Crucial Integral Evaluation Enabling Performance Trade-offs for a Two Confidence Region (CR2) Approach to Failure Detection*

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Abstract

A derivation is provided herein of the fundamental Integral and its evaluation using the Cauchy Residue Theorem to enable a rigorous setting of decision thresholds for the Two Confidence Region (CR2) approach to failure detection. This CR2 approach (for 1-D and 2-D) was historically used on board US Submarines in monitoring the SINS/ESGM Inertial Navigation Systems (INS) for the presence of ramp failures, a prominent failure mode observed to sometimes occur within the ESGM as it was originally being introduced on board these boats 30+ years ago. The resulting equations enable statistical analysis and associated performance trade-offs constituting the CR2 Receiver Operating Characteristics (ROC). CR2 continues to be rigorous even when reduced-order Kalman filter models are used in the implementation, which causes the associated filter residuals to no longer be purely white and unbiased in the unfailed situation. Related to this aspect of model mismatch, we also point out apparent weaknesses of GLR and IMM-KF, the latter being especially worrisome in its attempted generalization of associated probability calculations for nonlinear applications.

1 Introduction

A detailed derivation is provided here of the equations that provide a rigorous setting of decision thresholds for the Two Confidence Region (CR2) approach to failure detection triggered by the lack of overlap of certain ellipsoidal confidence regions (defined and described analytically in [1], with statistics derived in [2]and further refined in [3]-[6]). This CR2 failure detection approach was historically developed for use [1] with the hybrid SINS/ESGM Navigation system consisting of the existing Ships Inertial Navigation System (SINS) (a conventional INS with its rotating gyros and a dual version of it used as a *warm standby system* [16] as a backup) in conjunction with the newer Electrostatically Supported Gyro Monitor (ESGM), generally more accurate (but initially more susceptible to failure), [10]-[14].

As discussed in [1], "Failure detection and failure isolation are common problems in engineering systems. In general, failure detection requires continuous vigilant monitoring of the observable output variables of the system. Under normal conditions, the output variables follow certain known patterns of evolution within certain limits of uncertainty introduced by slight random system disturbances and measurement noise in the sensors. When failures occur, the output variables deviate from their nominal state space trajectories or evolutionary pattern. Most failure detection techniques are based on spotting these deviations from the usual in the observable output variables.

Whereas the detection of an unknown signal at a known time or the detection of a known signal at an unknown time are standard problems in communication theory, the problem in failure detection is to detect a signal of unknown magnitude which occurs at an unknown time. Failure detection is a more difficult problem that has been receiving attention" only since the 1970's, a representative sampling being [1]-[8], [17], [21],

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[1]Ultimately implemented by John Weiss (Sperry Systems Management) under managers: Dr. Hy Strell and Norman Zabb.

[22], [67], [76]-[78] and their bibliographies, especially the extensive one in [21]. The problem of tracking a maneuvering target is mathematically dual to that of failure detection [68], [88].

The Generalized Likelihood Ratio (GLR) [67], Residual Whiteness Tests [76], and Interactive Multiple Models (IMM) [77], [78]are failure detection approaches that will be considered further in Appendices A and B.) "Our CR2 approach to failure detection uses a different philosophical approach to solve the problem. It places a confidence region about the nominally unfailed trajectory corresponding to the H_0-hypothesis and a second confidence region about the Kalman estimate based on processing the actual measurements. When these two confidence regions are disjoint, implying a non-H_0 situation, a failure is declared.

The associated underlying integral evaluation of Eq. 29 (examined in detail here) enables a statistical analysis for the CR2 approach involving calculation of this detection algorithm's performance trade-off characterized by, what is known in the parlance of Statistical Communications or Information Theory [15], as Receiver Operating Characteristics (ROC) from which the operating point [2] is set by specifying the value of the decision threshold to be used. The underlying mathematical evaluations are tediously long exercises in appropriately substituting variables in creative ways [2, Eq. 40] and in constructively applying Cauchy's Residue Theorem and, although such operations are initially somewhat unwieldy, it still warrants documenting since the application is of considerable engineering significance and the rigorous analytical stepping stones that ultimately yield such useful results in [2], [3] are surprisingly tractable and eventually collapse to shorter more manageable more intuitive expressions that will likely be useful to others as well, as identified in [7], [8], [22], [52, Sec. III]. The CR2 failure detection approach is predicated on the system of interest being adequately described or modeled in continuous-time as a state variable representation (e.g., a system of coupled ordinary differential equations) [28], [29, Eq. 4-39]:

$$\frac{dx(t)}{dt} = f(x,t) + g_1(x,t)u(t) + g_2(x,t)w(t), \tag{1}$$

along with discrete-time sensor data measurements being available of the following algebraic form:

$$z(t_k) = h(x(t_k), t_k) + v(t_k), \tag{2}$$

where $w(t)$ and $v(t)$ in the above are independent zero mean Gaussian White Noises (GWN) of known, specified covariance intensity [3], Q and R, respectively, and also independent of the Gaussian initial condition [4] $x(t_o) \sim \mathcal{N}(\bar{(x)}(0), P_0)$ and $u(t)$ is a deterministic control or exogenous input, with technical regularity conditions of *observability* and *controllability* being satisfied by the system and its noises and control inputs in Eqs. 1 and 2 (or, at least conditions of *detectability* and *stabilizability* being satisfied). The functions $f(x,t)$, $g_1(x,t)$, and $g_2(x,t)$ are assumed to be *bounded* and *measurable* and satisfy a global Lipschitz condition and $h(x(t),t)$ is continuous in x and t.

The system is assumed to be outfitted or equipped with an adequately matched Kalman filter (or an Extended Kalman Filter or an Iterated Extended Kalman Filter [27] matched to a linearized version [5] of the system) since the CR2 failure detection approach is definitely Kalman filter-based. The CR2 algorithm itself makes use of (a proper subset [6] of) the state estimates, $\hat{x}$, and corresponding associated covariance

[2]The familiar concept of evaluating an ROC curve is essentially a delineation of the Pareto optimal solution for the two cost functions of Probability of False Alarm and Probability of Correct Detection that characterize this bicriteria optimization problem. The minimax solution would be to fix the decision threshold associated with the test statistic so that the operating point is at the knee of the curve. The more prevalent approach (and our approach) would be to set the decision threshold to meet a specified probability of false alarm that is set in system specifications, perhaps as a Constant False Alarm Probability (CFAR) [71].

[3]The noise covariance intensity matrices Q and R are symmetric and positive definite and can be time-varying for nonstationary GWN as long as the time-varying values are completely specified beforehand.

[4]The initial covariance is also symmetric and positive definite.

[5]For both the Extended Kalman Filter (EKF) and the Iterated Extended Kalman Filter (IEKF), the Filter's system model is relinearized using a Jacobian evaluated about the last available output state estimate at each new data point (and so reflects the information accumulated from the actual sensor measurements). For Inertial Navigation Systems (INS) involving a constellation of gyros and accelerometers, even though the mechanization itself is nonlinear (e.g., Space Stable, Local Level-wander azimuth, Local Level-free inertial, Strapdown) the underlying error model is linear [30], [32] and as such the optimal estimator is a linear Kalman filter, usually implemented in indirect feedback form (as one of the three possible mechanization options available) [29, Chap. 6].

[6]The subset constitutes the particular states being monitored for failures. For the ESGM, the states of interest were the three INS gyro drift rates.

of estimation error, $P_1(t)$, (computed from the underlying discrete-time Matrix Riccati Equation) that are available as outputs on-line from the Kalman filter at each discrete-time step, k. The covariance output of an associated Matrix Lyapunov Equation, $P_2(t)$, (corresponding to the covariance or uncertainty of the system without any sensor measurements being available) is also assumed to be available to (or calculated within) the CR2 algorithm. The CR2 algorithm essentially compares what can be inferred about system covariance behavior (centered about the filter estimate) by utilizing the available sensor measurement data (reflecting what is actually happening as being either the hypothesis of no failure H_0 holding or the possibly of a failure holding as H_1–see [2, Eqs. 8, 9]) to what can be inferred about system covariance behavior under the assumption of having only a purely unfailed system H_0 (without any measurements being available to test against and so the unconditional covariance is centered about an unconditional mean of zero in the case of monitoring the ESGM for anomalous gyro drift).

2 Overview of CR2 Approach to Failure Detection and Objectives of this Investigation

At a particular fixed discrete-time step k, the defining equations that need to be evaluated for the completely general n-dimensional case in order to specify the probability of false alarm, $P_{fa}(k)$ and probability of correct detection, $P_d(k)$, are provided in [2] as Eqs. 33 and 34, respectively. Both are expressed in terms of the underlying Gaussian densities assumed to be present under the mixed hypotheses H_0 and H_1 and are also expressed in terms of the measured Signal-to-Noise Ratio, having the following structural form [2, Eq. 35]:

$$SNR(k) \stackrel{\triangle}{=} \sqrt{d^T(k) P_{\tilde{x}\tilde{x}}^{-1}(k) d(k)}, \tag{3}$$

and in terms of the computed scalar CR2 test statistic and in its relationship to the (possibly time-varying) decision threshold, $K_1(k)$), where in the above:

$$d(k) \stackrel{\triangle}{=} \text{the mean deterministic response of the Kalman filter to an assumed specific failure mode } \bar{\nu}, \tag{4}$$

where the time evolution of the vector $d(k)$ may be evaluated explicitly from a joint system and filter simulation using the truth models for system and (possibly reduced-order) Kalman-like filter [23] but with the system and measurement noise sample functions zeroed out (i.e., $Q \equiv 0$, $R \equiv 0$) and only $\bar{\nu}$, as the particular failure mode of interest being activated. Details of the underlying structure of the system component failure detection problem are available in [22], [68].

These results are fundamental in detection theory and need to be evaluated for any practical application. We will treat the evaluation of $P_{fa}(k)$ and $P_d(k)$ separately in what follows below, corresponding to the different form of the underlying probability distribution functions (pdf's) under H_0 and H_1, respectively, as:

$$(\text{no}-\text{failure}) \quad \mathrm{H}_0 : \ \tilde{x}_i(k) \sim \mathcal{N}(0, [P_{\tilde{x}\tilde{x}}(k)]_{ii}) \quad (\text{zero mean}), \tag{5}$$

$$(\text{failure}) \quad \mathrm{H}_1 : \ \tilde{x}_i(k) \sim \mathcal{N}(d(k), [P_{\tilde{x}\tilde{x}}(k)]_{ii}) \quad (\text{non}-\text{zero mean}), \tag{6}$$

where, in the above:

$$\begin{aligned}
\tilde{x}(k) &= x(k) - \hat{x}(k), && (7)\\
\hat{x}(k) &\stackrel{\triangle}{=} E[x(k)|Z(k)] \text{ (the conditional mean being the optimal estimate [29])}, && (8)\\
E[x(k)] &= E\left[E[x(k)|Z(k)]\right], \text{ (a property of conditional expectation)}, && (9)\\
E[\tilde{x}(k)] &= E[x(k)] - E[\hat{x}(k)] = E[x(k)] - E\left[E[x(k)|Z(k)]\right] = E[x(k)] - E[x(k)] = 0, && (10)\\
0 = E[\tilde{x}(k)\hat{x}^T(k)] &= E[x(k)\hat{x}^T(k)] - E[\hat{x}(k)\hat{x}^T(k)] = E[x(k)\hat{x}^T(k)] - P_1(k), && (11)
\end{aligned}$$

which, when Eq. 7 is multiplied by its own transpose and unconditional expectations taken throughout yields:

$$P_{\tilde{x}\tilde{x}}(k) = P_2(k) - P_1(k), \tag{12}$$

where the cross-terms dropped out as a consequence of the Hilbert space projection theorem result of the left hand side of Eq. 11. The following has been rigorously established earlier in [1, Lemma 5.1] by taking the synchronous difference between the two respective matrix difference equations, Lyapunov and Riccati, which describe their evolution (in discrete-time) by demonstrating that the difference is always positive definite as it evolves in time:

$$P_2(k) - P_1(k) > 0 \text{ for all k} > 0. \tag{13}$$

In the expressions of conditional expectation above, the notation $Z(k)$ denotes the sigma algebra generated by the sequence of measurements received $z(i)$ for $0 \leq i \leq k$ (also interpreted as the subspace spanned by the discrete measurements received up to time k).

A mechanization of a failure detection solution using the CR2 approach is of the following form:

$$\text{Decide that no} - \text{failure occured (indicative of H}_0 \text{ holding) at time} - \text{step}, k, \text{ when}: \quad \ell(k) \leq K_1(k),$$

$$\text{Decide that a failure occured (indicative of H}_1 \text{ holding) at time} - \text{step}, k, \text{ when}: \quad \ell(k) > K_1(k). \tag{14}$$

3 Evaluation of $P_{fa}(k)$ for the CR2 detection test

For CR2, a detection is declared when the scalar test statistic $\ell(k)$ exceeds the decision threshold setting $K_1(k)$ as depicted in Eq. 14 so the probability of false alarm corresponds to the following situation for the n-dimensional case:

$$P_{fa}(k) \triangleq Prob[\ell(k) > K_1(k)|\text{H}_0] = \int \cdots \int_{\ell(k)>K_1(k)} \mathcal{N}(0, P_{\tilde{x}\tilde{x}}(k))\, d\tilde{x}. \tag{15}$$

The above expression is difficult to evaluate for general dimension n. However, it is relatively easy to evaluate for the scalar case (treated in Sec. 3.1) but very challenging even for the case of $n = 2$ (handled in Sec. 3.2). In both cases, simpler expressions are needed (and have already been obtained in [2]) for the test statistic in order to enable explicit evaluation of the integrals encountered corresponding to Eq. 15 and to ultimately enable specification of the requisite decision threshold, $K_1(k)$, that corresponds to a value of $P_{fa}(k)$ imposed as a constraint to satisfy system performance specifications.

For the application of interest that funded this investigation [10]-[14], only gyros and accelerometers with one or two input axes are involved so the corresponding version of CR2 only needs 1- and 2-dimensional CR2 mechanizations, respectively, to monitor their behavior. We therefore restrict attention here to evaluating the $P_{fa}(k)$ and $P_d(k)$ for just the 1-D and 2-D cases as the simplification in vogue rather than pursue the more general n-dimensional case (that remains an open question for later generations to tackle and solve). In a 3-D world, the ESGM had two gyros, each with two input axes, one of the four input axes being redundant, so the gyro with nonredundant input axes (i.e., both input axes participating in the computed navigation solution) used a 2-D version of CR2 and the other gyro (with only a single actively used input axis participating in the computed navigation solution) needing to be outfitted with only a 1-D version of CR2. The statistical analysis and calculation of the scalar CR2 test statistic for the 2-D case is much harder to handle than that for the 1-D case, as will become quite evident in what follows below in Secs. 3.2, 3.3, and 4.2.

3.1 The CR2 $P_{fa}(k)$ Calculations Simplify Nicely for the 1-D Case

From [2, Eq. 23], the scalar CR2 test statistic for the 1-D case is [7]:

$$\ell(k) \triangleq \frac{[\hat{x}_i(k)]^2}{(\sqrt{[P_2(k)]_{ii}} + \sqrt{[P_1(k)]_{ii}})^2}, \tag{16}$$

[7] The constrained optimization problem and associated scalar Lagrange multiplier that define the scalar CR2 test statistic both have a closed-form exact solution for the 1-D case as, respectively, Eqs. 20 and 21 of [2].

which, when substituted into Eq. 15 yields:

$$\begin{aligned} P_{fa}(k) &= \int_{\hat{x}_i^2 > K_1(k)\cdot(\sqrt{P_2}+\sqrt{P_1})^2} \mathcal{N}_{\hat{x}}(0, [P_2(k) - P_1(k)])\, d\hat{x}_i, \\ &= \int_{u^2 > K_1(k)\cdot(\sqrt{P_2}+\sqrt{P_1})/(\sqrt{P_2}-\sqrt{P_1})^2} \exp\left[\frac{-u^2}{\sqrt{\pi}}\right] du, \\ &= 1 - \frac{1}{2}\mathrm{erf}\left[\sqrt{\frac{K_1(k)}{2}} \cdot \sqrt{\frac{(\sqrt{P_2(k)} + \sqrt{P_1(k)})}{(\sqrt{P_2(k)} - \sqrt{P_1(k)})}}\right]. \end{aligned} \tag{17}$$

To obtain the decision threshold $K_1(k)$, given a fixed value of $P_{fa}(k)$ to be maintained at each check time, involves using tables to solve for the constant b in the following equation:

$$P_{fa}(k) = 1 - \frac{1}{2}\mathrm{erf}\left[\frac{b}{\sqrt{2}}\right], \tag{18}$$

and, by equating and substituting as depicted in [2, Eq. 39a], the CR2 decision threshold for the 1-D case is:

$$K_1(k) = b^2 \cdot \left[\frac{(\sqrt{[P_2(k)]_{ii}} - \sqrt{[P_1(k)]_{ii}})}{(\sqrt{[P_2(k)]_{ii}} + \sqrt{[P_1(k)]_{ii}})}\right]. \tag{19}$$

The above is a time-varying decision threshold that can be used to maintain a constant specified instantaneous false alarm rate. (A methodology is provided in [3] for specifying a decision threshold using random process level-crossing theory so that a particular probability of false alarm exists over an entire specified time interval and not just instantaneously at each discrete check time k.) Real-time on-line mechanization of CR2 for 1-D uses only Eqs. 16, 18, 19 and the two comparison tests of Eq. 14.

3.2 Evaluating CR2 $P_{fa}(k)$ for the Challenging 2-D Case

The expression for the scalar CR2 test statistic for $n = 2$ is considerably more complex than for the 1-D case. It is obtained by first solving the scalar iteration equation for the associated Lagrange multiplier [1, Eq. 34], [8, Eq. 1]:

$$\lambda_{n+1} = \frac{1}{1 + \frac{w^T(k)[(1-\lambda_n)P_2(k)+\lambda_n P_1(k)]^{-1}P_1(k)[(1-\lambda_n)P_2(k)+\lambda_n P_1(k)]^{-1}w(k)}{w^T(k)[(1-\lambda_n)P_2(k)+\lambda_n P_1(k)]^{-1}w(k)}} \text{ for } \lambda_0 = 0.75, \tag{20}$$

where $w(k) \stackrel{\triangle}{=} \hat{x}_1(k) - \bar{x}(k)$. This iteration equation [8] converges geometrically fast as a contraction mapping [1, Theorem 5.1] to a unique solution $\bar{\lambda}(k)$, which is then substituted back into the accompanying Lagrangian saddle point solution for the minimum $x^*(k)$ of the constrained optimization problem that serves as the scalar CR2 test statistic:

$$\ell(k) = \ell\left(\bar{\lambda}, x^*(k)\right) = \bar{\lambda}\left(1 - \bar{\lambda}\right)\left[\left(1 - \bar{\lambda}\right)P_2(k) + \bar{\lambda}P_1(k)\right]. \tag{21}$$

The above expression along with $K_1(k)$ is used in the limits of the integral representing the $P_{fa}(k)$. For the case of $n = 2$, the integrals of Eq. 15 become:

$$P_{fa}(k) = \int_{K_1(k)}^{\infty} p_{L|\mathrm{H}_0}(\ell)\, d\ell = \int_{K_1(k)}^{\infty} \left[\frac{1}{|a_1|}p_{\chi^2}\left(\frac{\cdot}{a_1}\right) * \frac{1}{|a_2|}p_{\chi^2}\left(\frac{\cdot}{a_2}\right)\right] dL, \tag{22}$$

$$= \int_{K_1(k)}^{\infty} \left[\frac{\exp\left(-L/2a_2\right)}{2\pi\sqrt{a_1 a_2}} \int_0^L \frac{e^{-bx}}{\sqrt{x(L-x)}}\, dx\right] dL, \tag{23}$$

$$= \int_{K_1(k)}^{\infty} \left[\left(\frac{1}{4\pi\sqrt{a_1 a_2}}\right) \int_{-\pi}^{\pi} \exp\{-\frac{1}{2}\left[\frac{1}{a_2} + b + b\sin\theta\right] L\}\, d\theta\right] dL, \tag{24}$$

[8] The best grouping to minimize the associated computational burden in terms of operation counts is also identified in [1], [8, Eqs. 4, 5].

$$= \left(\frac{1}{4\pi\sqrt{a_1 a_2}}\right)\int_{-\pi}^{\pi}\left[\int_{K_1(k)}^{\infty}\exp\{-\frac{1}{2}\left[\frac{1}{a_2}+b+b\sin\theta\right]L\}\,dL\right]d\theta, \quad (25)$$

$$= \left(\frac{\exp\{-K_1C/2\}}{2\pi C\sqrt{a_1 a_2}}\right)\int_{-\pi}^{\pi}\left[\frac{\exp\{-(bK_1/2)\sin\theta\}}{1+(b/C)\sin\theta}\right]d\theta, \quad (26)$$

$$= \left(\frac{\exp\{-K_1C/2\}}{2\pi C\sqrt{a_1 a_2}}\right)\int_{-\pi}^{\pi}\left[\sum_{i=0}^{\infty}\frac{(-bK_1/2)^i\,(\sin\theta)^i\,/i!}{1+(b/C)\sin\theta}\right]d\theta, \quad (27)$$

$$= \left(\frac{\exp\{-K_1C/2\}}{2\pi C\sqrt{a_1 a_2}}\right)\sum_{i=0}^{\infty}\left[(-bK_1/2)^i\,/i!\right]\cdot\int_{-\pi}^{\pi}\frac{(\sin\theta)^i}{1+(b/C)\sin\theta}\,d\theta, \quad (28)$$

$$= \left(\frac{\exp\{-K_1C/2\}}{\pi C\sqrt{a_1 a_2}}\right)\sum_{i=0}^{\infty}\left[(K_1 b)^i\,/i!\right]\int_{-\infty}^{\infty}\frac{x^i}{\left[(x^2+2(b/C)x+1)\,(1+x^2)^i\right]}\,dx, \quad (29)$$

$$= \left(\frac{\exp\{-K_1C/2\}}{e_0}\right)\left[e_0+e_1K_1+e_2K_1^2+e_3K_1^3+e_4K_1^4+e_5K_1^5+\cdots\right], \quad (30)$$

where in the above [2, Eqs. B.1-11 to B.1-13]:

$$a_i > 0 \text{ for } i = 1,\ 2, \quad (31)$$

$$b \triangleq \frac{a_2 - a_1}{2a_2a_1} > 0, \quad (32)$$

$$C \triangleq \frac{1}{a_2}+b = \frac{1}{a_2}+\frac{a_2-a_1}{2a_2a_1} = \frac{a_2+a_1}{2a_2a_1} > 0, \quad (33)$$

$$C > b \quad (34)$$

$$1 > (b/C)^2 > 0, \quad (35)$$

and $*$ denotes the operation of convolution. Since the integrand of Eq. 22 is positive as are those of Eqs. 23 and 24 [9] as probability density functions (pdf's) and, as such, have a finite integral when integrated over $K_1(k)$ to ∞, use of Fubini's theorem [55] allows the rigorous interchange of the order of integration, resulting in Eq. 25. After integrating out the variable L in Eq. 25 to obtain Eq. 26, Eq. 26 may be rewritten, using the series expansion of the exponential, as Eq. 27. Since the resulting series of continuous functions in Eq. 27 is a uniformly convergent series by the Weierstrass M-test [57], the order of integration and summation can be rigorously (i.e., validly) interchanged in Eq. 27 to result in Eq. 28. Using the half-angle substitution [56] of $x = \tan(\theta/2)$ for which $\theta = 2\arctan x$ and $d\theta = \frac{2\,dx}{1+x^2}$ in Eq. 28 yields Eq. 29. Going from Eq. 29 to obtain the result of Eq. 30 is very challenging and tedious since it involves fairly long intermediate expressions but they eventually collapse into shorter more manageable expressions, as summarized here. The details of this challenging evaluation is provided next in Sec. 3.3.

3.3 Obtaining a Tractable Series Needed for Handling the 2-D Case of CR2

The following integral that arose as Eq. 29:

$$P_{fa}(k) = \left(\frac{\exp\{-K_1C/2\}}{\pi C\sqrt{a_1 a_2}}\right)\sum_{i=0}^{\infty}\left[(K_1 b)^i\,/i!\right]\int_{-\infty}^{\infty}\frac{x^i}{\left[(x^2+2(b/C)x+1)\,(1+x^2)^i\right]}\,dx, \quad (36)$$

can be evaluated over a closed path involving a semi-circle and the real axis in the complex plane using the Cauchy Residue Theorem [58] in conjunction with some limiting arguments to make the upper half-plane semi-circle have a radius that goes to infinity (and the corresponding real axis segment go from $-\infty$ to ∞), as explained below.

[9] The inner integral in Eq. 24 was originally to be integrated from $-\pi/2$ to $\pi/2$ but that is equivalent to the more convenient version from $-\pi$ to π when divided by two.

Notice that the general integral of Eq. 29 has poles at the values of z where the following two polynomials have zeroes:

$$
\begin{aligned}
0 &= z^2 + 2(b/C)z + 1, && (37)\\
0 &= \left(z^2+1\right)^i, && (38)
\end{aligned}
$$

specifically, the roots of interest, which fall within a closed infinite semi-circle in the upper half complex plane, occur at the following values of z:

$$
\begin{aligned}
z &= -\frac{b}{C} + \jmath\sqrt{1-(b/C)^2}, && (39)\\
z &= +\jmath\,(\text{of multiplicity } i), && (40)
\end{aligned}
$$

and the negative imaginary roots of both of these quadratic polynomials lie outside of the closed upper semi-circle and therfore play no role in the numerical evaluation via a sum of the enclosed residues in the counter-clockwise direction. The contribution of the path integral along the infinite semi-circle is zero because the degree of the denominator is more than two greater than that of the numerator so the real integrals of Eq. 29 (Eq. 36) are equivalent to the complex path integrals over the simply connected region enclosed:

$$
\begin{aligned}
2\pi\jmath\sum \text{Res} = \oint \frac{z^i\,dz}{\left[\left(z^2+2(b/C)z+1\right)\left(1+z^2\right)^i\right]} &= \lim_{R\to\infty}\int_0^{\pi} \frac{\jmath R^{i+1}e^{\jmath(i+1)\theta}\,d\theta}{\left[\left(R^2e^{\jmath2\theta}+2(b/C)Re^{\jmath\theta}+1\right)\left(1+R^2e^{\jmath2\theta}\right)^i\right]}\\
+\lim_{R\to\infty}\int_{-R}^{+R} \frac{x^i\,dx}{\left[\left(x^2+2(b/C)x+1\right)\left(1+x^2\right)^i\right]} &= 0+\int_{-\infty}^{\infty}\frac{x^i\,dx}{\left[\left(x^2+2(b/C)x+1\right)\left(1+x^2\right)^i\right]}, \qquad (41)
\end{aligned}
$$

where the integrand is analytic within the contour described except at the above mentioned simple poles that are enclosed.

Evaluation of the integrals of Eq. 36 for the first five terms, using the Cauchy Residue Theorem yields the first three easy evaluations that demonstrate how the evaluations will be performed for the remaining two harder cases:

$$
\begin{aligned}
I_0 &= \int_{-\infty}^{\infty}\frac{dx}{\left[\left(x^2+2(b/C)x+1\right)\right]} = 2\pi\jmath\cdot\frac{1}{\jmath 2\sqrt{1-(b/C)^2}} = \frac{\pi}{\sqrt{1-(b/C)^2}}, && (42)\\
I_1 &= \int_{-\infty}^{\infty}\frac{x\,dx}{\left[\left(x^2+2(b/C)x+1\right)\left(1+x^2\right)\right]} = 2\pi\left[\frac{1}{2\sqrt{1-(b/C)^2}(-2b/C)}+\frac{1}{2\,(2b/C)}\right], && (43)\\
I_2 &= \int_{-\infty}^{\infty}\frac{x^2\,dx}{\left[\left(x^2+2(b/C)x+1\right)\left(1+x^2\right)^2\right]} = 2\pi\jmath\left[\frac{1}{2\jmath\sqrt{1-(b/C)^2}(-b/C)^2}\right]\\
&+\ 2\pi\jmath\left[\frac{d}{dx}\left(\frac{x^2}{[x^2+2(b/C)x+1](x+1\jmath)^2}\right)\right]\bigg|_{x=1\jmath} = 2\pi\jmath\left[\frac{1}{2\jmath\sqrt{1-(b/C)^2}(-b/C)^2}\right]+\\
&2\pi\jmath\left[\frac{-2x^4-2(b/C)x^3+\jmath\left(2(b/C)x^2+2x\right)}{[x^2+2(b/C)x+1](x+1\jmath)^3}\right]\bigg|_{x=1\jmath} = 2\pi\left[\frac{1}{2\sqrt{1-(b/C)^2}(2b/C)^2}-\frac{1}{2\,(2b/C)^2}\right] && (44)
\end{aligned}
$$

Continuing in like manner, but sparing the reader much of the long unwieldy intermediate expressions, yields:

$$
\begin{aligned}
I_3 &= \int_{-\infty}^{\infty}\frac{x^3\,dx}{\left[\left(x^2+2(b/C)x+1\right)\left(1+x^2\right)^3\right]}\\
&= 2\pi\jmath\left[\frac{1}{2\jmath\sqrt{1-(b/C)^2}(-2b/C)^3}+\frac{1}{2}\left[\frac{d^2}{dx^2}\left(\frac{x^3}{[x^2+2(b/C)x+1](x+1\jmath)^3}\right)\right]\bigg|_{x=1\jmath}\right]
\end{aligned}
$$

$$= \quad 2\pi\left[\frac{1}{2\sqrt{1-(b/C)^2}\,(-2b/C)^3} + \frac{1}{2}\left[\frac{1}{(2b/C)^3} + \frac{1}{8\,(2b/C)}\right]\right] \tag{45}$$

and

$$\begin{aligned}
I_4 &= \int_{-\infty}^{\infty} \frac{x^4\,dx}{\left[(x^2+2(b/C)x+1)\,(1+x^2)^4\right]} \\
&= 2\pi\jmath\left[\frac{1}{2\jmath\sqrt{1-(b/C)^2}(-2b/C)^4} + \frac{1}{6}\left[\frac{d^3}{dx^3}\left(\frac{x^4}{[x^2+2(b/C)x+1](x+1\jmath)^4}\right)\right]\Big|_{x=1\jmath}\right] \\
&= 2\pi\left[\frac{1}{2\sqrt{1-(b/C)^2}\,(-2b/C)^4} - \frac{1}{2^4}\left[\frac{1}{(2b/C)^2} + \frac{8}{(2b/C)^4}\right]\right] \qquad (46) \\
I_5 &= \int_{-\infty}^{\infty} \frac{x^5\,dx}{\left[(x^2+2(b/C)x+1)\,(1+x^2)^5\right]} \\
&= 2\pi\jmath\left[\frac{1}{2\jmath\sqrt{1-(b/C)^2}(-2b/C)^5} + \frac{1}{24}\left[\frac{d^4}{dx^4}\left(\frac{x^5}{[x^2+2(b/C)x+1](x+1\jmath)^5}\right)\right]\Big|_{x=1\jmath}\right] \\
&= 2\pi\left[\frac{1}{2\sqrt{1-(b/C)^2}\,(-2b/C)^5} + \frac{1}{2^4}\left[\frac{1}{(2b/C)^5\,2} + \frac{1}{(2b/C)^3\,2^4} + \frac{3}{(2b/C)\,2^8}\right]\right]. \qquad (47)
\end{aligned}$$

Although the above results were originally derived by long hand, they fortuitously possessed a type of internal error cross-check by the imaginary contribution of the enclosed residues collapsing to be identically zero. Nowadays, these expressions can be confirmed using a symbol manipulation routine such as Maple©.

As derived and defined in [2, Lemma 1, Eq. B.1-2], a useful auxiliary matrix is:

$$S(\bar{\lambda}, k) \triangleq [P_2(k) - P_1][(1-\bar{\lambda})P_2(k) + \bar{\lambda}P_1] \tag{48}$$

that can be used as an intermediary in specifying the requisite parameters a_1, a_2, b^2, and C, from which ultimately the parameters e_0, e_1, e_2, e_3, e_4, e_5 are defined [2, Eqs. 41-44]. It is also established in [2, Eqs. B.2-12 to B.2-13], by the simple algebraic manipulation of inequalities, that:

$$\begin{aligned}
C &> 0 \qquad (49) \\
e_i &> 0 \text{ for all } i = 0, \cdots, 5. \qquad (50)
\end{aligned}$$

In order to solve Eq. 30 for the unknown $K_1(k)$, a useful contrivance is to decompose it into two separate algebraic equations to be solved simultaneously as:

$$\begin{aligned}
y_1(K_1) &\triangleq e_0 + e_1K_1 + e_2K_1^2 + e_3K_1^3 + e_4K_1^4 + e_5K_1^5, \qquad (51) \\
y_2(K_1) &\triangleq e_0P_{fa}(k)\cdot\exp\left[\frac{CK_1}{2}\right]. \qquad (52)
\end{aligned}$$

Notice that the vertical intercept of the two equations are rordered as $e_0 > P_{fa}\cdot e_0$ and the exponentially increasing term initially starts below the quintic at $K_1 = 0$ but ultimately grows to intersect it since the exponent is purely positive and the exponential will eventually dominate the quintic polynomial even though it starts below it. A successive approximations implementation of these two equations [59] can be used to easily solve this problem evaluation for the unknown $K_1(k)$, as depicted in [2, Fig. 4]. Convergence is obvious from the figure cited. This successive approximations approach is iterated to convergence for each successive k to yield a time-varying decision threshold that yields a CFAR implementation of the CR2 test (cf., [71]). Real-time on-line mechanization of CR2 for 2-D uses only Eqs. 20, 21, 51, 52 and the two comparison tests of Eq. 14.

4 Evaluation of $P_d(k)$ for the CR2 detection test

For CR2, the probability of correct detection corresponds to the following situation for the n-dimensional case:

$$P_d(k) \stackrel{\triangle}{=} Prob[\ell(k) > K_1(k)|\mathrm{H}_1] = \int \cdots \int_{\ell(k)>K_1(k)} \mathcal{N}(d(k), P_{\tilde{x}\tilde{x}}(k))\, d\tilde{x}. \tag{53}$$

Similar to the situation for evaluation of $P_{fa}(k)$, the above expression is difficult to evaluate for general dimension n. However, it is easy to evaluate for the scalar case (treated in Sec. 4.1) and tractable but more challenging for the case of $n = 2$ (handled in Sec. 4.2).

4.1 The CR2 $P_d(k)$ Evaluation Simplifies Nicely for the 1-D Case

In complete analogy to what was done in Sec. 3.1 and the simplifications that accrued for the 1-D case, the integral of Eq. 53 reduces to the following closed form (with constituent parts that are known and easy to evaluate):

$$\begin{aligned} P_d(k) \quad = \quad & 1 - \frac{1}{2}\mathrm{erf}\left[\left(\frac{\mathrm{SNR}(k)}{\sqrt{2}}\right) + \sqrt{\frac{K_1(k)}{2}} \cdot \sqrt{\frac{(\sqrt{P_2(k)} + \sqrt{P_1(k)})}{(\sqrt{P_2(k)} - \sqrt{P_1(k)})}}\right] \\ & -\frac{1}{2}\mathrm{erf}\left[\left(\frac{\mathrm{SNR}(k)}{\sqrt{2}}\right) - \sqrt{\frac{K_1(k)}{2}} \cdot \sqrt{\frac{(\sqrt{P_2(k)} + \sqrt{P_1(k)})}{(\sqrt{P_2(k)} - \sqrt{P_1(k)})}}\right], \end{aligned} \tag{54}$$

where in the above, the expression for the signal to noise ration of Eq. 3 simplifies as:

$$\mathrm{SNR}(k) = \frac{|d(k)|}{\sqrt{P_2(k) - P_1(k)}}. \tag{55}$$

4.2 Evaluating CR2 $P_d(k)$ for the Challenging 2-D Case

For the 2-D case, after performing an offset by the indicated mean and scaling by the covariance matrix present in the Gaussian distribution, Eq. 53 simplifies as:

$$P_d(k) = 1 - \int\int_G \mathcal{N}_u(0, I)\, du, \tag{56}$$

where G is the following elliptical region:

$$\left(u + [P_2 - P_1]^{-1/2} d(k)\right)^T \bar{A}^{-1}(\bar{\lambda}) \left(u + [P_2 - P_1]^{-1/2} d(k)\right) \le \frac{K_1(k)}{\bar{\lambda}(1-\bar{\lambda})}, \tag{57}$$

where the integral here represents the volume under the circular (independent) bivariate Gaussian surface enclosed by an offset ellipse and can be evaluated using existing tables [36]. A circular approximation to the above elliptical region is offered in [2, Eq. 54] and enables these integrals of a circular bivariate Gaussian surface to be evaluated over an offset circle, as found in more generally available tables [35], [34].

5 Summary and Conclusions

We have summarized the rigorous mathematics underlying the CR2 approach to failure detection, with particular attention being given here to the evaluation of the complex integral of Eq. 29 which, up to now, had received short shrift in other associated CR2 documentation. This evaluation was crucial in order to evaluate $P_{fa}(k)$ and $P_d(k)$, as arise in characterizing the CR2 performance in terms of ROC (and CFAR values of P_{fa} and associated P_d (incurred for failure magnitudes to be protected against) are used as parameters in the non-ideal 3 state switches that arise in associated system Reliability/Availability diagrams

[6]). An example of another approach to evaluating ROC (for a Schweppe likelihood ratio detection test also obtained from further operations on the outputs of a Kalman filter) uses Chernoff upper bounds, which are then optimized to be as tight as possible, is demonstrated in [9]. There are detailed figures intuitively depicting all aspects of CR2 diagrammatically in [1]-[3]. The fundamental characterization of CR2 found in [1], [2] is particularly amenable to being depicted graphically since the underlying test for the overlap of two ellipsoidal confidence region sheaths at a particular check time k is geometrical in character and is solved by embedding the n-dimensional problem within an (n+1)-dimensional space. A recent test for n-dimensional ellipsoid overlap test [83] (which avoids CR2's restrictive hypothesized condition of Eq. 13) also embeds the problem in (n+1)-dimensions in order to elegantly solve the general n-dimensional overlap problem, as pointed out in [54]. This recent overlap test can also be applied within the complete Failure Detection, Identification, and Reconfiguration (FDIR) approach advocated in [21], which previously avoided using CR2. GPS/INS Integrity monitoring, where consideration of GPS interaction with an INS is currently absent in GPS Reciever Autonomous Intergrity Monitoring (RAIM) [53, p. 598], can be based on the joint approaches of [21]with [83]. The results of applying CR2 to real SINS/ESGM sensor data is depicted in [2, Fig. 3], [4], [5]. Only failure magnitudes corresponding to SNR = $\tilde{1}2.5$ dBm or more above the background noise of the coarser SINS will have good detection behavior, a standard benchmark number.

Since we have a working perspective into many other aspects of Kalman filtering [18]-[21], [23], [24], [51]; nonlinear estimation [25]-[27], [74]; and its related concerns [37]-[54], [68], [73]; we use this forum to also point out perceived weaknesses that exist in several other alternative approaches [10] to failure detection (treated in Appendices A and B) that have not been publicized hithertofore. Such considerations arise in reducing mere theory to a final practical implementation instead of continuing to dwell on ideal starting points without facing the realities of the constraints that exist in real implementations (one such being the standard use of reduced-order filters [11], where filter residuals are no longer white and unbiased [72] thus foiling or corrupting the original approach of [76] which relies on whiteness of residuals as a gauge of normal unfailed behavior, as GLR also relies [67]). An in depth understanding of a system's principles of operation must be known [10]-[14], [50], [73] before one knows how to break it [49], [87] or better yet defend it, as in developing counter-countermeasures C^5I (as *Old Crows* know full well).

A Status of GLR Approach to Event Detection

While Generalized Likelihood Ratios (GLR) (where maximum likelihood estimates of unknown parameters are utilized within the H_1 to H_0 likelihood ratios in lieu of the parameters being unknown) are presented and developed by Davenport and Root [60], Root went further [61] to investigate applicability of GLR techniques in the radar detection problem of resolving closely spaced targets in a background of either known arbitrary correlated Gaussian noise or in Gaussian white noise. However, Root [61] obtained explicit criteria that could be applied to indicate conditions under which one could expect to not resolve two known signals (of unknown amplitudes and parameters) and additionally pointed out a difficulty of using GLR for this purpose.

Selin [62] found that some of the unknown parameters (such as unknown relative carrier phase) must also be estimated in order to maximize the *a posteriori* probability in the estimation of two similar signals in white Gaussian noise. Selin further identified four standard caveats [63, p. 106] associated with use of a maximum likelihood estimate of the unknown parameters in a likelihood ratio (as utilized in GLR).

McAulay and Denlinger [66] advocated use of GLR in conjunction with a Kalman filter in decision-directed adaptive control applications. Finally, Stuller [64] defined an M-ary GLR test that ostensibly overcame Root's original objections [61] to GLR for this type of application. (Ref. [64] also provides a limited history of GLR developments for radar, excepting no mention of [66], which possibly eluded him.

[10] This tact was pursued here since illustrative counterexamples are tracked far less diligently in engineering literature than in the field of mathematics (e.g., [84], [85]).

[11] Gaussian confidence regions still persist as Gaussians when reduced-order filters are inserted in the application and certain reduced-order filters still avail exact covariances on-line in real-time [23] so CR2 is therefore robust with respect to this aspect. The truth models for SINS and ESGM are 34 and over 100 states, respectively, while the dimensions of the associated Nav filter models were 15 and 18 states. Moreover, Gaussian confidence regions arise even when the pdfs are from an exponential family, where the important conditionals and marginals are still Gaussian [86, Chaps. 1-4].

The use of GLR for failure detection was pioneered by Willsky and Jones [67] using an identical GLR formulation as presented by McAulay and Denlinger [66]. While both Willsky and McAulay claim optimality of the GLR, they never explicitly specify a criteria by which it may be judged optimal nor do they supply a proof or reference where such a claim is demonstrated (specifically, [66] references the proof to be in a English translation of an identified German textbook but diligent follow-up on our part here revealed no such substantiation located there).

On [15, p. 92], attention is called to the fact that GLR is not a Uniformly Most Powerful (UMP) test, while [15, p. 96] offers recognition that cases exist where use of GLR can give bad results. That a maximum likelihood estimate (MLE) is not necessarily statistically consistent in general is explicitly demonstrated in a counterexample in [65, p. 146].

GLR is again being advocated for use in radar applications but those that do appear to ignore the historical objections for use of GLR in these types of applications as well as the explicit counterexamples in [17, 968 ff, App. A, pp. 973-974] that, apparently, have never been refuted. The new version of GLR (called "Ed Kelly's GLR") is of a different form than used by the others mentioned above [88] and is apparently a pseudo-GLR but useful none-the-less for radar tracking but evidently somewhat lacking in its present form for failure detection since it ignores the onset time of the detection event. Use of the Entropy Maximization (E-M) algorithm may placate Selin's and Roots' concerns above but is a large computational burden that may defy a real-time implementation.

B Status of Interactive Multiple Model (IMM) Parallel Bank-of-Kalman filters for Nonlinear Applications

While, by now, it is routine to consider the generalization of Kalman filter estimation techniques from mere linear systems (for which Kalman filters are optimal estimators [28], [29], [32]) to nonlinear systems (for which Extended Kalman filters or Iterated Extended Kalman filters [27]are frequently useful, tractable approximations to nonlinear filtering [41, Sec. 12]), as also discussed in [28], [29, Vol. 2, 1982], [32]. Similar ideas should successfully generalize each of the Kalman filters arising in the bank-of-Kalman filters that occur in Interactive Multiple Model (IMM) mechanizations as IMM is generalized beyond the exclusively linear case for which it was rigorously derived as merely a two level approximation (even in the purely linear case [12]); however, the associated IMM probability calculations are more suspect in an attempted generalization to the nonlinear case. In each of the following three references [75, before Eq. 4], [77, after Eq. 2], [78, after Eq. 6], "the critical mixture is *assumed* to be a sum of Gaussians [13], then the prior pdf is a Gaussian mixture and can be approximated (via moment matching) with a single Gaussian...." Our objection here is four-fold. First, for nonlinear systems, the estimates outputted by an EKF is not Gaussian in general (unlike the assumption). Secondly, there are already existing analytic results [82] which caution that a single subsuming Gaussian pdf [14] is not always possible (or not usually possible) even if the individual participating pdf's were in fact Gaussian when the means of the various contributing pdf's are not in close enough proximity, as gauged by the spread of the umbrella of associated covariances. This topic has been an issue since the historically well known Gaussian-Sums approach of [80], [81] also used a bank-of-Kalman-filters structure (which also did not match "expectations", so to speak). Indeed, nonlinear filtering situations frequently exhibit multimodal output estimates as a fact of life, as discussed in [79]. Thirdly, the "moment matching" called for in [75], [77], [78] is not explained there nor is there an opportunity to do so within the algorithm for each time-step k, as needed. Fourthly, what is to be matched in "moment matching" by what and to what and by what gauge will it be determined that it matches closely enough. Nothing about these aspects has been explained in the three references cited above.

[12] The sojourn times and Markov chain transition probabilities are a new contrivance within IMM, useful by providing additional parameters to tune to better match potential application situations by keeping alternative models more actively viable than they were for the Magill bank-of-Kalman-filters (1965).

[13] While sums of Gaussian random variables or sums of random processes are always Gaussian, that is not the issue or situation here. The topic here is of the resulting pdf's of the output which are claimed to be a weighted sum of the Gaussian pdf's called a "Gaussian mixture".

[14] This aspect is not subsumed under the Central Limit Theorem nor under its more recent generalizations of the last 38 years.

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