{\underline{x}}_0, \underline{\Gamma}_0]$

a.) Derive the Fokker-Planck equation for the above vector case:

$\Delta p(\underline{x},t|\underline{y},s) = \Delta p = p(\underline{z},t+\Delta|\underline{y},s) - p(\underline{x},t|\underline{y},s)$

where $\underline{z} = \underline{x}(t+\Delta)$ at $(t+\Delta)$

Then $\Delta\underline{x} = \underline{x}(t+\Delta) - \underline{x}(t) = \underline{z} - \underline{x}$

Let $S: R^n \rightarrow R$
Let $S(\underline{x})$ be an arbitrary, non-negative function from $R^n \rightarrow R$ with $\left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x})$ and $\left(\frac{\partial}{\partial \underline{x}}\right)^T\left(\frac{\partial}{\partial \underline{x}}\right) S(\underline{x})$ existing and being continuous in $\underline{x}$, $\forall \underline{x} \in \{\underline{x}: \sum_{i=1}^{n} x_i^2 \le a^2\}$ for some $a \in R$, $a^2 > 0$; and $S(\underline{x}) = 0$, $\left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x}) = \underline{0}$, $\left(\frac{\partial}{\partial \underline{x}}\right)^T\left(\frac{\partial}{\partial \underline{x}}\right) S(\underline{x}) = \underline{0}$ for $\sum_{i=1}^{n} x_i^2 \ge a^2$.

Since $\underline{x}(t)$ is a vector Markov process, the Chapman-Komogorov equation is valid:

$$p(\underline{z},t+\Delta|\underline{y},s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{z},t+\Delta|\underline{x},t)\,p(\underline{x},t|\underline{y},s)\,d\underline{x}$$

$$\underbrace{\int_{-a}^{a}\int_{-a}^{a}\cdots\int_{-a}^{a}}_{n \text{ fold integral}} S(\underline{x})\,\Delta p\, d\underline{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\,\Delta p\, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\left[p(\underline{z}, t+\Delta|\underline{y}, s) - p(\underline{x}, t|\underline{y}, s)\right] d\underline{x}$$

Change the variable of integration from x to z since the ranges is infinite.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\, p(\underline{z}, t+\Delta|\underline{y}, s)\, d\underline{x} - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\, p(\underline{x}, t|\underline{y}, s)\, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{z})\, p(\underline{z}, t+\Delta|\underline{y}, s)\, d\underline{z} - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\, p(\underline{x}, t|\underline{y}, s)\, d\underline{x}$$

Utilizing the Chapman-Komogorov eqn.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{z})\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{z}, t+\Delta|\underline{x}, t)\, p(\underline{x}, t|\underline{y}, s)\, d\underline{x}\right] d\underline{z}$$
$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\, p(\underline{x}, t|\underline{y}, s)\, d\underline{x}$$

Interchanging the order of integration under the safe assumption that the integrals are uniformly convergent

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x}, t|\underline{y}, s)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{z})\, p(\underline{z}, t+\Delta|\underline{x}, t)\, d\underline{z}\right] d\underline{x}$$
$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\, p(\underline{x}, t|\underline{y}, s)\, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x}, t|\underline{y}, s)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{z})\, p(\underline{z}, t+\Delta|\underline{x}, t)\, d\underline{z} - S(\underline{x})\right] d\underline{x}$$

Expanding $S(\underline{z})$ in a Taylor series about $\underline{z} = \underline{x}$ yields:

$$S(\underline{z}) = S(\underline{x}) + \left[\frac{\partial}{\partial \underline{x}}\right]^T S(z)\Big|_{\underline{z}=\underline{x}} [\underline{z} - \underline{x}] + \tfrac{1}{2}[\underline{z}-\underline{x}]^T\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T\left(\frac{\partial}{\partial \underline{x}}\right) S(\underline{z})\Big|_{\underline{z}=\underline{x}}\right][\underline{z}-\underline{x}]$$
$$+\, o\left([\underline{z}-\underline{x}]^T[\underline{z}-\underline{x}]\right)$$

So:

$$S(\underline{z}) = S(\underline{x}) + \left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x}) [\underline{z}-\underline{x}] + \frac{1}{2}[\underline{z}-\underline{x}]^T \left[\left(\frac{\partial}{\partial \underline{x}}\right)^T \left(\frac{\partial}{\partial \underline{x}}\right) S(\underline{x})\right][\underline{z}-\underline{x}]$$

$$+ \; o\left([\underline{z}-\underline{x}]^T[\underline{z}-\underline{x}]\right) \ldots$$

Substituting the Taylor series expansion about $\underline{z}=\underline{x}$ for $S(\underline{z})$ of eqn II into eqn I:

$$\int_{-a}^{a}\int_{-a}^{a}\cdots\int_{-a}^{a} S(\underline{x})\,\Delta p \, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s)\Bigg[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Big\{S(\underline{x}) + \left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x})[\underline{z}-\underline{x}] + \frac{1}{2}[\underline{z}-\underline{x}]\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T\left(\frac{\partial}{\partial \underline{x}}\right)S(\underline{x})\right][\underline{z}-\underline{x}]$$

$$+ o\left([\underline{z}-\underline{x}]^T[\underline{z}-\underline{x}]\right)\Big\} p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z} - S(\underline{x})\Bigg]\,d\underline{x}$$

Since $S(\underline{x})$ is not function of $\underline{z}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s)\Bigg[S(\underline{x})\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}}_{\equiv 1}$$

$$+ \left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{z}-\underline{x}]\, p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{z}-\underline{x}]^T\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T\left(\frac{\partial}{\partial \underline{x}}\right)S(\underline{x})\right][\underline{z}-\underline{x}]\, p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} o\left([\underline{z}-\underline{x}]^T[\underline{z}-\underline{x}]\right) p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}$$

$$- S(\underline{x})\Bigg]\,d\underline{x}$$

$$\int_{-a}^{a}\int_{-a}^{a}\cdots\int_{-a}^{a} S(\underline{x})\,\Delta p\, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t\,|\,\underline{y},s)\Big[\, S(\underline{x}) + \left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{z}-\underline{x}]\, p(\underline{z},t+\Delta\,|\,\underline{x},t)\, d\underline{z}$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{z}-\underline{x}]^T\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T\left(\frac{\partial}{\partial \underline{x}}\right) S(\underline{x})\right][\underline{z}-\underline{x}]\, p(\underline{z},t+\Delta\,|\,\underline{x},t)\, d\underline{z}$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} o\left([\underline{z}-\underline{x}]^T[\underline{z}-\underline{x}]\right) p(\underline{z},t+\Delta\,|\,\underline{x},t)\, d\underline{z}$$

$$- S(\underline{x})\,\Big]\, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t\,|\,\underline{y},s)\Big[\left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{z}-\underline{x}]\, p(\underline{z},t+\Delta\,|\,\underline{x},t)\, d\underline{z}$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{z}-\underline{x}]^T\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T\left(\frac{\partial}{\partial \underline{x}}\right) S(\underline{x})\right][\underline{z}-\underline{x}]\, p(\underline{z},t+\Delta\,|\,\underline{x},t)\, d\underline{z}$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} o\left([\underline{z}-\underline{x}]^T[\underline{z}-\underline{x}]\right) p(\underline{z},t+\Delta\,|,\underline{x},t)\, d\underline{z}\,\Big]\, d\underline{x}$$

The first incremental moment is:

$$\underline{A}(\underline{x},t) \triangleq \lim_{\Delta\to 0}\frac{1}{\Delta}\left\langle [\underline{x}(t+\Delta)-\underline{x}(t)]\,\middle|\,\underline{x}(t)=\underline{x}\right\rangle = \lim_{\Delta\to 0}\frac{1}{\Delta}\left\langle \Delta\underline{x}(t)\,\middle|\,\underline{x}(t)=\underline{x}\right\rangle$$

$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\left\langle [\underline{f}(\underline{x},t)\Delta + \underline{G}(\underline{x},t)\Delta\underline{\xi}]\,\middle|\,\underline{x}(t)=\underline{x}\right\rangle$$

$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\left\langle \underline{f}(\underline{x},t)\Delta\,\middle|\,\underline{x}(t)=\underline{x}\right\rangle + \lim_{\Delta\to 0}\frac{1}{\Delta}\left\langle \underline{G}(\underline{x},t)\Delta\underline{\xi}\,\middle|\,\underline{x}(t)=\underline{x}\right\rangle$$

$$= \lim_{\Delta\to 0} \underline{f}(\underline{x},t)\left\langle \frac{\cancel{\Delta}}{\cancel{\Delta}}\,\middle|\,\underline{x}(t)=\underline{x}\right\rangle + \lim_{\Delta\to 0} \underline{G}(\underline{x},t)\frac{1}{\Delta}\left\langle \Delta\underline{\xi}\,\middle|\,\underline{x}(t)=\underline{x}\right\rangle$$

$$= \underline{f}(\underline{x},t)$$

The second incremental moment is:

$$\underline{B}(\underline{x},t) \triangleq \lim_{\Delta\to 0} \frac{1}{\Delta} \langle \Delta\underline{x}\, \Delta\underline{x}^T | \underline{x}(t) = \underline{x} \rangle$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta} \langle [\underline{f}(\underline{x},t)\Delta + \underline{G}(\underline{x},t)\Delta\underline{\xi}][\underline{f}(\underline{x},t)\Delta + \underline{G}(\underline{x},t)\Delta\underline{\xi}]^T | \underline{x}(t) = \underline{x} \rangle$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta} \langle [\underline{f}(\underline{x},t)\underline{f}^T(\underline{x},t)\Delta^2 + \underline{G}(\underline{x},t)\Delta\underline{\xi}\,\underline{f}^T(\underline{x},t)\Delta$$

$$+ \underline{f}(\underline{x},t)\Delta\underline{\xi}^T\underline{G}^T(\underline{x},t)\Delta + \underline{G}(\underline{x},t)\Delta\underline{\xi}\Delta\underline{\xi}^T\underline{G}^T(\underline{x},t)] | \underline{x}(t) = \underline{x} \rangle$$

$$= \lim_{\Delta\to 0} \underline{f}(\underline{x},t)\underline{f}^T(\underline{x},t) \langle \frac{\Delta^2}{\Delta} | \underline{x}(t) = \underline{x} \rangle$$

$$+ \lim_{\Delta\to 0} \frac{1}{\Delta} \underline{G}(\underline{x},t) \langle \Delta\underline{\xi} \rangle \underline{f}^T(\underline{x},t)\Delta$$

$$+ \lim_{\Delta\to 0} \frac{1}{\Delta} \underline{f}(\underline{x},t) \langle \Delta\underline{\xi}^T \rangle \underline{G}^T(\underline{x},t)\Delta$$

$$+ \lim_{\Delta\to 0} \frac{1}{\Delta} \underline{G}(\underline{x},t) \langle \Delta\underline{\xi}\Delta\underline{\xi}^T \rangle \underline{G}^T(\underline{x},t)$$

$$= \underline{f}(\underline{x},t)\underline{f}^T(\underline{x},t) \cdot \lim_{\Delta\to 0} \Delta$$

$$+ \underline{G}(\underline{x},t)\, \underline{0} \cdot \underline{f}^T(\underline{x},t)$$

$$+ \underline{f}(\underline{x},t) \cdot \underline{0} \cdot \underline{G}^T(\underline{x},t)$$

$$+ \lim_{\Delta\to 0} \frac{1}{\Delta} \underline{G}(\underline{x},t)\, Q(t)\Delta\, \underline{G}^T(\underline{x},t)$$

$$= 0 + 0 + 0 + \underline{G}(\underline{x},t)\, Q(t)\, \underline{G}^T(\underline{x},t)$$

$$= \underline{G}(\underline{x},t)\, Q(t)\, \underline{G}^T(\underline{x},t)$$

$$\int_{-a}^{a}\int_{-a}^{a}\cdots\int_{-a}^{a} S(\underline{x})\,\Delta p\,d\underline{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\,\Delta p\,d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t\,|\,\underline{y},s)\Big\{\Big[\Big(\frac{\partial}{\partial \underline{x}}\Big)^T S(\underline{x})\Big]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[\underline{z}-\underline{x}]\,p(\underline{z},t+\Delta\,|\,\underline{x},t)\,d\underline{z}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[\underline{z}-\underline{x}]^T\Big[\Big(\frac{\partial}{\partial \underline{x}}\Big)^T\Big(\frac{\partial}{\partial \underline{x}}\Big)S(\underline{x})\Big][\underline{z}-\underline{x}]\,p(\underline{z},t+\Delta\,|\,\underline{x},t)\,d\underline{z}\Big\}\,d\underline{x}$$

From the first and second incremental moments:

$$A(\underline{x},t) \triangleq \lim_{\Delta\to 0}\frac{1}{\Delta}\left\langle [X(t+\Delta)-X(t)]\,\middle|\,\underline{X}(t)=\underline{x}\right\rangle = \lim_{\Delta\to 0}\frac{1}{\Delta}\left\langle [\underline{z}(t)-\underline{X}(t)]\,\middle|\,\underline{X}(t)=\underline{x}\right\rangle$$

$$= \underline{f}(\underline{x},t)$$

so $\left\langle [\underline{z}-\underline{X}(t)]\,\middle|\,\underline{X}(t)=\underline{x}\right\rangle = (\Delta)\,\underline{f}(\underline{x},t)$

$$\underline{B}(\underline{x},t) \triangleq \lim_{\Delta\to 0}\frac{1}{\Delta}\left\langle \Delta\underline{X}\,\Delta\underline{X}^T\,\middle|\,\underline{X}(t)=\underline{x}\right\rangle$$

$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\left\langle [\underline{X}(t+\Delta)-\underline{X}(t)][\underline{X}(t+\Delta)-\underline{X}(t)]^T\,\middle|\,\underline{X}(t)=\underline{x}\right\rangle$$

$$= \underbrace{\underline{G}}_{n\times m}(\underline{x},t)\,\underbrace{\underline{Q}}_{m\times m}(t)\,\underbrace{\underline{G}^T}_{m\times n}(\underline{x},t)$$

$$n\times n$$

so $\left\langle \Delta\underline{X}\,\Delta\underline{X}^T\,\middle|\,\underline{X}(t)=\underline{x}\right\rangle = \Delta\,\underline{G}(\underline{x},t)\,\underline{Q}(t)\,G^T(\underline{x},t)$

$$= \underline{G}(\underline{x},t)\left[\underline{Q}(t)\Delta\right]\underline{G}^T(\underline{x},t)$$

or $\left\langle [z(t)-x(t)][z(t)-x(t)]^T\,\middle|\,\underline{X}(t)=\underline{x}\right\rangle = \underline{G}(\underline{x},t)\left[\underline{Q}(t)\Delta\right]\underline{G}^T(\underline{x},t)$

Continuing to evaluate the integrals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x}) \;\Delta p \quad d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s) \left[\left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x})\right]\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{z}-\underline{x}]\, p(\underline{z},t+\Delta|\underline{x},t)\, d\underline{z}\right\}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s)\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{z}-\underline{x}]^T\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T\left(\frac{\partial}{\partial \underline{x}}\right)S(\underline{x})\right][\underline{z}-\underline{x}]\, p(\underline{z},t+\Delta|\underline{x},t)\, d\underline{z}\right\}d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s) \left[\left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x})\right] (\Delta)\, \underline{f}(\underline{x},t)\; d\underline{x}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s)\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} (z_i-x_i)\left[\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}S(\underline{x})\right](z_j-x_j) \cdot p(\underline{z},t+\Delta|\underline{x},t)\, d\underline{z}\right\}d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s) \left[\left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x})\right] (\Delta) \cdot \underline{f}(\underline{x},t)\, d\underline{x}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s) \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}S(\underline{x})\right]\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (z_i-x_i)(z_j-x_j)\, p(\underline{z},t+\Delta|\underline{x},t)\, d\underline{z}\right\}d\underline{x}$$

Therefore

L.H.S.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x}) \;\Delta p \quad d\underline{x}$$

R.H.S. term 1

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s) \left[\left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x})\right] (\Delta)\, \underline{f}(\underline{x},t)\, d\underline{x}$$

R.H.S. term 2

$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s) \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\underline{S}(x)\right]\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (z_i-x_i)(z_j-x_j)\, p(\underline{z},t+\Delta|\underline{x},t)\, d\underline{z} \cdot\right\}d\underline{x}$$

Integrating by parts, the 1st term on the R.H.S. yields:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s)\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x})\right](\Delta)\ \underline{f}(\underline{x},t)\,d\underline{x}$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\left(\frac{\partial}{\partial \underline{x}}\right)^T S(\underline{x})\right](\Delta)\,\underline{f}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\,d\underline{x}$$

$$= [1,1,\ldots,1]\ S(\underline{x})(\Delta)\underline{f}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\Delta\Big|_{\underline{x}^T=(-\infty,-\infty,\ldots,-\infty)}^{\underline{x}^T=(\infty,\infty,\ldots,\infty)}$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(\Delta)\,S(\underline{x})\left(\frac{\partial}{\partial \underline{x}}\right)^T\left\{\underline{f}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right\}d\underline{x}$$

$$= 0-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(\Delta)\,S(\underline{x})\left(\frac{\partial}{\partial \underline{x}}\right)^T\left\{\underline{f}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right\}d\underline{x}$$

Integrating by parts, the 2nd term on the R.H.S. yields:

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(\underline{x},t|\underline{y},s)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}S(\underline{x})\right]\cdot$$

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(z_i-x_i)(z_j-x_j)\,p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}\right\}d\underline{x}$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg\{p(\underline{x},t|\underline{y},s)\left[\frac{\partial}{\partial x_j}S(\underline{x})\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(z_i-x_i)(z_j-x_j)\,p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}\,\Big|_{x=(-\infty,\ldots)}^{x=(\infty,\ldots)}$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial x_j}S(\underline{x})\right]\frac{\partial}{\partial x_i}\left(p(\underline{x},t|\underline{y},s)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(z_i-x_i)(z_j-x_j)\,p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}\right)d\underline{x}\Bigg\}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big\{-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Big[\frac{\partial}{\partial x_j} S(\underline{x})\Big]\cdot$$

$$\frac{\partial}{\partial x_i}\Big(p(\underline{x},t|\underline{y},s)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(z_i-x_i)(z_j-x_j)\,p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}\Big)$$

$$d\underline{x}\Big\}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big\{- S(\underline{x})\frac{\partial}{\partial x_i}\Big(p(\underline{x},t|\underline{y},s)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(z_i-x_i)(z_j-x_j)\,p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}\Big)\Big|_{\underline{x}^T=(-\infty,\ldots}^{\underline{x}^T=(\infty,\infty,\ldots}$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\cdot$$

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\Big(p(\underline{x},t|\underline{y},s)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(z_i-x_i)(z_j-x_j)\,p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}\Big)\,d\underline{x}\Big\}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\cdot$$

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(z_i-x_i)(z_j-x_j)\,p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}\;p(\underline{x},t|\underline{y},s)\Big)\,d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x})\cdot$$

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(z_i-x_i)(z_j-x_j)\,p(\underline{z},t+\Delta|\underline{x},t)\,d\underline{z}\;p(\underline{x},t|\underline{y},s)\Big)\,d\underline{x}$$

So having evaluated the R.H.S. by parts the equation is:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x}) \;\Delta p \, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x}) \cdot$$

$$\Big\{-(\Delta)\left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\right\} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (z_i - x_i)(z_j - x_j)$$
$$\cdot\, p(\underline{z}, t+\Delta|\underline{x},t)\, d\underline{z}\; p(\underline{x},t|\underline{y},s)\Big)\Big\}\, d\underline{x}$$

or

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\underline{x}) \Big\{ \Delta p + (\Delta)\left(\frac{\partial}{\partial \underline{x}}\right)^T\left\{\underline{f}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\right\}$$

$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (z_i - x_i)(z_j - x_j)\, p(\underline{z}, t+\Delta|\underline{x},t)\, d\underline{z}\; p(\underline{x},t|\underline{y},s)\Big)\Big\}\, d\underline{x} = 0$$

for arbitrary $S(\underline{x})$.

Therefore by the fundamental lemma of variational calculus

$$\Delta p + (\Delta)\left(\frac{\partial}{\partial x}\right)^T\left\{\underline{f}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\right\}$$
$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (z_i - x_i)(z_j - x_j)\, p(\underline{z}, t+\Delta|\underline{x},t)\, d\underline{z}\; p(\underline{x},t|\underline{y},s)\Big) = 0$$

$$\forall\, \underline{x}$$

Rearranging and dividing both sides by Δ:

$$\frac{\Delta p}{\Delta} \equiv \frac{p(\underline{x}(t+\Delta), t+\Delta \mid y, s) - p(\underline{x}, t \mid y, s)}{\Delta}$$

$$= -\left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x}, t)\, p(\underline{x}, t \mid y, s) \right\}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left(\frac{1}{\Delta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (z_i - x_i)(z_j - x_j)\, p(\underline{z}, t+\Delta \mid \underline{x}, t)\, d\underline{z}\, p(\underline{x}, t \mid y, s) \right)$$

Taking the limit as $\Delta \to 0$:

$$\frac{\partial p(\underline{x}, t \mid y, s)}{\partial t} \triangleq \lim_{\Delta \to 0} \frac{\Delta p}{\Delta} =$$

$$-\left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x}, t)\, p(\underline{x}, t \mid y, s) \right\}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left(\lim_{\Delta \to 0} \frac{1}{\Delta} \left\langle (z_i - x_i)(z_j - x_j) \right\rangle p(\underline{x}, t \mid y, s) \right)$$

Therefore

$$\frac{\partial p(\underline{x}, t \mid y, s)}{\partial t} = -\left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x}, t)\, p(\underline{x}, t \mid y, s) \right\}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left(B_{ij}(\underline{x}, t)\, p(\underline{x}, t \mid y, s) \right)$$

where $\underline{B}(x, t) = \{ B_{ij}(\underline{x}, t) \}$ is the second incremental moment or, more compactly:

$$\frac{\partial p(\underline{x},t|\underline{y},s)}{\partial t} = -\left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x},t)\, p(\underline{x},t|\underline{y},s) \right\}$$

$$+ \frac{1}{2}\, \mathrm{trace}\left\{ \left(\frac{\partial}{\partial \underline{x}}\right)\left(\frac{\partial}{\partial \underline{x}}\right)^T \left[G(\underline{x},t)\, Q(t)\, G^T(\underline{x},t)\, p(\underline{x},t|\underline{y},s) \right] \right\}$$

b.) Find the differential equations for the mean $\hat{x}(t) = \langle x(t) \rangle$ and the covariance $\Gamma(t) = \langle [x - \hat{x}(t)][x - \hat{x}(t)]^T \rangle$ of the process.

$$\frac{\partial p(\underline{x}, t | \underline{y}, s)}{\partial t} = -\left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x}, t)\, p(\underline{x}, t | \underline{y}, s) \right\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[B_{ij}(\underline{x}, t)\, p(\underline{x}, t | \underline{y}, s) \right]$$

Multiplying both sides by $\underline{x}(t)$:

$$\underline{x}(t) \frac{\partial p(\underline{x}, t | \underline{y}, s)}{\partial t} = -\underline{x}(t) \left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x}, t)\, p(\underline{x}, t | \underline{y}, s) \right\} + \underline{x}(t) \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[B_{ij}(\underline{x}, t)\, p(\underline{x}, t | \underline{y}, s) \right]$$

$$\dot{\hat{x}}(t) \triangleq \frac{d}{dt} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underline{x}(t)\, p(\underline{x}, t | \underline{y}, s)\, d\underline{x}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underline{x}(t) \frac{\partial p(\underline{x}, t | \underline{y}, s)}{\partial t}\, d\underline{x}$$

$$= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underline{x}(t) \left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x}, t)\, p(\underline{x}, t | \underline{y}, s) \right\} d\underline{x}$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underline{x}(t) \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[B_{ij}(\underline{x}, t)\, p(\underline{x}, t | \underline{y}, s) \right] d\underline{x}$$

$$\dot{\hat{\underline{X}}}(t) = -\underline{X}(t)[1,1,\dots,1]\,\underline{f}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\Big|_{x=(-\infty,-\infty,\dots,-\infty)}^{x=(\infty,\infty,\dots,\infty)}$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \underline{f}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\,d\underline{x}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} \underline{X}(t)\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\left[B_{ij}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]}_{\text{scalar}}d\underline{x}$$

So:

$$\dot{\hat{\underline{X}}}(t) = \langle \underline{f}(\underline{x},t)\rangle + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\underline{X}(t)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\left[B_{ij}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]d\underline{x}$$

Integrating the second term on the R.H.S. by parts:

$$\frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}}_{n\text{-fold integrals}}\underline{X}(t)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\left[B_{ij}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]d\underline{x}$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\begin{bmatrix} x_1\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\left[B_{ij}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]\\ x_2\sum_{i=1}^{n}\sum_{j=1}^{n}\ \text{etc}\\ \vdots\\ x_n\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\ \text{etc.}\end{bmatrix}d\underline{x}$$

$$=\frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}}_{(n-1)\text{ fold integrals}}\begin{bmatrix} x_1\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[B_{1j}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]\\ x_2\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[B_{2j}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]\\ \vdots\\ x_n\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[B_{nj}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]\end{bmatrix}\Bigg|_{\underline{x}^T=(-\infty,-\infty,\dots,-\infty)}^{\underline{x}^T=(\infty,\infty,\dots,\infty)} d\underline{x}_{(n-1)}$$

$$-\frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}}_{n\text{-fold}}\begin{bmatrix}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[B_{1j}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]\\ \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[B_{2j}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]\\ \vdots\\ \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[B_{nj}(\underline{x},t)\,p(\underline{x},t|\underline{y},s)\right]\end{bmatrix}d\underline{x}$$

since $\frac{\partial}{\partial x_j}[B_{ij}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)] = \frac{\partial B_{ij}(\underline{x},t)}{\partial x_j} p(\underline{x},t|\underline{y},s) + B_{ij} \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_j}$

and $\lim_{|x_i|\to\infty} p(\underline{x},t|\underline{y},s) = 0$ and $\lim_{|x_i|\to\infty} \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_j} = 0$,

the first integrated by parts term is zero, $\underline{0}$.

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \underline{X}(t) \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[B_{ij}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$= -\frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n\text{-fold integral}} \begin{bmatrix} \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[B_{1j}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)] \\ \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[B_{2j}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)] \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[B_{nj}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)] \end{bmatrix} d\underline{x}$$

$$= -\frac{1}{2}\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{(n-1)\text{ fold integral}} \begin{bmatrix} \sum_{j=1}^{n} B_{1j}(\underline{x},t)\, p(\underline{x},t|\underline{y},s) \\ \vdots \\ \sum_{j=1}^{n} B_{nj}(\underline{x},t)\, p(\underline{x},t|\underline{y},s) \end{bmatrix} \Bigg|_{\underline{x}^T=(-\infty,-\infty,\ldots-\infty)}^{\underline{x}^T=(\infty,\infty,\ldots\infty)} d\underline{x}_{(n-1)}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \begin{bmatrix} 0 \cdot \sum_{j=1}^{n} B_{1j}\, p \\ 0 \cdot \sum_{j=1}^{n} B_{2j}\, p \\ \vdots \\ 0 \cdot \sum_{j=1}^{n} B_{nj}\, p \end{bmatrix} d\underline{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So the second term on the R.H.S contributes nothing.

The differential equation for the mean is

$$\dot{\hat{x}}(t) = \langle \underline{f}(\underline{x}, t) \rangle$$

b.) Obtain the differential equation for the covariance:

$$\frac{\partial p(\underline{x},t|\underline{y},s)}{\partial t} = -\left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\right\} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[B_{ij}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\right]$$

Multiplying both sides by $[\underline{x}-\underline{\hat{x}}(t)][\underline{x}-\underline{\hat{x}}(t)]^T$:

$$[\underline{x}-\underline{\hat{x}}(t)][\underline{x}-\underline{\hat{x}}(t)]^T \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial t}$$

$$= -[\underline{x}-\underline{\hat{x}}(t)][\underline{x}-\underline{\hat{x}}(t)]^T \left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\right\}$$

$$+ \frac{1}{2}[\underline{x}-\underline{\hat{x}}(t)][\underline{x}-\underline{\hat{x}}(t)]^T \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[B_{ij}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\right]$$

And integrating both sides:

$$\dot{\underline{\Gamma}}(t) \triangleq \frac{d}{dt}\left\langle [\underline{x}-\underline{\hat{x}}(t)][\underline{x}-\underline{\hat{x}}(t)]^T \right\rangle = \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[\underline{x}-\underline{\hat{x}}(t)][\underline{x}-\underline{\hat{x}}(t)]^T p(\underline{x},t|\underline{y},s)\,d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{x}-\underline{\hat{x}}(t)][\underline{x}-\underline{\hat{x}}(t)]^T \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial t}\,d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{x}-\underline{\hat{x}}(t)][\underline{x}-\underline{\hat{x}}(t)]^T \left(\frac{\partial}{\partial \underline{x}}\right)^T \left\{ \underline{f}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\right\} d\underline{x}$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{x}-\underline{\hat{x}}(t)][\underline{x}-\underline{\hat{x}}(t)]^T \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[B_{ij}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\right] d\underline{x}$$

Integrating the first term on the R.H.S. by parts:

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{x}-\hat{\underline{x}}(t)][\underline{x}-\hat{\underline{x}}(t)]^T\left(\frac{\partial}{\partial \underline{x}}\right)^T\{\underline{f}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\}\, d\underline{x}$$

$$=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \begin{bmatrix} (x_1-\hat{x}_1)^2 & (x_1-\hat{x}_1)(x_2-\hat{x}_2)\cdots(x_1-\hat{x}_1)(x_n-\hat{x}_n) \\ (x_2-\hat{x}_2)(x_1-\hat{x}_1) & (x_2-\hat{x}_2)^2 \cdots (x_2-\hat{x}_2)(x_n-\hat{x}_n) \\ \vdots & \vdots \\ (x_n-\hat{x}_n)(x_1-\hat{x}_1) & (x_n-\hat{x}_n)(x_2-\hat{x}_2) \cdots (x_n-\hat{x}_n)^2 \end{bmatrix} \underbrace{\frac{\partial}{\partial \underline{x}}\{\underline{f}(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\}}\cdot d\underline{x}$$

scalar

$$=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \begin{bmatrix} (x_1-\hat{x}_1)^2\left(\frac{\partial f_1 p}{\partial x_1}+\frac{\partial f_2 p}{\partial x_2}+\frac{\partial f_3 p}{\partial x_3}+\cdots\right) & (x_1-\hat{x}_1)(x_2-\hat{x}_2)\left(\frac{\partial f_1 p}{\partial x_1}+\text{etc.}\right) \\ (x_2-\hat{x}_2)(x_1-\hat{x}_1)\left(\frac{\partial f_1 p}{\partial x_1}+\frac{\partial f_2 p}{\partial x_2}+\frac{\partial f_3 p}{\partial x_3}+\cdots\right) & (x_2-\hat{x}_2)^2\left(\frac{\partial f_1 p}{\partial x_1}+\text{etc.}\right) \\ \vdots & \\ (x_n-\hat{x}_n)(x_1-\hat{x}_1)\left(\frac{\partial f_1 p}{\partial x_1}+\frac{\partial f_2 p}{\partial x_2}+\frac{\partial f_3 p}{\partial x_3}+\cdots\right) & \cdots \ \text{etc.} \end{bmatrix}\cdot d\underline{x}$$

(and the integrated parts term is zero since $\lim_{|x_i|\to\infty} p(\underline{x},t|\underline{y},s)=0$)

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \begin{bmatrix} 2(x_1-\hat{x}_1)f_1 p & (x_1-\hat{x}_1)f_2 p+(x_2-\hat{x}_2)f_1 p & \cdots \\ (x_1-\hat{x}_1)f_2 p+(x_2-\hat{x}_2)f_1 p & 2(x_2-\hat{x}_2)^2 f_2 p & \vdots \\ \vdots & & \\ (x_n-\hat{x}_n)f_1 p+(x_1-\hat{x}_1)f_n p & \cdots & 2(x_n-\hat{x}_n)f_n p \end{bmatrix} d\underline{x}$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \begin{bmatrix} 2(x_1-\hat{x}_1)f_1 & (x_1-\hat{x}_1)f_2+(x_2-\hat{x}_2)f_1 & \cdots \\ (x_1-\hat{x}_1)f_2+(x_2-\hat{x}_2)f_1 & 2(x_2-\hat{x}_2)f_2 & \\ \vdots & & \\ (x_n-\hat{x}_n)f_1+(x_1-\hat{x}_1)f_n & \cdots & \cdots \end{bmatrix} p(\underline{x},t|\underline{y},s)\,d\underline{x}$$

Integrating the second term on the R.H.S.:

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\underline{x}-\hat{\underline{x}}(t)][\underline{x}-\hat{\underline{x}}(t)]^T \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}[B_{ij}(\underline{x},t)p(\underline{x},t|\underline{y},s)]}_{\text{scalar}} d\underline{x}$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\begin{bmatrix}(x_1-\hat{x}_1)^2\sum_{i=1}^{n}\sum_{j=1}^{n}\text{etc.} & (x_1-\hat{x}_1)(x_2-\hat{x}_2)\sum_{i=1}^{n}\sum_{j=1}^{n}\text{etc.} & \cdots \\ (x_2-\hat{x}_2)(x_1-\hat{x}_1)\sum_{i=1}^{n}\sum_{j=1}^{n}\text{etc.} & \cdots & \cdots \\ \vdots & & \\ (x_n-\hat{x}_n)(x_1-\hat{x}_1)\sum_{i=1}^{n}\sum_{j=1}^{n}\text{etc.} & \cdots & \cdots\end{bmatrix} d\underline{x}$$

Integrating by parts twice, both integrated by parts terms vanish since $\lim_{|x_i|\to\infty} p(\underline{x},t|\underline{y},s)=0$, $\lim_{|x_i|\to\infty}\frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i}=0$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\begin{bmatrix}2B_{11} & B_{12}+B_{21} & \cdots & B_{1n}+B_{n1} \\ B_{12}+B_{21} & 2B_{22} & \cdots & B_{2n}+B_{n2} \\ \vdots & & & \vdots \\ B_{1n}+B_{n1} & B_{n2}+B_{2n} & \cdots & 2B_{nn}\end{bmatrix} p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

but $B_{ij}=B_{ji}$ since $\underline{B}(\underline{x},t)=\underline{G}(\underline{x},t)\underline{Q}(t)\underline{G}^T(\underline{x},t)$ and is therefore symmetric.

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\begin{bmatrix}B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & & & \vdots \\ B_{n1} & & \cdots & B_{nn}\end{bmatrix} p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \underline{G}(\underline{x},t)\,\underline{Q}(t)\,\underline{G}^T(\underline{x},t)\, p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$=\langle \underline{G}(\underline{x},t)\,\underline{Q}(t)\,\underline{G}^T(\underline{x},t)\rangle$$

Therefore

$$\dot{\underline{\Gamma}}(t) = \left\langle \begin{bmatrix} 2(x_1-\hat{x}_1)f_1 & (x_1-\hat{x}_1)f_2+(x_2-\hat{x}_2)f_1 & \cdots & (x_1-\hat{x}_1)f_n+(x_n-\hat{x}_n)f_1 \\ (x_1-\hat{x}_1)f_2+(x_2-\hat{x}_2)f_1 & 2(x_2-\hat{x}_2)f_2 & \cdots & \\ \vdots & & & \\ (x_n-\hat{x}_n)f_1+(x_1-\hat{x}_1)f_n & \cdots & & 2(x_n-\hat{x}_n)f_n \end{bmatrix} \right\rangle$$

$$+ \langle \underline{G}(\underline{x},t)\, \underline{Q}(t)\, \underline{G}^T(\underline{x},t) \rangle$$

or for each component

$$\{\dot{\Gamma}_{ij}\} = \{\langle (x_i-\hat{x}_i(t))f_j(\underline{x},t)+(x_j-\hat{x}_j(t))f_i(\underline{x},t)\rangle + \langle B_{ij}(\underline{x},t)\rangle\}$$

or more compactly,

$$\dot{\Gamma}(t) = \langle [x(t)-\hat{x}(t)]f^T(x(t),t)\rangle + \langle f(x(t),t)[x(t)-\hat{x}(t)]^T\rangle +$$

$$+ \langle G(x(t),t)\,Q(t)\,G^T(x(t),t)\rangle$$

$$\Gamma(t_0) = \Gamma_0 = \langle (x(t_0)-\hat{x}_0)(x(t_0)-\hat{x}_0)^T\rangle$$

c.) Find the diff. equations for $\underline{\hat{x}}(t)$ and $\underline{\Gamma}(t)$ from part (b.) when the system equations are linear with additive noise.

$$d\underline{x}(t) = \underline{A}(t)\,\underline{x}(t)\,dt + \underline{G}(t)\,d\underline{\xi}(t)$$

$$\underline{x}(t_0) = \underline{x}_0 \quad , \quad p[\underline{x}_0, t_0] = N[\underline{\hat{x}}_0, \underline{\Gamma}_0]$$

$$\langle d\underline{\xi}(t)\rangle = \underline{0} \quad , \quad \langle d\underline{\xi}(t)\, d\underline{\xi}^T(t)\rangle = \underline{Q}(t)\,dt$$

From part (b): $\dot{\underline{\hat{x}}}(t) = \langle \underline{f}(\underline{x},t)\rangle$

So for this linear case it reduces to:

$$\dot{\underline{\hat{x}}} = \langle \underline{A}(t)\,\underline{x}(t)\rangle = \underline{A}(t)\langle \underline{x}(t)\rangle = \underline{A}(t)\,\underline{\hat{x}}(t)$$

or $\quad \dot{\underline{\hat{x}}} = \underline{A}(t)\,\underline{\hat{x}}(t) \qquad$ with $\underline{\hat{x}}(t_0) = \langle \underline{x}(t_0)\rangle = \hat{x}_0$

From part (b):

$$\dot{\underline{\Gamma}}(t) = \left\langle \begin{bmatrix} 2(x_1-\hat{x}_1)f_1 & (x_1-\hat{x}_1)f_2+(x_2-\hat{x}_2)f_1 & \cdots & (x_1-\hat{x}_1)f_n+(x_n-\hat{x}_n)f_1 \\ (x_2-\hat{x}_2)f_1+(x_1-\hat{x}_1)f_2 & 2(x_2-\hat{x}_2)f_2 & & \vdots \\ \vdots & & & \\ (x_n-\hat{x}_n)f_1+(x_1-\hat{x}_1)f_n & \cdots & & 2(x_n-\hat{x}_n)f_n \end{bmatrix} \right\rangle$$

$$+ \langle \underline{G}(\underline{x},t)\,\underline{Q}(t)\,\underline{G}^T(\underline{x},t)\rangle$$

So for this linear case it reduces to:

$$\dot{\underline{\Gamma}}(t) = \left\langle \begin{bmatrix} 2(x_1-\hat{x}_1)f_1 & (x_1-\hat{x}_1)f_2+(x_2-\hat{x}_2)f_1 & \cdots \\ \vdots & & \\ (x_n-\hat{x}_n)f_1+(x_1-\hat{x}_1)f_n & \cdots & \text{etc} \end{bmatrix} \right\rangle$$

$$+ \langle \underline{G}(t)\,\underline{Q}(t)\,\underline{G}^T(t)\rangle \qquad \text{where } f_i = \sum_{j=1}^{n} a_{ij} x_j$$

Since $\langle \underline{G}(t)\,\underline{Q}(t)\,\underline{G}^T(t) \rangle = \underline{G}(t)\,\underline{Q}(t)\,\underline{G}^T(t) \langle I \rangle = \underline{G}(t)\underline{Q}(t)\,\underline{G}^T(t)$

$$\dot{\underline{\Gamma}}(t) = \left\langle \begin{bmatrix} 2(x_1-\hat{x}_1)f_1 & (x_1-\hat{x}_1)f_2+(x_2-\hat{x}_2)f_1 & \cdots \\ (x_1-\hat{x}_1)f_2+(x_2-\hat{x}_2)f_1 & 2(x_2-\hat{x}_2)f_2 & \cdots \\ \vdots & & \vdots \\ (x_n-\hat{x}_n)f_1+(x_1-\hat{x}_1)f_n & \cdots & 2(x_n-\hat{x}_n)f_n \end{bmatrix} \right\rangle + \underline{G}(t)\,Q(t)\,\underline{G}^T$$

where $f_i(\underline{x},t) = \sum_{j=1}^{n} a_{ij}(t)x_j$

or for each component

$$\{\dot{\Gamma}_{ij}(t)\} = \{\langle (x_i-\hat{x}_i(t))f_j(\underline{x},t) + (x_j-x_j(t))f_i(\underline{x},t)\rangle + B_{ij}(t)\}$$

$$f_i = \sum_{j=1}^{n} a_{ij}(t)\,x_j$$

$$\{\dot{\Gamma}_{ij}(t)\} = \{\langle [x_i-\hat{x}_i(t)]\sum_{l=1}^{n} a_{jl}x_l + [x_j-x_j(t)]\sum_{k=1}^{n} a_{ik}x_k\rangle + B_{ij}(t)\}$$

$$= \{\langle \sum_{l=1}^{n} a_{jl}(t)\; x_l(x_i-\hat{x}_i(t)) + \sum_{k=1}^{n} a_{ik}(t)\, x_k(x_j-\hat{x}_j(t))\rangle + B_{ij}(t)\}$$

$$= \{\langle \sum_{l=1}^{n} a_{jl}(t)\;(x_l-\hat{x}_l)(x_i-\hat{x}_i) + \sum_{l=1}^{n} a_{jl}(t)\;\hat{x}_l(x_i-\hat{x}_i) + \sum_{k=1}^{n} a_{ik}(x_k-\hat{x}_k)(x_j-\hat{x}_j) + \sum_{k=1}^{n} a_{ik}\,\hat{x}_k\,(x_j-\hat{x}_j)\rangle + B_{ij}(t)\}$$

add and subtract the same terms

$$= \{ \sum_{l=1}^{n} a_{jl}(t)\langle (x_l-\hat{x}_l)(x_i-\hat{x}_i)\rangle + \sum_{l=1}^{n} a_{jl}(t)\hat{x}_l\langle x_i-\hat{x}_i\rangle + \sum_{k=1}^{n} a_{ik}(t)\langle (x_k-\hat{x}_k)(x_j-\hat{x}_j)\rangle + \sum_{k=1}^{n} a_{ik}(t)\,\hat{x}_k\langle x_j-\hat{x}_j\rangle + B_{ij}(t)\}$$

$$\{\dot{\Gamma}_{ij}(t)\} = \Big\{ \sum_{l=1}^{n} a_{jl}(t) \langle (x_l-\hat{x}_l)(x_i-\hat{x}_i)\rangle$$
$$+ \sum_{k=1}^{n} a_{ik}(t) \langle (x_k-\hat{x}_k)(x_j-\hat{x}_j)\rangle + B_{ij}(t)\Big\}$$

$$= \Big\{ \sum_{l=1}^{n} a_{jl}(t)\, \Gamma_{li} + \sum_{k=1}^{n} a_{ik}(t)\, \Gamma_{kj} + B_{ij}(t)\Big\}$$

So the differential equation for the covariance for the linear system equation componentwise is:

$$\{\dot{\Gamma}_{ij}(t)\} = \Big\{ \sum_{l=1}^{n} a_{jl}(t)\, \Gamma_{li}(t) + \sum_{k=1}^{n} a_{ik}(t)\, \Gamma_{kj}(t) + B_{ij}(t)\Big\}$$

or

$$\dot{\Gamma}(t) = A(t)\,\Gamma(t) + \Gamma(t)\,A^T(t) + \{B_{ij}\}$$

but $\{B_{ij}\} = \underline{G}(t)\,\underline{Q}(t)\,\underline{G}^T(t)$

so $$\dot{\Gamma}(t) = A(t)\,\Gamma(t) + \Gamma(t)\,A^T(t) + \underline{G}(t)\,\underline{Q}(t)\,\underline{G}^T(t)$$

with $\underline{\Gamma}(t_0) = \langle \underline{X}(t_0)\,\underline{X}^T(t_0)\rangle = \underline{\Gamma}_0$

for the linear case.

$$\left\{\begin{array}{l}\dot{\hat{\underline{X}}} = A(t)\,\hat{\underline{X}} \\ \hat{\underline{X}}(t_0) = \hat{\underline{X}}_0\end{array}\right\}$$ has the solution $\hat{\underline{X}} = \Phi_A(t,t_0)\,\hat{\underline{X}}_0$

where $\dot{\Phi}_A(t,t_0) = A(t)\,\Phi_A(t,t_0)$, $\Phi_A(t_0,t_0) = I$ is the transition matrix associated with $\underline{A}(t)$

$$\left\{\begin{array}{l}\dot{\underline{\Gamma}}(t) = \underline{A}(t)\underline{\Gamma}(t) + \underline{\Gamma}(t)\underline{A}^T(t) + \underline{G}(t)\underline{Q}(t)\underline{G}^T(t) \\ \underline{\Gamma}(t_0) = \langle \underline{X}(t_0)\,\underline{X}^T(t_0)\rangle = \underline{\Gamma}_0\end{array}\right\}$$

has the solution

$$\underline{\Gamma}(t) = \underline{\Phi}_A(t,t_0)\,\underline{\Gamma}_0\,\underline{\Phi}_A^T(t,t_0) + \int_{t_0}^{t} \underline{\Phi}_A(t,\tau)\,\underline{G}(\tau)\underline{Q}(\tau)\underline{G}^T(\tau)\underline{\Phi}^T(t,\tau)\,d\tau$$

which can be checked by differentiating and substituting back into the differential equation and verifying that L.H.S. = R.H.S.

$$\dot{\underline{\Gamma}}(t) = \dot{\underline{\Phi}}_A(t,t_0)\,\underline{\Gamma}_0\,\underline{\Phi}_A^T(t,t_0) + \underline{\Phi}_A(t,t_0)\,\underline{\Gamma}_0\,\dot{\underline{\Phi}}_A^T(t,t_0)$$

$$+ \underline{\Phi}_A(t,t)\,\underline{G}(t)\,\underline{Q}(t)\,\underline{G}^T(t)\,\underline{\Phi}_A^T(t,t)$$

$$+ \int_{t_0}^{t} \dot{\underline{\Phi}}_A(t,t_0)\,\underline{G}(\tau)\underline{Q}(\tau)\underline{G}^T(\tau)\underline{\Phi}^T(t,\tau)\,d\tau$$

$$+ \int_{t_0}^{t} \underline{\Phi}_A(t,t_0)\,\underline{G}(\tau)\,\underline{Q}(\tau)\,\underline{G}^T(\tau)\,\dot{\underline{\Phi}}^T(t,\tau)\,d\tau$$

$$= \underline{A}(t)\,\underline{\Phi}_A(t,t_0)\underline{\Gamma}_0\underline{\Phi}_A^T(t,t_0) + \underline{\Phi}_A(t,t_0)\underline{\Gamma}_0\,\underline{\Phi}_A^T(t,t_0)\,A^T(t)$$

$$+ \underline{G}(t)\,\underline{Q}(t)\,\underline{G}^T(t)$$

$$+ \underline{A}(t)\int_{t_0}^{t} \underline{\Phi}_A(t,t_0)\,\underline{G}(\tau)\,\underline{Q}(\tau)\,\underline{G}^T(\tau)\,\underline{\Phi}^T(t,\tau)\,d\tau$$

$$+ \int_{t_0}^{t} \underline{\Phi}_A(t,t_0)\,\underline{G}(\tau)\,\underline{Q}(\tau)\,\underline{G}^T(\tau)\,\underline{\Phi}^T(t,\tau)\,d\tau\,\underline{A}^T(t)$$

$$= A(t)\,\Gamma(t) + \Gamma(t)\,A^T(t) + G(t)\,Q(t)\,G^T(t)$$

And the initial condition checks

$$\underline{\Gamma}(t_0) = \underline{\Phi}_A(t_0,t_0)\,\underline{\Gamma}_0\,\underline{\Phi}_A^T(t_0,t_0) + \int_{t_0}^{t_0} \underline{\Phi}_A(t_0,\tau)G(\tau)Q(\tau)G^T(\tau)\underline{\Phi}^T(t_0,\tau)\,d\tau$$

$$= \Gamma_0$$

In summary, the equation for $\underline{\hat{x}}(t)$

is $\underline{\hat{x}} = \underline{\Phi}_A(t,t_0)\,\underline{\hat{x}}_0$

and the equation for the covariance is

$$\underline{\Gamma}(t) = \underline{\Phi}_A(t,t_0)\,\underline{\Gamma}_0\,\underline{\Phi}_A^T(t,t_0) + \int_{t_0}^{t} \underline{\Phi}_A(t,\tau)\underline{G}(\tau)\underline{Q}(\tau)\underline{G}^T(\tau)\underline{\Phi}^T(t,\tau)\,d\tau$$

where $\underline{\Phi}_A(t,t_0)$ is the transition matrix associated with $\underline{A}(t)$.

d.) Find the differential equation for the characteristic function $C(t)$ when $d\underline{x}(t) \triangleq \underline{A}(t)\underline{x}(t)\,dt + \underline{G}(t)\,d\underline{\xi}(t)$

$$\underline{x}(t_0) = \underline{x}_0\ ,\ p[\underline{x}_0, t_0] = N[\hat{\underline{x}}_0, \underline{\Gamma}_0]$$

$$\langle d\underline{\xi}(t)\rangle = \underline{0}\ ,\ \langle d\underline{\xi}(t)\, d^T\underline{\xi}(t)\rangle = \underline{Q}(t)dt$$

$$C(t) \triangleq E[e^{j\underline{v}^T\underline{x}}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\ p(\underline{x},t|\underline{y},s)\,d\underline{x}$$

where $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

The Fokker-Planck eqn. for the present case of a linear system is

$$\frac{\partial p(\underline{x},t|\underline{y},s)}{\partial t} = -\left(\frac{\partial}{\partial \underline{x}}\right)^T\left[\ A(t)\underline{x}\ p(\underline{x},t|\underline{y},s)\right]$$

$$+\ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}(t)\ \frac{\partial^2 p(\underline{x},t|\underline{y},s)}{\partial x_i\,\partial x_j}$$

Note that

$$\dot{C}(t) = \frac{d}{dt}C(t) = \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\ p(\underline{x},t|\underline{y},s)\,d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\ \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial t}\,dx$$

Now using the Fokker-Planck eqn. to evaluate the integral!

$$\dot{C}(t) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \left(\frac{\partial}{\partial \underline{x}}\right)^T \left[\underline{A}(t)\,\underline{x}\,p(\underline{x},t|\underline{y},s)\right] d\underline{x}$$
$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}(t)\frac{\partial^2 p(\underline{x},t|\underline{y},s)}{\partial x_i \partial x_j}\right) d\underline{x}$$

where $e^{j\underline{v}^T\underline{x}} = e^{j(v_1x_1+v_2x_2+\ldots+v_nx_n)} = e^{j\sum_{i=1}^{n} v_ix_i} = e^{jv_1x_1}\cdot e^{jv_2x_2}\cdots e^{jv_nx_n}$
$= \prod_{i=1}^{n} e^{jv_ix_i}$

Integrating the first term on the R.H.S.:

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \left(\frac{\partial}{\partial \underline{x}}\right)^T \left[\underline{A}(t)\,\underline{x}\,p(\underline{x},t|\underline{y},s)\right] d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right] \begin{bmatrix} \{a_{11}(t)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n\}p(\underline{x},t|\underline{y},s) \\ \{a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n\}p(\underline{x},t|\underline{y},s) \\ \vdots \\ \{a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n\}p(\underline{x},t|\underline{y},s) \end{bmatrix} d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right] \begin{bmatrix} \left(\sum_{j=1}^{n} a_{1j}x_j\right) p(\underline{x},t|\underline{y},s) \\ \left(\sum_{j=1}^{n} a_{2j}x_j\right) p(\underline{x},t|\underline{y},s) \\ \vdots \\ \left(\sum_{j=1}^{n} a_{nj}x_j\right) p(\underline{x},t|\underline{y},s) \end{bmatrix} d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n} (a_{ij}(t)x_j)\,p(\underline{x},t|\underline{y},s)\right] d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \sum_{i=1}^{n}\sum_{j=1}^{n}\left[a_{ij}(t)\frac{\partial x_j}{\partial x_i}\,p(\underline{x},t|\underline{y},s) + a_{ij}(t)x_j\frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i}\right] \cdot d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)\,p(\underline{x},t|\underline{y},s)\,\delta_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)x_j\frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i}\, d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \left\{ \sum_{i=1}^{n} a_{ii}(t)\, p(\underline{x},t|\underline{y},s) + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)\, x_j \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i} \right\} d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \left\{ \sum_{i=1}^{n} a_{ii}(t)\, p(\underline{x},t|\underline{y},s) + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)\, x_j \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i} \right\} d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \left(\sum_{i=1}^{n} a_{ii}(t) \right) p(\underline{x},t|\underline{y},s)\, d\underline{x}$$
$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)\, x_j \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i}\, d\underline{x}$$

$$= -\left(\sum_{i=1}^{n} a_{ii}(t) \right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\, p(\underline{x},t|\underline{y},s)\, d\underline{x}$$
$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)\, x_j \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i}\, d\underline{x}$$

$$= -\left[\sum_{i=1}^{n} a_{ii}(t) \right] C(t)$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{l=1}^{n} e^{j v_l x_l} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)\, x_j \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i}\, d\underline{x}$$

$$= -\left[\sum_{i=1}^{n} a_{ii}(t) \right] C(t)$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\prod_{l=1}^{n} e^{j v_l x_l} \right) a_{ij}(t)\, x_j \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i}\, d\underline{x}$$

R.H.S. ~~eq~~ (I-b)

Evaluating R.H.S. eqn (I-b) by integrating by parts:

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\prod_{l=1}^{n} e^{jv_l x_l}\right) a_{ij}(t)\, x_j \frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_i}\right] d\underline{x}$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_i}\left[\prod_{l=1}^{n} e^{jv_l x_l} a_{ij}(t)\, x_j\right] p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

integrated by parts term is zero

$$= +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} jv_l\left(\prod_{l=1}^{n} e^{jv_l x_l}\right)\delta_{li}\, a_{ij}(t)\, x_j\right] p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\prod_{l=1}^{n} e^{jv_l x_l}\right) a_{ij}(t)\ \delta_{ji}\right] p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} jv_i\left[\prod_{l=1}^{n} e^{jv_l x_l}\right] a_{ij}(t)\, x_j\right] p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\left[\sum_{i=1}^{n} a_{ii}(t)\right] p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\sum_{i=1}^{n} jv_i\left(e^{j\underline{v}^T\underline{x}}\right)\sum_{j=1}^{n} a_{ij}(t)\, x_j\right] p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$+\left[\sum_{i=1}^{n} a_{ii}(t)\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\, p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\, j\,[v_1, v_2, \ldots, v_n]\begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t)\\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t)\\ \vdots & & \ddots & \\ a_{n1}(t) & \cdot & \cdot & a_{nn}(t)\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} p(\underline{x},t|\underline{y},s)\, d\underline{x}$$

$$+\left[\sum_{i=1}^{n} a_{ii}(t)\right] C(t) = j\underline{v}^T\langle e^{j\underline{v}^T x}\underline{A}(t)\underline{X}(t)\rangle + \left[\sum_{i=1}^{n} a_{ii}(t)\right] C(t)$$

So the first term on the R.H.S. when evaluated is:

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\left(\frac{\partial}{\partial \underline{x}}\right)^T\left[\underline{A}(t)\underline{x}\,p(\underline{x},t|\underline{y},s)\right]d\underline{x}$$

$$= -\left[\sum_{i=1}^{n} a_{ii}(t)\right]C(t) + \left[\sum_{i=1}^{n} a_{ii}(t)\right]C(t) + j\underline{v}^T\langle e^{j\underline{v}^T\underline{x}}\underline{A}(t)\underline{x}(t)\rangle$$

$$= j\,\underline{v}^T\underline{A}(t)\langle e^{j\underline{v}^T\underline{x}}\,\underline{x}\rangle = j\,\langle e^{j\underline{v}^T\underline{x}}\,\underline{v}^T\underline{A}(t)\underline{x}\rangle$$

For scalar, it is equal to its transpose

$$= j\,\langle e^{j\underline{v}^T\underline{x}}\,\underline{x}^T\underline{A}^T(t)\,\underline{v}\rangle = j\,\langle e^{j\underline{v}^T\underline{x}}\,\underline{x}^T\rangle\,\underline{A}^T(t)\,\underline{v}$$

$$= \left[\left(\frac{\partial}{\partial \underline{v}}\right)^T\langle e^{j\underline{v}^T\underline{x}}\rangle\right]\underline{A}^T(t)\,\underline{v} = \left[\left(\frac{\partial}{\partial \underline{v}}\right)^T C(t)\right]\underline{A}^T(t)\,\underline{v}$$

Now evaluating the second term on the R.H.S.:

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}(t)\frac{\partial^2 p(\underline{x},t|\underline{y},s)}{\partial x_i\,\partial x_j}\right)d\underline{x}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}(t)\left(\prod_{l=1}^{n} e^{jv_l x_l}\right)\frac{\partial^2 p(\underline{x},t|\underline{y},s)}{\partial x_i\,\partial x_j}\right|d\underline{x}$$

Integrated by parts, integrated parts terms vanish.

$$= -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}(t)\,j v_l\,\delta_{li}\left(\prod_{l=1}^{n} e^{jv_l x_l}\right)\frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_j}\,d\underline{x}$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} j\sum_{i=1}^{n} v_i\sum_{j=1}^{n} B_{ij}(t)\left(\prod_{l=1}^{n} e^{jv_l x_l}\right)\frac{\partial p(\underline{x},t|\underline{y},s)}{\partial x_j}\,d\underline{x}$$

Integrated by parts, the integrated parts term vanishes.

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left| j\sum_{i=1}^{n} v_i\sum_{j=1}^{n} B_{ij}(t)\,j v_l\left(\prod_{l=1}^{n} e^{jv_l x_l}\right)\delta_{lj}\;p(\underline{x},t|\underline{y},s)\right|d\underline{x}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}} \underbrace{(j)^2}_{-1} \sum_{i=1}^{n}\sum_{j=1}^{n} v_i B_{ij}(t) v_j \; p(\underline{x},t|\underline{y},s)\,d\underline{x}$$

$$= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} v_i B_{ij} v_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\underline{v}^T\underline{x}}\, p(\underline{x},t|\underline{y},s)\,d\underline{x}$$

$$= -\frac{1}{2}\underline{v}^T\underline{B}\,\underline{v}\,C(t) = -\frac{1}{2}\underline{v}^T\underline{G}(t)Q(t)\underline{G}^T(t)\underline{v}\;C(t)$$

In conclusion, the differential equation for the characteristic function is:

$$\begin{cases} \dot{C}(t) = \left[\left(\frac{\partial}{\partial \underline{v}}\right)^T C(t)\right] A^T(t)\,\underline{v} - \frac{1}{2}\underline{v}^T\underline{G}(t)Q(t)G^T(t)\underline{v}\,C(t) \\ \text{with } C(t_0) = \exp\left[j\,\underline{v}^T\hat{\underline{x}}_0 - \frac{1}{2}\underline{v}^T\Gamma_0\underline{v}\right] \end{cases}$$

no "j"

✓

2.) $$dx(t) = \frac{-x(t)}{1+x^2(t)}\,dt + g(t)\,d\xi(t)$$

$$p[x_0, t_0] = N[\hat{x}_0, \gamma_0]$$

where $\langle d\xi(t)\rangle = 0$ and $\langle [d\xi(t)]^2\rangle = q(t)\,dt$

a.) The Fokker-Planck equation is

$$\frac{\partial p(x,t|y,s)}{\partial t} = -\frac{\partial}{\partial x}\left[a(x,t)\,p(x,t|y,s)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[b(x,t)\,p(x,t|y,s)\right]$$

where

$$a(x,t) = \frac{-x}{1+x^2}$$

$$b(x,t) = g^2(t)\,q(t)$$

So the Fokker-Planck eqn. for this example is:

$$\frac{\partial p(x,t|y,s)}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{-x}{(1+x^2)}\,p(x,t|y,s)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[g^2(t)\,q(t)\,p(x,t|y,s)\right]$$

$$\frac{\partial p(x,t|y,s)}{\partial t} = \left[\frac{1+x^2-2x\cdot x}{(1+x^2)^2}\right]p(x,t|y,s) + \frac{x}{1+x^2}\frac{\partial p(x,t|y,s)}{\partial x} + \frac{1}{2}g^2(t)\,q(t)\,\frac{\partial^2 p(x,t|y,s)}{\partial x^2}$$

$$\frac{\partial p(x,t|y,s)}{\partial t} = \frac{1-x^2}{(1+x^2)^2}\,p(x,t|y,s) + \frac{x}{1+x^2}\frac{\partial p(x,t|y,s)}{\partial x} + \frac{1}{2}g^2(t)\,q(t)\,\frac{\partial^2 p(x,t|y,s)}{\partial x^2}$$

b.) $\dot{\hat{x}} = \hat{f}(x,t)$

$$\dot{\hat{x}} = \left\langle \frac{-x}{1+x^2} \right\rangle = \int_{-\infty}^{\infty} \frac{-x}{1+x^2}\, p(x,t|y,s)\, dx$$

$$\dot{v}(t) = 2\left\langle [x-\hat{x}(t)]\, f(x,t) \right\rangle + \left\langle g^2(t)\, q(t) \right\rangle$$

$$v(t_0) = v_0$$

$$\dot{v}(t) = 2\left\langle [x-\hat{x}(t)] \left(\frac{-x}{1+x^2}\right) \right\rangle + g^2(t)\, q(t)$$

$$\dot{v}(t) = 2\left\langle \frac{-x^2+\hat{x}(t)\,x}{1+x^2} \right\rangle + g^2(t)\, q(t)$$

$$\dot{v}(t) = 2\left\langle \frac{-[1+x^2]}{1+x^2} + \frac{1+\hat{x}(t)\,x}{1+x^2} \right\rangle + g^2(t)\, q(t)$$

$$\dot{v}(t) = -2 + 2\left\langle \frac{1+\hat{x}(t)\,x}{1+x^2} \right\rangle + g^2(t)\, q(t)$$

c.) $\frac{\partial p(x,t|y,s)}{\partial t} = 0 \qquad \forall\, t, x$

$\Longrightarrow \quad p(x,t|y,s) = u(x)$, a function of x alone.

From the Fokker-Planck eqn for this problem:

$$0 = \frac{1-x^2}{(1+x^2)^2}\, p(x,t|y,s) + \frac{x}{1+x^2}\frac{\partial p(x,t|y,s)}{\partial x} + \frac{1}{2} g^2(t)\, q(t)\, \frac{\partial^2 p(x,t|y,s)}{\partial x^2}$$

but since $p(x,t|y,s) = u(x)$

$$0 = \frac{1-x^2}{(1+x^2)^2}\, u(x) + \frac{x}{1+x^2}\frac{\partial u(x)}{\partial x} + \frac{1}{2} g^2(t)\, q(t)\, \frac{\partial^2 u(x)}{\partial x^2}$$

since $u(x)$ is a function of x alone, the partials can be replaced by ordinary derivatives

$$0 = \frac{1-x^2}{(1+x^2)^2}\, u(x) + \frac{x}{1+x^2} \frac{du(x)}{dx} + \frac{1}{2} g^2(t)\, q(t) \frac{d^2 u(x)}{dx^2}$$

$$0 = \frac{d}{dx}\left[\frac{x}{1+x^2}\, u(x)\right] + \frac{1}{2} g^2(t)\, q(t) \frac{d}{dx}\left[\frac{du(x)}{dx}\right]$$ ✓

$$0 = \frac{d}{dx}\left[\frac{x}{1+x^2}\, u(x) + \frac{1}{2} g^2(t)\, q(t) \frac{du(x)}{dx}\right]$$

and the above is an exact differential which can be directly integrated to yield:

$$c_1 = \frac{x}{1+x^2}\, u(x) + \frac{1}{2} g^2(t)\, q(t) \frac{du(x)}{dx}$$

From the boundary condition that

$$\lim_{x\to\infty} p(x,t|y,s) = 0 \quad ; \quad \lim_{x\to\infty} \frac{\partial p(x,t|y,s)}{\partial x} = 0$$

It follows that for our case

$$\lim_{x\to\infty} u(x) = \lim_{x\to\infty} p(x,t|y,s) = 0$$

and $$\lim_{x\to\infty} \frac{du(x)}{dx} = \lim_{x\to\infty} \frac{\partial p}{\partial x}(x,t|y,s) = 0$$

And since the sum of the limits is equal to the limit of the sum when they exist (similarly for products)

$$c_1 = \lim_{x\to\infty}\left[\frac{x}{1+x^2}\, u(x) + \frac{1}{2} g^2(t)\, q(t) \frac{du(x)}{dx}\right]$$

$$= \left(\lim_{x\to\infty} \frac{x}{1+x^2}\right)\left(\lim_{x\to\infty} u(x)\right) + \frac{1}{2} g^2(t)\, q(t) \left(\lim_{x\to\infty} \frac{du(x)}{dx}\right)$$

$$c_1 = \left(\lim_{x\to\infty} \frac{x}{1+x^2}\right)\left(\lim_{x\to\infty} u(x)\right) + \frac{1}{2} g^2(t)\, q(t) \left(\lim_{x\to\infty} \frac{du(x)}{dx}\right)$$

applying L'Hospital's rule

$$= \left(\lim_{x\to\infty} \frac{1}{2x}\right)\cdot\left(\lim_{x\to\infty} u(x)\right) + \frac{1}{2} g^2(t)\, q(t)\cdot\left(\lim_{x\to\infty} \frac{du(x)}{dx}\right)$$

$$= 0\cdot 0 + \frac{1}{2} g^2(t)\, q(t)\cdot 0 = 0$$

$$c_1 = 0$$

So

$$\frac{1}{2} g^2(t)\, q(t) \frac{du(x)}{dx} + \frac{x}{1+x^2} u(x) = 0$$

$$\frac{du(x)}{dx} + \frac{2}{g^2(t)\, q(t)} \cdot \frac{x}{1+x^2} u(x) = 0$$

$$\frac{du(x)}{u(x)} = \frac{-2}{g^2(t)\, q(t)} \cdot \frac{x\,dx}{1+x^2}$$

$$\int \frac{du(x)}{u(x)} = \frac{-2}{g^2(t)\, q(t)} \int \frac{x\,dx}{1+x^2}$$

$$\ln u(x) = \frac{-1}{g^2(t)\, q(t)} \ln(1+x^2) + \ln c_2$$

$$\ln u(x) = \ln \frac{1}{(1+x^2)^{\frac{1}{g^2(t) q(t)}}} + \ln c_2$$

$$e^{\ln u(x)} = e^{\ln\left[\frac{c_2}{(1+x^2)^{\frac{1}{g^2(t) q(t)}}}\right]}$$

$$u(x) = \frac{c_2}{(1+x^2)^{\frac{1}{g^2(t) q(t)}}}$$

So $p(x,t|y,s) = u(x) = \dfrac{c_2}{(1+x^2)^{\frac{1}{g^2(t)\,q(t)}}}$

Evaluate c_2 to make it satisfy the requirement that

$$1 = \int_{-\infty}^{\infty} p(x,t|y,s)\,dx = \int_{-\infty}^{\infty} \frac{c_2}{(1+x^2)^{\frac{1}{g^2(t)q(t)}}}\,dx$$

$$\therefore \quad c_2 = \frac{1}{\left[\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{\frac{1}{g^2(t)q(t)}}}\right]}$$

$$\therefore p(x,t|y,s) = \left[\frac{1}{\left(\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{\frac{1}{g^2(t)q(t)}}}\right)}\right] \cdot \frac{1}{(1+x^2)^{\frac{1}{g^2(t)q(t)}}}$$

Note that as $t \to \infty$

$$p(x,t|y,s) \longrightarrow p(x,\infty|y,s) \longrightarrow p(x)$$

So that $p(x,t|y,s)$ depends on x only

$$\left(g^2(t)\,q(t) \longrightarrow g^2 q \right)$$

steady-state value

3.) Considering the Lienard equation:

$$dx_1(t) = x_2(t)\,dt$$

$$dx_2(t) = -[2x_1(t) + 3x_1^2(t) + x_2(t)]\,dt + g(t)\,d\xi(t)$$

where $\langle d\xi(t)\rangle = 0$; $\langle d\xi(t)^2\rangle = q(t)dt$

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -2x_1(t) - 3x_1^2(t) - x_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ g(t) \end{bmatrix} d\xi(t)$$

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

a.) The first incremental moment is:

$$\underline{A}(\underline{x}, t) = \begin{bmatrix} x_2 \\ -2x_1 - 3x_1^2 - x_2 \end{bmatrix} \checkmark$$

The second incremental moment is:

$$\underline{B}(\underline{x}, t) = \begin{bmatrix} 0 \\ g(t) \end{bmatrix} [q(t)] [0 \quad g(t)] = \begin{bmatrix} 0 & 0 \\ 0 & g^2(t)q(t \end{bmatrix} \checkmark$$

The Fokker-Planck eqn for the above diffusive Markov process is

$$\frac{\partial p(\underline{x}, t | \underline{y}, s)}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[f_i(\underline{x}, t)\, p(\underline{x}, t | \underline{y}, s)\right] + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j}\left[B_{ij}(\underline{x}, t)\, p(\underline{x}, t|\right.$$

$$\frac{\partial p(\underline{x},t|\underline{y},s)}{\partial t} = -\frac{\partial}{\partial x_1}\left(x_2\, p(x_1,x_2,t|y_1,y_2,s)\right) + \frac{\partial}{\partial x_2}\left([2x_1+3x_1^2+x_2]\, p(x_1,x_2,t|y_1,y_2,s)\right)$$

$$+\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}\left[B_{ij}(\underline{x},t)\, p(x_1,x_2,t|y_1,y_2,s)\right]$$

$\therefore$

$$\frac{\partial p(x_1,x_2,t|y_1,y_2,s)}{\partial t} = -x_2\,\frac{\partial p(x_1,x_2,t|y_1,y_2,s)}{\partial x_1}$$

$$+\ p(x_1,x_2,t|y_1,y_2,s) + (2x_1+3x_1^2+x_2)\frac{\partial p(x_1,x_2,t|y_1,y_2,s)}{\partial x_2}$$

$$+\ \frac{1}{2}\, g^2(t)\, q(t)\ \frac{\partial^2 p(x_1,x_2,t|y_1,y_2,s)}{\partial x_2^2}$$

b.) Find the differential equation for the mean and for the covariance.

From problem I, part (b):

$$\dot{\hat{\underline{X}}}(t) = \left\langle \underline{f}(\underline{x}, t) \right\rangle$$

$$\dot{\underline{\underline{\Gamma}}}(t) = \left\langle \begin{bmatrix} 2(x_1-\hat{x}_1)f_1 & (x_1-\hat{x}_1)f_2+(x_2-\hat{x}_2)f_1 & \cdots \\ (x_2-\hat{x}_2)f_1+(x_1-\hat{x}_1)f_2 & 2(x_2-\hat{x}_2)f_2 & \vdots \\ \vdots & & \\ (x_n-\hat{x}_n)f_1+(x_1-\hat{x}_1)f_n & \cdots & 2(x_n-\hat{x}_n)f_n \end{bmatrix} \right\rangle$$

$$+ \left\langle \underline{G}(\underline{x}, t)\, Q(t)\, \underline{G}^T(\underline{x}, t) \right\rangle$$

Or for each component

$$\left\{ \dot{\Gamma}_{ij}(t) \right\} = \left\{ \left\langle (x_i-\hat{x}_i) f_j(\underline{x}, t) + (x_j-\hat{x}_j) f_i(\underline{x}, t) \right\rangle + \left\langle B_{ij}(\underline{x}, t) \right\rangle \right\}$$

So for the above Lienard's equation

$$\dot{\hat{\underline{X}}}(t) = \left\langle \begin{bmatrix} x_2 \\ -2x_1-3x_1^2-x_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x_2 \\ -2x_1-3(x_1-\hat{x}_1)^2-x_2-6x_1\hat{x}_1+3\hat{x}_1^2 \end{bmatrix} \right\rangle$$

$$= \begin{bmatrix} \hat{x}_2 \\ -2\hat{x}_1-3\Gamma_{11}-\hat{x}_2-6\hat{x}_1^2+3\hat{x}_1^2 \end{bmatrix} = \begin{bmatrix} \hat{x}_2 \\ -2\hat{x}_1-\hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3\Gamma_{11}-3\hat{x}_1^2 \end{bmatrix}$$

$$\dot{\Gamma}_{12}(t) = -2\Gamma_{11} - 3\langle (x_1-\hat{x}_1)^3\rangle - 6\hat{x}_1\langle (x_1-\hat{x}_1)^2\rangle$$

$$- 6\hat{x}_1^2\langle x_1-\hat{x}_1\rangle - \Gamma_{12} + \Gamma_{22}$$

$$= -2\Gamma_{11} - 6\hat{x}_1\Gamma_{11} - \Gamma_{12} + \Gamma_{22} - 3\langle (x_1-\hat{x}_1)^3\rangle$$

a term from the third central moment.

$$\dot{\Gamma}_{21}(t) = \dot{\Gamma}_{12}(t)$$

$$\dot{\Gamma}_{22}(t) = \langle 2(x_2-\hat{x}_2)(-2x_1-3x_1^2-x_2)\rangle + g^2(t)q(t)$$

$$= -4\langle (x_2-\hat{x}_2)(x_1-\hat{x}_1)\rangle - 4\hat{x}_1\langle (x_2-\hat{x}_2)\rangle$$

$$- 6\langle x_1^2(x_2-\hat{x}_2)\rangle - 2\langle (x_2-\hat{x}_2)^2\rangle$$

$$-2\hat{x}_2\langle (x_2-\hat{x}_2)\rangle + g^2(t)q(t)$$

$$= -4\Gamma_{12} - 6\langle x_1^2(x_2-\hat{x}_2)\rangle - 2\Gamma_{22} + g^2(t)q(t)$$

$$= -4\Gamma_{12} - 2\Gamma_{22} + g^2(t)q(t)$$

$$- 6\langle (x_1-\hat{x}_1)^2(x_2-\hat{x}_2)\rangle - 12\hat{x}_1\langle x_1(x_2-\hat{x}_2)\rangle$$

$$+ 6\hat{x}_1^2\langle x_2-\hat{x}_2\rangle$$

$$= -4\Gamma_{12} - 2\Gamma_{22} + g^2(t)q(t) - 6\langle (x_1-\hat{x}_1)(x_2-\hat{x}_2)(x_1-\hat{x}_1)\rangle$$

$$- 12\hat{x}_1\Gamma_{12} - 12\hat{x}_1^2\langle x_2-\hat{x}_2\rangle$$

$$= -4\Gamma_{12} - 2\Gamma_{22} - 12\hat{x}_1\Gamma_{12} + g^2(t)q(t) - 6\langle (x_1-\hat{x}_1)^2(x_2-\hat{x}_2)\rangle$$

a term from the [illegible] moment.

$$\dot{\hat{\underline{X}}} = \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} \hat{x}_2 \\ -2\hat{x}_1 - \hat{x}_2 - 3\hat{x}_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3\Gamma_{11} \end{bmatrix}$$

$$\{\dot{\Gamma}_{ij}(t)\} = \left\{ \langle (x_i - \hat{x}_i) f_j(\underline{x}, t) + (x_j - \hat{x}_j) f_i(\underline{x}, t) \rangle + \langle B_{ij}(\underline{x}, t) \rangle \right\}$$

$$\dot{\Gamma}_{11}(t) = \langle (x_1 - \hat{x}_1) x_2 + (x_1 - \hat{x}_1) x_2 \rangle + \langle 0 \rangle$$

$$= \langle 2x_2 (x_1 - \hat{x}_1) \rangle = 2\langle (x_2 - \hat{x}_2)(x_1 - \hat{x}_1) + \hat{x}_2 (x_1 - \hat{x}_1) \rangle$$

$$= 2\langle (x_2 - \hat{x}_2)(x_1 - \hat{x}_1) \rangle + 2\hat{x}_2 \langle x_1 - \hat{x}_1 \rangle$$

$$= 2\,\Gamma_{12}(t)$$

so $\dot{\Gamma}_{11}(t) = 2\,\Gamma_{12}(t)$

$$\dot{\Gamma}_{12}(t) = \langle (x_1 - \hat{x}_1)(-2x_1 - 3x_1^2 - x_2) + (x_2 - \hat{x}_2) x_2 \rangle + \langle 0 \rangle$$

$$= \langle -2x_1(x_1 - \hat{x}_1) - 3x_1^2 (x_1 - \hat{x}_1) - x_2 (x_1 - \hat{x}_1)$$
$$+ (x_2 - \hat{x}_2)^2 + \hat{x}_2 (x_2 - \hat{x}_2) \rangle$$

$$= -2\langle (x_1 - \hat{x}_1)^2 \rangle - 2\hat{x}_1 (x_1 - \hat{x}_1) - 3\langle x_1^2 (x_1 - \hat{x}_1) \rangle$$
$$- \langle (x_2 - \hat{x}_2)(x_1 - \hat{x}_1) \rangle - \hat{x}_2 \langle x_1 - \hat{x}_1 \rangle + \langle (x_2 - \hat{x}_2)^2 \rangle$$
$$+ \hat{x}_2 \langle x_2 - \hat{x}_2 \rangle$$

$$= -2\Gamma_{11} - 3\langle x_1^2 (x_1 - \hat{x}_1) \rangle - \Gamma_{12} + \Gamma_{22}$$

$$= -2\Gamma_{11} - 3\langle (x_1 - \hat{x}_1)^3 \rangle - 6\hat{x}_1 \langle x_1 (x_1 - \hat{x}_1) \rangle + 3\hat{x}_1^2 \langle x_1 - \hat{x}_1 \rangle$$

$$\dot{\hat{\underline{x}}} = \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} \hat{x}_2 \\ -2\hat{x}_1 - \hat{x}_2 - 3\hat{x}_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3\Gamma_{11} \end{bmatrix}$$

$$\dot{\underline{\Gamma}}(t) = \begin{bmatrix} \dot{\Gamma}_{11} & \dot{\Gamma}_{12} \\ \dot{\Gamma}_{21} & \dot{\Gamma}_{22} \end{bmatrix} = \begin{bmatrix} 2\Gamma_{12} & -2\Gamma_{11} - \Gamma_{12} + \Gamma_{22} \\ -2\Gamma_{11} - \Gamma_{12} + \Gamma_{22} & -4\Gamma_{12} - 2\Gamma_{22} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -6\hat{x}_1\Gamma_{11} \\ -6\hat{x}_1\Gamma_{11} & -12\hat{x}_1\Gamma_{12} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -3\langle (x_1-\hat{x}_1)^3 \rangle \\ -3\langle (x_1-\hat{x}_1)^3 \rangle & -6\langle (x_1-\hat{x}_1)^2 (x_2-\hat{x}_2) \rangle \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & g^2(t)\,q(t) \end{bmatrix}$$

$$
\dot{\underline{\Gamma}}(t) = \begin{bmatrix} \dot{\Gamma}_{11} & \dot{\Gamma}_{12} \\ \dot{\Gamma}_{21} & \dot{\Gamma}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} + \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 & 0 \\ 0 & g^2(t)q(t) \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 & 0 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \hat{x}_1
$$

$$
+ \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} 0 & -6 \\ 0 & 0 \end{bmatrix} \hat{x}_1
$$

$$
+ \begin{bmatrix} 0 & -3\langle (x_1-\hat{x}_1)^3 \rangle \\ -3\langle (x_1-\hat{x}_1)^3 \rangle & -6\langle (x_1-\hat{x}_1)^2 (x_2-\hat{x}_2) \rangle \end{bmatrix}
$$

Written more compactly, the mean and covariance eqns for Lienard's eqn are:

$$\dot{\underline{\Gamma}}(t) = \begin{bmatrix} \dot{\Gamma}_{11} & \dot{\Gamma}_{12} \\ \dot{\Gamma}_{21} & \dot{\Gamma}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (-2-6\hat{x}_1) & -1 \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} + \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} 0 & (-2-6\hat{x}_1) \\ 1 & -1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & g^2(t)q(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -3\langle(x_1-\hat{x}_1)^3\rangle \\ -3\langle(x_1-\hat{x}_1)^3\rangle & -6\langle(x_1-\hat{x}_1)^2(x_2-\hat{x}_2)\rangle \end{bmatrix}}_{\text{terms from the third central moment}}$$

$$\dot{\hat{\underline{X}}}(t) = \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} \hat{x}_2 \\ -2\hat{x}_1 - 3\hat{x}_1^2 - \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3\underbrace{\Gamma_{11}}_{\text{term from the covariance}} \end{bmatrix}$$

V. Good