COMPUTATIONAL TECHNIQUES FOR THE MATRIX PSEUDOINVERSE IN MINIMUM VARIANCE REDUCED-ORDER FILTERING AND CONTROL

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There have been two new algorithms of fairly recent origin offered for the calculation of the matrix pseudoinverse. Unfortunately, nonpathological counterexamples can be constructed, as offered herein, that demonstrate the questionable nature of these two algorithms; however, a resolution is offered here to help prevent possible uncritical propagation of the questionable algorithms. As a rigorous alternative, a well-established technique (endorsed by numerical analysts) is reviewed for calculating the correct matrix pseudoinverse using a computer. Additionally, this technique possesses existent independently verified/validated and accessible software code for a convenient implementation. However, historical loose ends in calculating the associated condition number are singled out here as cause for concern and as a topic for future resolution and refinement. Although an optimal control application of pseudoinverses is also presented here, the primary motivation for considering these pseudoinverse issues is offered in an application example from estimation theory in the implementation and analysis of a minimum variance reduced-order (MVRO) filter, having proper performance that critically hinges on the correct computation of the matrix pseudoinverse. While examples of applying MVRO to navigation applications were provided almost a decade ago, a clear indication of the somewhat restrictive conditions of applicability were wanting, and so are elucidated here, since there appears to be a resurgence of interest in this analytic technique. Another contribution is in providing a tally of the *drawbacks* to be incurred in using MVRO, as well as its previously publicized benefits. This is done in order that a balanced view be offered on what should be considered in a *fair* tradeoff to assess the utility of using MVRO for a particular application.

I. INTRODUCTION

Although fairly widely utilized in select areas of control and estimation theory (notably in forming bounds and in numerical calculation of the transition matrix and the discrete-time process noise covariance matrix for time-invariant linear systems [19–21]), it is fairly well known that a previous measure thought to be a norm for over twelve years (as evidenced in Refs. [19–21, 39, 110]) has now been demonstrated to be invalid [1–3]. Additional, more severe, analytical misconceptions currently exist in related areas and have been somewhat pervasively propagated (as representatively identified in Refs. [4, 5, 22, 23, 46, 49, 115, 148] for correction). Some counterexamples are presented herein to point out weaknesses in two approaches offered recently for calculating the matrix pseudoinverse. Previously unacknowledged limitations in even the preferred computational technique are offered here as representative of what hurdles are to be encountered in attempting to implement an MVRO filter for navigation applications. These caveats are provided in the same constructive vein as the above-mentioned corrections. A brief theoretical overview is provided in Appendix A of what constitutes a valid pseudoinverse.

Counterexamples are offered in Section II to two recent approaches suggested by researchers, and some textbooks (as identified) for calculating and propagating the matrix pseudoinverse. A technique, currently becoming the standard solution approach, as endorsed by numerical analysts, is reviewed in Section III, along with a brief consideration of implementation details, validation history, and acknowledged although not well-known loose ends in the associated "condition number" estimation (related to the ratio of the largest to smallest eigenvalue encountered in the matrix of concern, being a measure of the degree of ill conditioning encountered). While some varied applications of pseudoinverse calculation are offered in Section IV, Section V concentrates on its impact in minimum variance reduced-order filtering. Section V also provides new insights and explicit restrictions or conditions for valid implementation of an MVRO filter and for the subsequent engineering utility of such an MVRO filter as the main contribution of this article. A brief overall summary is provided in Section VI. Certain augmenting details and illustrative examples are relegated to Appendices A, B, and C.

II. COUNTEREXAMPLES TO TWO QUESTIONABLE PSEUDOINVERSE ALGORITHMS

Two recent apparently fundamental misconceptions pertaining to the calculation of the matrix pseudoinverse are now identified by means of transparently simple counterexamples.

A. PSEUDOINVERSE ALGORITHM NUMBER 1

In an application of the generalized likelihood ratio (GLR) to failure detection and other event detections, the telescoping property inherent in the following definition [6, Eq. (46); 7, Eq. (29); 8, Eq. (12)]:

$$C(k;\theta) \triangleq \sum_{j=N_0}^{k} G^T(j;\theta)V^{-1}(j)G(j;\theta), \tag{1}$$

where θ is a fixed scalar variable representing an unknown event time; j is the time index; k is the current time; $G(j;\theta)$ is an $m \times n$ matrix not necessarily of full rank; $V(j)$ is an $m \times m$ symmetric positive definite matrix; T as a superscript represents a vector or matrix transpose, and Eq. (1) is used in Refs. [6–8] to establish one means of recursively generating $C(k;\theta)$ as in [6, Eq. (55)]:

$$C(k;\theta) = G^T(k;\theta)V^{-1}(k)G(k;\theta) + Ck-1;\theta). \tag{2}$$

In Ref. [6, Eq. (56)], the matrix inversion lemma is applied to (2) when each $G(\bullet;\bullet)$ is of full rank, to result in:

$$\begin{aligned} C^{-1}(k;\theta) &= \left[G^T(k;\theta)V^{-1}(k)G(k;\theta) + C(k-1;\theta)\right]^{-1} \\ &= C^{-1}(k-1;\theta) - C^{-1}(k-1;\theta)G^T \\ &\quad \times \left[GC^{-1}(k-1;\theta)G^T + V\right]^{-1}GC^{-1}(k-1;\theta). \end{aligned} \tag{3}$$

However, in Ref. [6, p. 17], it is *asserted* that the matrix inversion lemma can also be used to propagate the pseudoinverse recursively in case all the $G(\bullet;\bullet)$ are *not* of full rank [such that the strict inverse of $C(k-1;\theta)$ is *not* guaranteed to exist]. However, this assertion was never proved nor referenced in Ref. [6], nor properly qualified as being only a conjecture, and no such matrix-inversion lemma-like property is acknowledged to exist in either of the three recent encyclopedic references [9], [10], or [29], or in other recent specialized discussions ([123, Section 5.2]; [124, Chapter 3]). Indeed, a counterexample to the property asserted in Ref. [6], that the pseudoinverse (denoted herein by a superscript dagger) may be recursively propagated via (3), is offered below.

B. COUNTEREXAMPLE TO ALGORITHM NUMBER 1

$$C(k-1;\theta) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}; \quad C^{\dagger}(k-1;\theta) = \frac{1}{25}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{4}$$

$$G(k;\theta) = I_2; \quad V(k) = V^{-1}(k) = I_2, \tag{5}$$

where I_2 is the identity matrix, which when used in (2) yields

$$C(k;\theta) = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \tag{6}$$

having a valid pseudoinverse (which in this case is the same as the inverse) being

$$C^{\dagger}(k;\theta) = C^{-1}(k;\theta) = \frac{1}{6}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}. \tag{7}$$

However, the following *erroneous result*

$$C^{\dagger}(k;\theta) = \frac{1}{30}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{8}$$

is obtained when the matrix inversion lemma of the form of (3) is utilized in an *attempt* to recursively generate the pseudoinverse according to the path advocated in Refs. [6–8]. (The algorithm described above, as offered for pseudoinverse updating in Ref. [6], was encountered in the GLR approach to failure detection and other event detection; however, there are other alternative approaches to failure detection in navigation systems, such as in Refs. [40–46, 55–57, 62], or as surveyed in Refs. [62, 90–92], that do not require any calculation of the pseudoinverse as well as the GLR modification in Ref. [132].) Recent insights that obliquely relate the matrix inversion lemma with calculating pseudoinverses is as provided in Ref. [52, Theorem 5] and Ref. [122, Theorem 3.1]. However, use of the matrix inversion lemma in performing recursive calculations (rather than merely to provide theoretical and structural insight) has been demonstrated to be numerically unstable in general [98, p. 1038]. Valid generalizations of the matrix inversion lemma are in Refs. [130, 131]. Another slightly different, but commonly encountered, theoretical misconception relating to the matrix pseudoinverse in discussed next.

C. PSEUDOINVERSE ALGORITHM NUMBER 2

In the textbook [11, p. 19], it is *asserted* that the matrix pseudoinverse always takes one or the other of the following two forms:

$$C^{\dagger} = C^{T}(CC^{T})^{-1} \tag{9}$$

or

$$C^{\dagger} = (C^{T}C)^{-1}C^{T}. \tag{10}$$

It is emphasized here as a *warning of the possible theoretical oversight or definite oversimplification* in the above dichotomy asserted for pseudoinverse calculation in the textbook [11] that the above two forms are appropriate to represent pseudoinverse calculation *only if* either (CC^T) or (C^TC) is nonsingular, respectively.

D. COUNTEREXAMPLES TO ALGORITHM NUMBER 2

A simple example that illustrates a frequently encountered general case when neither of the above two simple forms is appropriate to represent the pseudoinverse is (Ref. [12, p. 168, Exercise 20], with the solution provided here being original, as derived in Appendix B as an example of the requisite calculations to be performed in determining the pseudoinverse via hand computation for even simple low-dimensional examples):

$$C_1 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \tag{11}$$

having pseudoinverse

$$C_1^{\dagger} = \begin{bmatrix} \frac{-1}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} \tag{12}$$

(as can be verified by showing that it satisfies the necessary and sufficient Properties 1–9 of a pseudoinverse [12, p. 165] or as found in Ref. [51, Chapter 4], as previously referenced by Ref. [50]); yet

$$C_1^{T}C_1 = \begin{bmatrix} 3 & 4 & 1 \\ 4 & 6 & 2 \\ 1 & 2 & 1 \end{bmatrix}; \quad C_1C_1^{T} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix} \tag{13}$$

are both singular (since for $C_1{}^TC_1$ the sum of the first and third columns equals the second, and for $C_1C_1{}^T$ the second and third columns are identical). The word-

ing in the dichotomous pseudoinverse definition of Ref. [11] indicates that when C has more columns than rows, then there is no worry that (CC^T) could be singular. The following example demonstrates that this is not the case. Another representative nonsquare matrix that does not fit into either dichotomous alternative offered in Ref. [11, p. 19] of the above Eqs. (9) or (10) is

$$C_2 = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 2 & 5 & 3 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix} \tag{14}$$

being of rank 2. That both (9) and (10) fall short of adequately handling every case is not just a consequence of the matrix being used for the counterexample being square, since the counterexample of (14) is a rectangular matrix.

This apparently fundamental conceptual error, overlooked in the methodology advocated in Ref. [11] for pseudoinverse calculation, persists in the first through eighth (current 1984) printing.

As indicated in Ref. [100, Sections 7.1 and 7.2], if the general rectangular $m \times n$ matrix C is of rank $r(>0)$, then it admits to a rank factorization of the form

$$C^{(m\times n)} = D^{(m\times r)} E^{(r\times n)} \tag{15}$$

and the corresponding Moore–Penrose generalized inverse (or as simply designated, the pseudoinverse of C) is

$$C^\dagger = E^T(EE^T)^{-1}(D^TD)^{-1}D^T \tag{16}$$

or simply

$$C^\dagger = E^\dagger D^\dagger. \tag{17}$$

The results of (15)–(17) are what is neglected in the textbook of Ref. [11] that would make the story complete. Otherwise, the expressions of (9) and (10) merely represent the right and left inverses of C, respectively, when they exist [124, p. 70]. However, finding the necessary factorization/decomposition indicated in (15) and unequivocally establishing the true rank r are computationally relegated to the use of the SVD, as discussed further in Section III.

While some matrix pseudoinverse examples can be calculated easily because the structure degenerates to a very simple form (as illustrated in the block upper triangular examples of Section V), the calculation of the matrix pseudoinverse in the general case is well known to be quite a formidable problem, as simply discussed in Refs. [51] and [136] for hand calculations. However, for the higher-dimensional matrices to be encountered in most practical applications that are intractable by hand calculation, an accepted computer algorithm is apparently available for pseudoinverse calculation, as now reviewed.

III. PROPER CALCULATION OF MATRIX PSEUDOINVERSE BY VALIDATED SOFTWARE: A STATUS REVIEW

An approach that has been endorsed by numerical analysts [13, pp. 257–258; 14, p. 171; 101] for calculating the matrix pseudoinverse of an arbitrary $n \times m$ matrix C is to utilize the well-known singular value decompositions (SVD), represented as indicated below for the following three (exhaustive) cases (where the asterisk denotes the conjugate transpose of the matrix).

Case 1 (m = n).

$$C = U\begin{bmatrix} \Lambda_r & 0 \\ 0 & 0_{n-r} \end{bmatrix} V^* \tag{18a}$$

for r such that $1 < r < n$.

Case 2 (n > m).

$$C = U\begin{bmatrix} \Lambda_m \\ 0 \end{bmatrix} V^*. \tag{18b}$$

Case 3 (n < m).

$$C = U[\Lambda_n \mid 0]V^*. \tag{18c}$$

In the above, U (the orthogonal eigenvectors of C*C, denoted as the left singular vectors of C) and V (orthogonal eigenvectors of CC*, denoted as the right singular vectors of C) are unitary:

$$UU^* = I_n \tag{19}$$

$$VV^* = I_m \tag{20}$$

and Λ_i is a diagonal matrix (not necessarily nonsingular). The pseudoinverse is then available [13, 14] as the corresponding case below.

Case 1' (m = n).

$$C^\dagger = V\begin{bmatrix} \Lambda_r^\dagger & 0 \\ 0 & 0_{n-r} \end{bmatrix} U^*. \tag{21a}$$

Case 2' (n > m).

$$C^{\dagger} = V \begin{bmatrix} \Lambda_m^{\dagger} \\ \hline 0 \end{bmatrix} U^*. \tag{21b}$$

Case 3' (n < m).

$$C^{\dagger} = V \left[\Lambda_n^{\dagger} \mid 0 \right] U^*. \tag{21c}$$

where any *i-dimensional diagonal* matrix of the form appearing above,

$$\Lambda_i = \begin{bmatrix} \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_j) & 0 \\ \hline 0 & 0_{i-j} \end{bmatrix} \tag{22}$$

for some fixed j ($1 \le j \le i$) and $\sigma_1 > \ldots > \sigma_j > 0$, the corresponding pseudoinverse, is *defined* to be

$$\Lambda_i^{\dagger} = \begin{bmatrix} \operatorname{diag}\left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_j}\right) & 0 \\ \hline 0 & 0_{i-j} \end{bmatrix}. \tag{23}$$

If r = n in Case 1', then the original square matrix is nonsingular and the pseudoinverse is identical to the standard inverse; consequently, the SVD is not really required where a standard matrix inversion routine will suffice, unless particular caution is being exercised in situations of possible numerical ill conditioning, as can occur in some Kalman filtering applications. {In the calculation of the covariance of estimation error for some Kalman filter applications, the additional expense incurred in utilizing the less severely affected SVD-based procedure is justified to enable added insight into pinpointing any sources of ill conditioning due to pathological circumstances not initially anticipated to be encountered in the application, and to allow the "robustness" of being able to continue the numerical evaluations of covariances unhindered by indicated underflows or by zeros occurring on the principal diagonal that would otherwise halt computations without use of an SVD-based (or UDU^T-based [45]) procedure.} This particular detailed structural specification in terms of the above three separate cases was presented here to avail the reader of the exact form of the correct solution (for ease in actual computational verification) and to avoid any possible slight confusion on where the zeros should occur within the solution matrix, as has arisen in the past in some references (e.g., Ref. [14, Theorem, Section III, p. 166] trying to include the slightly different structural forms encountered in each of these three cases in one apparently

overly compact statement). A brief historical summary of the evolution of the correct SVD solution procedures for each of the cases of square/rectangular matrices having real/complex elements is provided in Ref. [14, introduction to Section III]; thus, no further comments on this topic are warranted here.

Decompositions of the form illustrated in Cases 1, 2, and 3, as appropriate, can be computationally accomplished using a commercially available implementation as an EISPACK [15] software routine (as recommended in Ref. [14, p. 167] as being perhaps the best version for SVD currently available). The LINPACK [26] implementation of SVD is similar to that of EISPACK; however, in the years following [14], it became apparent that there may be a slight error in the LINPACK version. The IMSL version of SVD (as LSVDF) is an implementation based on the routine SVDRs, written by Charles Lawson and Richard Hanson [95], and should be similar to that available with EISPACK. Details of the EISPACK validation and, in particular, the validation of the SVD routine by seventeen cooperative but independent universities and government laboratories across the country, are available in Ref. [15] (which additionally serves to compare the efficiencies of different machines for the same test problems). The practical decision as to which of the σ_i's should be effectively considered to be zero in computer calculations (affected by roundoff and truncation errors) is usually accomplished for so-called "equilibrated" matrices C, i.e., scaled such that $\sigma_1 = 1$, by a simple comparison to a tolerance threshold consisting of the larger of the following two quantities: (1) the square root of the particular machine's precision or, (2) a constant reflecting the uncertainty in the data comprising the most uncertain element of the matrix C [14, p. 171]. Alternative suggestions also exist (e.g., Ref. [51, p. 71]) for the proper choice of a decision tolerance threshold to determine an effective zero.

Additionally, a so-designated "backwards error analysis" has been previously performed by Wilkinson and Reinsh for the SVD implementation utilized in EISPACK so that an approximate measure of the "condition number" [13] is ostensibly available (as asserted in Refs. [15, p. 78; 27]) for user monitoring as a gauge of the degree of numerical ill conditioning encountered during the computations that consequently dictate the degree of confidence to be assigned to the final "answer" that is ultimately output. (Less reassuring open research questions pertaining to SVD condition numbers are divulged in Refs. [33, 38], indicating that some aspects of its calculation are still open questions, even though the earlier user manual [15] offers only reassurances of its validity.) An update to the condition number calculation has recently become available [53; cf. Ref. [121, pp. 289–301].

Upon computationally completing the indicated decomposition for the particular case of (18) via an SVD software routine, the reciprocals indicated in (23) should be performed; then the corresponding recombining matrix multiplications of (21) carried to completion to yield a valid matrix pseudoinverse (of minimum norm, denoted as a Moore–Penrose generalized inverse), as the primary goal. While other approaches to matrix pseudoinverse computation are available (e.g., Refs. [19, 109]), only the SVD-based approach has been successfully validated to this author's knowledge. However, other approaches to SVD computation have

recently emerged [47, 48] and are undergoing further independent testing/corroboration, including a version that is feasible as a systolic array [71], implementable using VHSIC or other commercially available multiprocessor chips such as the NCR45CG72 geometric arithmetic parallel processor (GAPP).

IV. A SURVEY OF PSEUDOINVERSE APPLICATIONS

As further motivation warranting such detailed consideration of the above SVD computational issues, consider the following reasonable prediction in Ref. [14, p. 166]: "It is likely that within five or ten years SVD will be one of the most important and fundamental working tools for the control/systems community, particularly in the area of linear systems." Evidence substantiating the validity of this prediction is in Ref. [128]. Indeed, the usefulness of the SVD in accomplishing a factorization generally required in order to apply the so-called surely locally unbiased (SLU) decentralized filtering approach [24, 32, p. 17], but apparently not previously recognized, has recently been demonstrated in Ref. [25, Section 2.2.2]. New high-resolution signal detection also exploits use of SVD [99, 104–106, 116]. {However, some complaints by way of counterexamples do exist to the use of the pseudoinverse in lieu of the true inverse in the calculation of array gains in the direction of jammers when using the sampled matrix inversion (SMI) technique for adaptive beamforming and in estimating jammer bearings via the maximum likelihood (ML) method in spectral analysis [106]. Perhaps these counterexamples should be more thoroughly re-examined.} Additionally, SVD variants constitute the preferred method of verifying or establishing positive definiteness or semidefiniteness of matrices encountered in realistically dimensioned computer problem simulations and typical industrial implementations [93]. The SVD has been referred to in Refs. [14, 96] as the only reliable method for determining the rank of a matrix. However, for large symmetric matrices of dimension 200, and even up to 500, Lanczos techniques [97] may be the preferred approach for eigenvalue/eigenvector determination. Practical applications of the Moore–Penrose generalized inverse (which is a continuous operator [29]) already abound in statistics [28], electrical engineering [29, Chapter 5], and linear programming [29, Chapter 11]. Reference [29] also offers multitudinous applications of the extremely useful, alternative Drazin inverse in such areas as Markov chains and linear systems theory [29, Chapters 8 and 9]. Generalized inverses of polynomial matrices of the form of transfer function matrices (normalized by multiplication throughout by the least common denominator), as encountered in linear systems theory in the frequency/transform domain, have also been considered [30]. The topic of generalized inverses has also been considered in the design of Luenberger observers [50, 119] and in image restoration and pattern recognition [129]. The importance of this pseudoinverse topic is illustrated in a particularly lucid application example appearing in Ref. [12, p. 162], where employing the pseudoinverse neatly provides the explicit minimum energy optimum control solution for a

linear system with specified initial conditions, final conditions, and final time, as demonstrated in Appendix C. The expert system symbolic manipulation program MACSYMA, developed over approximately ten years at MIT (now available from Symbolics, Inc., and from National Energy Software/Argonne National Laboratory) has recently been successfully used by Charles Stark Draper Laboratory in providing explicit pseudoinverse representations for concatenations of direction cosine matrices representing successive rotations or the referencing of one coordinate frame to another, as occurs with robotic linkages and navigation systems. The matrices encountered in this application routinely involve numerous trigonometric functions that must be manipulated and ultimately simplified, using identities. Estimation theory examples, where the correct computation of the matrix pseudoinverse is *critical* for proper performance, are discussed in Section V.

V. DEPENDENCE OF MINIMUM VARIANCE REDUCED-ORDER FILTERING ON THE PSEUDOINVERSE

The standard linear dynamical system for which Kalman-type filters are designed has a discrete-time representation consisting of an n-dimensional state vector x_k and a p-dimensional measurement vector z_k of the following well-known form:

$$x_{k+1} = \Phi x_k + w_k \tag{24}$$

$$z_k = Hx_k + v_k, \tag{25}$$

where w_k and v_k are zero-mean, white Gaussian process and measurement noises (independent of the Gaussian initial condition) of covariance level Q and R, respectively. The usual conditions of observability/controllability (or less restrictive detectability/stabilizability conditions [82, p. 82], or even nondetectable [134]) are assumed to be satisfied here by the system of (24) and (25). [Only the standard unadorned form of the linear estimation/filtering problem is addressed here [81]. Straightforward modifications of what is offered here can be routinely accommodated according to existing accepted techniques in order to handle the less standard, but still tractable, variations of linear filtering, such as the cases of having a singular R (i.e., some measurements being uncorrupted by the measurement noise), of a system (24) failing to be process noise controllable/stabilizable and having cross-correlated process and measurement noises, time or serially correlated Gaussian noises, nonindependent initial condition, non-Gaussian initial condition and noises (but distributed according to an "elliptical" family). There can be comprehensive handling of filter replicates for multitarget tracking, the Magill–Laniotis bank-of-filter multimode hypothesis discrimination, use of noises that are centered white Poisson processes rather than Gaussian, and

use of mixed combinations of Gaussian and point processes for tracking. These variations are also of significance in seeking something other than LQG feedback control of stochastic systems where filtering is to be performed first to obtain adequate estimates of the current state of the system before corrective control is applied.]

Use of a reduced-order suboptimal filter model of a smaller dimension m (<n) is frequently necessitated to meet constraints on the computational capacity available for the filtering function on board ships or aircraft in aided navigation applications [18, 34, 43, 62, 65, 92]. However, standard *a priori* covariance analysis can still be carried out to account for this expedient reduced-order approximation by establishing the anticipated realistic error of estimation as obtained by *both* (1) acknowledging use of a reduced-order filter model as required as a compromise fit to computer resources available for the particular application, and (2) utilizing any known higher-order real-world model of dimension n ($m < n$). Standard conventional methodology for accomplishing such an exacting evaluation [11, Chapter 7; 17; 18, pp. 325–341; 63, 88] usually involves working with an augmented state vector of the form

$$x'(t) \triangleq \begin{bmatrix} Tx(t) - \hat{x}(t) \\ - - - - - \\ x(t) \end{bmatrix} \tag{26}$$

where, in the above, $T^{(m \times n)}$ (usually consisting of only zeros and ones) is used to circumvent a dimensional incompatibility that would otherwise exist. The second moment of the augmented vector x', being $E[x'(t)x'(t)^T]$, satisfies a so-called time-varying Lyapunov or variance equation that constitutes a computer burden that goes as $(n + m)^3$ following the completion of a first pass of order m^3 needed to specify (and store for use in the final pass) the suboptimal filter gains as calculated by the reduced-order filter (cf. operations counts in [94]).

A self-contained original derivation of the so-called minimum variance reduced-order (MVRO) filter introduced in [16, 35, 36, 111; cf. Ref. 118] is provided below by following the basic steps of Ref. [36], but by further augmenting them here to include subtle crucial intermediate steps and to explicitly feature critical assumptions (left implicit in Refs. [16, 35, 36, 111]), and to further elucidate previously unacknowledged limitations in the MVRO approach as it currently stands.

Suppose that the reduced-order filter to be utilized models a subset of m states of the full n states, x_k, as represented by

$$y_k = T^{(m \times n)} x_k \tag{27}$$

where the transformation T (possibly time varying as treated in Ref. [36, Eq. (5)] and Ref. [16, Eq. (29)] but *not* assumed to be so here for simplicity, and as the prevalent case to be encountered in practice of having fixed constant state selection) effectively serves to delete undesired less important states from consideration in the filter model.

The following three assumptions are implicitly utilized in Refs. [16, 35, 36].

Assumption 1 (state subset selection).

$$TT^{\dagger} = I_{m \times m} \tag{28}$$

[(28) can usually be met when availed of complete freedom in selecting T. As noted following [16, Eq. (2)], $T^{\dagger}T$ is an $n \times n$ matrix which is *not* the identity matrix.]

Assumption 2 (extrapolate structure).

$$\hat{y}_{k+1|k} = T\Phi T^{\dagger} \hat{y}_{k|k}. \tag{29}$$

Assumption 3 (update structure).

$$\hat{y}_{k|k} = \hat{y}_{k|k-1} + K_k [z_k - HT^{\dagger} \hat{y}_{k|k-1}]. \tag{30}$$

(Assumptions 2 and 3 can also be interpreted as computational constraints being complied with in using an MVRO filter.)

If the residual in (30) were in fact $[z_k - H\hat{x}_{k|k-1}]$, then the filter gain in (30) should be

$$K_k = TK_k^* \tag{31}$$

(where K_k^* is the standard well-known Kalman gain), as could be justified by premultiplying the standard Kalman filtering mechanization equations throughout by T. However, since the residual term in (30) *is* nonstandard, K_k is open for specification, as pursued in the MVRO approach to minimize the variance of estimation error as reflected in the following criterion:

$$\mathrm{tr}\tilde{P}_{k|k} \triangleq \mathrm{tr}\, E\left[\tilde{e}_{k|k}\tilde{e}_{k|k}^T\right], \tag{32}$$

where

$$\tilde{e}_{k|k} \triangleq \hat{y}_{k|k} - Tx_k. \tag{33}$$

It is frequently more convenient to work instead with

$$e_{k|k} \triangleq T^{\dagger}\hat{y}_{k|k} - x_k, \tag{34}$$

which may be premultiplied by T throughout to yield

$$Te_{k|k} = T\left(T^{\dagger}\hat{y}_{k|k} - x_k\right) \tag{35a}$$

$$= TT^{\dagger}\hat{y}_{k|k} - Tx_k. \tag{35b}$$

By Assumption 1 in (28), (35b) simplifies to yield

$$Te_{k|k} = \hat{y}_{k|k} - Tx_k \triangleq \tilde{e}_{k|k}. \tag{36}$$

Now, utilizing the constrained structural form of (30) to substitute for $\hat{y}_{k|k}$, (34) can be re-expressed as

$$e_{k|k} = T^{\dagger}\left(\hat{y}_{k|k-1} + K_k[z_k - HT^{\dagger}\hat{y}_{k|k-1}]\right) - x_k \tag{37a}$$

$$= (I - T^{\dagger}K_kH)e_{k|k-1} + (I - T^{\dagger}K_kH)x_k - x_k + T^{\dagger}K_kHx_k + T^{\dagger}K_kv_k \tag{37b}$$

$$= (I - T^{\dagger}K_kH)e_{k|k-1} + T^{\dagger}K_kv_k, \tag{37c}$$

where a fairly obvious notation consistent with (34) as

$$e_{k|k-1} \triangleq T^{\dagger}\hat{y}_{k|k-1} - x_k \tag{38}$$

is utilized in (37). It should be noted that a term $T^{\dagger}K_kHx_k$ was both added and subtracted in order to obtain (37b).

Using (37c) to form $[e_{k|k}e_{k|k}^T]$, and taking the total expectation throughout, yields:

$$P_{k|k} = [I - T^{\dagger}K_kH]P_{k|k-1}[I - T^{\dagger}K_kH]^T + T^{\dagger}K_kR_kK_k^T(T^{\dagger})^T \tag{39}$$

with

$$P_{k|k-1} \triangleq E\left[e_{k|k-1}e_{k|k-1}^T\right]. \tag{40}$$

Now combining the criterion of (32) to be minimized with (36), (39), and (40) yields:

$$\mathrm{tr}[\tilde{P}_{k|k}] = \mathrm{tr}\left\{E\left[(\hat{y}_{k|k} - Tx_k)(\hat{y}_{k|k} - Tx_k)^T\right]\right\} \tag{41a}$$

$$= \mathrm{tr}\left[T(I - T^{\dagger}K_kH)P_{k|k-1}(I - T^{\dagger}K_kH)^TT^T + TT^{\dagger}K_kRK_k^T(T^{\dagger})^TT^T\right]. \tag{41b}$$

By using standard matrix gradient formulas [37], a necessary condition (that is additionally a sufficient condition for the specific nonnegative definite quadratic form within the trace) for achieving the minimization of the criterion of (32) [re-expressed as (41b)] with respect to the filter gain K_k is

$$0 = \frac{\partial}{\partial K_k}\mathrm{tr}[\tilde{P}_{k|k}] = -2(T^{\dagger})^TT^TTP_{k|k-1}H^T + 2(T^{\dagger})^TT^TT(T^{\dagger})K_k\left[HP_{k|k-1}H^T + R\right] \tag{42a}$$

$$= -2TP_{k|k-1}H^T + 2K_k\left[HP_{k|k-1}H^T + R\right], \tag{42b}$$

where the simplification of (42b) is obtained via Assumption 1 as (28). It then comes immediately from (42b) that the MVRO filter gain that minimizes the error in estimation for filters of the form of Assumptions 2 and 3 is

$$K_k = TP_{k|k-1}H^T\left[HP_{k|k-1}H^T + R\right]^{-1}. \tag{43}$$

To complete the MVRO filter specification, a recursive expression for the time evolution of $P_{k|k-1}$ is needed. Returning to (38) and substituting structural Assumption 2 from (29), yields

$$e_{k|k-1} = T^\dagger T\Phi T^\dagger \hat{y}_{k-1|k-1} - x_k \tag{44a}$$

$$= T^\dagger T\Phi T^\dagger \hat{y}_{k-1|k-1} - \Phi x_{k-1} - w_{k-1} \tag{44b}$$

$$= T^\dagger T\Phi T^\dagger \hat{y}_{k-1|k-1} + \Phi T^\dagger \hat{y}_{k-1|k-1} - x_{k-1} - \Phi T^\dagger \hat{y}_{k-1|k-1} - w_{k-1} \tag{44c}$$

$$= \Phi e_{k-1|k-1} + (T^\dagger T - I)\Phi T^\dagger \hat{y}_{k-1|k-1} - w_{k-1}. \tag{44d}$$

For systems having the following structural form {identified in Ref. [36, following Eq. (17)], but not explicitly identified in Ref. [16] except for a passing allusion in Ref. [16, Eq. (54)]}:

Assumption 4.

$$(T^\dagger T - I)\Phi T^\dagger \hat{y}_{k-1|k-1} = 0. \tag{45}$$

The middle term of (44d) is zero; so (44) reduces to

$$e_{k|k-1} = \Phi e_{k-1|k-1} - w_{k-1}. \tag{46}$$

from which

$$P_{k|k-1} \triangleq E\left[e_{k|k-1}e^T_{k|k-1}\right] \tag{47a}$$

$$= E\left[\Phi e_{k-1|k-1}e^T_{k-1|k-1}\Phi^T\right] + E\left[w_{k-1}w^T_{k-1}\right] \tag{47b}$$

$$= \Phi P_{k-1|k-1}\Phi^T + Q \tag{47c}$$

since cross terms between $e_{k-1|k-1}$ and w_{k-1} are uncorrelated and have zero mean. Use of (47c) for covariance propagation in conjunction with (39) for updating the covariance to reflect measurement incorporation completes the specification of the MVRO filter of (29), (30), and (43), as summarized in Table 1 in juxtaposition with the standard mechanization equations of Kalman filtering to facilitate a later comparison.

If the condition of Assumption 4 in (45) does not hold, then MVRO filtering can still be done by state augmentation, as offered in Ref. [16, Eqs. (42)–(52)], but the computational burden is then of the order of $(n + m)^3$, and thus of no significant computational benefit over the conventional approaches of Refs. [11, 17, 18]. This is a point that had not previously been adequately emphasized.

There are generally two well-known ways that this rather severe structural constraint of (45) can be satisfied. One way this constraint is satisfied is when

$$(T^\dagger T - I)\Phi T^\dagger = 0, \tag{48}$$

as occurs when the structure of the original system of (24) is block upper-triangular *and* the filter model states constitute a *precise proper subset* of the upper truth model states with a line of demarcation that matches exactly the block partitioning of the original system (as obtained in exhibiting its block upper-triangular structure). The other way the constraint of (45) can be easily satisfied, even if (48) does not hold, is if control compensation is being performed, such as subtracting off the estimates from the system at the end of each filter measurement incorporation cycle (as is frequently done for navigation applications in the resetting of gyro drift rates and in the correcting of platform tilts through torquing, as discussed in Ref. [18, pp. 306–307]), so that

$$\hat{y}_{k-1|k-1} = 0 \tag{49}$$

and the condition of (45) is thus trivially satisfied. However, all of the states of the filter must be reset to result in (49), but not all applications offer this flexibility or capability. A third milder situation that is less well known (and

TABLE 1. Explicit Comparison Between MVRO and Standard Kalman Filter Implementation Equations[a]

	Standard Kalman filter mechanization	MVRO filter mechanization[b,c,d]
Covariance of estimation error		
Propagate step	$P_{k+1}^{-} = \Phi_{k+1} P_k^{+} \Phi_{k+1}^{T} + Q_k$	$P_{k+1}^{-} = \Phi_{k+1} P_k^{+} \Phi_{k+1}^{T} + Q_k$
Update step	$P_k^{+} = [I - \tilde{K}_k H] P_k^{-} [I - \tilde{K}_k H]^{T} + \tilde{K}_k R_k \tilde{K}_k^{T}$	$P_k^{+} = [I - T^{\dagger} K_k H] P_k^{-} [I - T^{\dagger} K_k H]^{T} + T^{\dagger} K_k R K_k^{T} (T^{\dagger})^{T}$
Filter		
Propagate step	$\hat{x}_{k+1}^{-} = \Phi_{k+1} \hat{x}_k^{+}$	$\hat{y}_{k+1} = T \Phi_{k+1} T^{\dagger} \hat{y}_k^{+}$
Update step	$\hat{x}_k^{+} = \hat{x}_k^{-} + \tilde{K}_k [z_k - \tilde{H}_k^{-}]$	$\hat{y}_k^{+} = \hat{y}_k^{-} + K_k [z_k - H T^{\dagger} \hat{y}_k]^{d}$
Filter gain calculation	$\tilde{K}_k = P_k^{-} H^{T} [H P_k H^{T} + R]^{-1}$	$K_k = T P_k^{-} H^{T} [H P_k H^{T} + R]^{-1}$

[a]Use of plus (+) and minus (–) superscript convention for update and propagate steps, respectively, is the notation popularized in Ref. [11].
[b]Even when the condition of (45) holds, the computational burden of the n-dimensional on-line real-time covariance calculation is seldom acceptable in filtering applications (unless filter gains are precalculated and stored on magnetic tape or disk).
[c]If assumption 4 does *not* hold, then MVRO mechanization is considerably more complex, as explicitly stated in Ref. [16, Eqs. (42)–(51)].
[d]Please note that $y_k = T^{(m \times n)} x_k$.

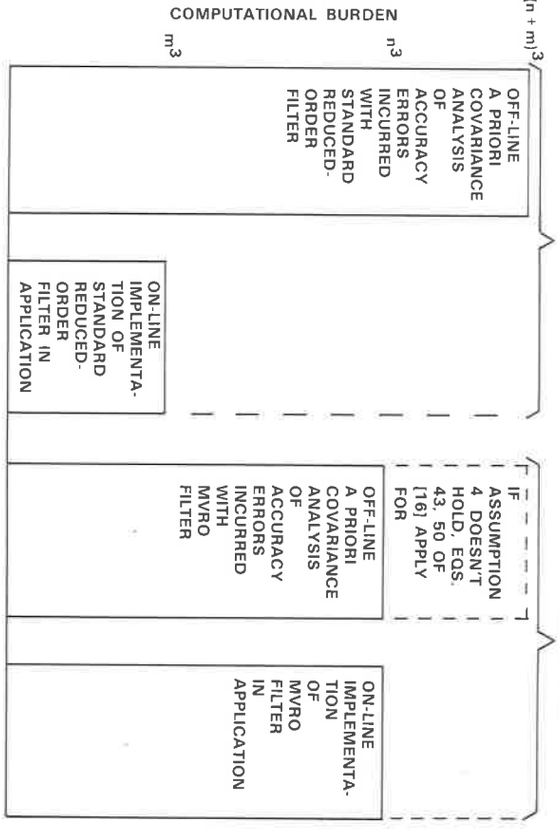

Fig. 1. A comparison of the allocation of computational burden for conventional application filters versus MVRO filters.

seldom occurs, but is indeed a possibility) is for neither (48) nor (49) to be satisfied exactly, but for each to be partially satisfied in such a way that the existing zeros of each exactly cancel the nonzero terms of the other in a complementary fashion in the indicated multiplication of the left-hand side of (48) with the left-hand side of (49) to yield the left-hand side of (45), being appropriately zero to satisfy the condition.

A fundamental comparison of MVRO filtering versus conventional filtering approaches is available from Fig. 1. The penalty associated with an on-line implementation of an MVRO filter is that the covariance calculations are n dimensional (rather than m dimensional for a conventional reduced-order filter). The reduced computational burden of exact *a priori* covariance analysis available with MVRO filters is appropriately consistent only if an MVRO filter is actually used in the physical application. Before an MVRO filter is selected for use in an application, the benefit of the *less* burdensome *a priori* MVRO covariance analysis should be balanced with a consideration of the *greater* burden to be encountered in an on-line MVRO implementation (that could perhaps even preclude satisfying the required constraint of being a real-time mechanization). The point that is being made here for the first time is that the computational burden of MVRO in the on-line implementation is greater than that of a conventional reduced-order filter implementation of the same number of states. This revelation is substantiated by the detailed consideration of the underlying MVRO derivation, as recounted herein with emphasis on previously unstated structural assumptions.

A direct explicit comparison between the mechanization equations of a standard Kalman filter (of dimension n) and an MVRO filter (of dimension m, m < n) is provided in Table 1. Please note that the covariance calculations of both approaches are of dimension n. Also notice that any standard filter mechanization (or any conventional suboptimal filtering analysis mechanization) may be expediently converted to an MVRO mechanization simply by the additional inclusion or insertion of the appropriate transformation matrices T and $T^\dagger$, as indicated in the bottom line of Table 1. Notice the ample opportunity of MVRO results to be adversely affected by any errors in the pseudoinverse calculation since, as clearly portrayed in Table 1, three of the five MVRO mechanization equations rely explicitly upon employing the correct pseudoinverse. While only the case of a time-invariant transformation matrix T was considered here (since most physical applications use a constant subset of truth model states as the filter model states for the duration of a mission), where only *one* pseudoinverse computation of $T^\dagger$ suffices for this MVRO mechanization, the presentations in Ref. [36, following Eq. (5)] and Ref. [16, following Eq. (29)] deal with a time-varying transformation T_k that therefore exhibits increased MVRO sensitivity to pseudoinverse calculation. This increased sensitivity occurs because a new $T_k^\dagger$ is needed at each time step (see Ref. [49] also). In addition, a further consideration is that pseudoinverse computations are fairly time-consuming calculations, except for extremely simple degenerate cases.

Since structural Assumption 1 was utilized twice in developing the equations describing the MVRO framework, this condition of (28) is now examined to demonstrate that it is sometimes satisfied by coincidence or by objective selection in specific applications. To motivate how the condition of (28) could be invoked so readily, please consider the following example.

A. A BLOCK UPPER-TRIANGULAR EXAMPLE

Consider the following time-invariant continuous-time block upper-triangular system of the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & F_{22} & F_{23} \\ 0 & 0 & F_{33} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix} \tag{50}$$

(where the discrete-time formulation has the corresponding block upper-triangular structure) that is to be tracked from its measurements by a filter that estimates the block subset of states

$$y(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{51}$$

where x_1, x_2,.and x_3 are of dimension n_1, n_2, and n_3, respectively. A straightforward observation is that

$$y(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \tag{52}$$

From (52), the transformation T is therefore implicitly defined to be

$$T = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \end{bmatrix} \tag{53}$$

and, consequently,

$$T^\dagger = T^T(TT^T)^{-1} \tag{54a}$$

$$= \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \\ 0 & 0 \end{bmatrix} \right)^{-1} \tag{54b}$$

$$= \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \\ 0 & 0 \end{bmatrix} \tag{54c}$$

so structural Assumption 1 is indeed satisfied, since

$$TT^\dagger = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \\ 0 & 0 \end{bmatrix} \tag{55a}$$

$$= \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} = I_{n_1 + n_2}. \tag{55b}$$

Even if the constraint of the application is that no resets can be performed to satisfy (49), a system structure such as (50) and (53) also leads to the condition of (45) holding, since the transition matrix corresponding to the dynamics matrix of (50) is of the form

$$\Phi = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \tag{56}$$

which simply yields the following calculation:

$$(T^{\dagger}T - I)\Phi T^{\dagger} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_{n_3} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \\ 0 & 0 \end{bmatrix} \tag{57a}$$

$$= 0. \tag{57b}$$

B. AN EXAMPLE THAT IS NOT BLOCK UPPER-TRIANGULAR

A different selection for a filter model that is also a proper subset of the block upper-triangular truth model is

$$y(k) = \begin{bmatrix} x_2(k) \\ \text{- - - -} \\ x_3(k) \end{bmatrix} \tag{58}$$

as represented by the transformation

$$T = \begin{bmatrix} 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}. \tag{59}$$

For this example, the critical Assumption 1 [as (28)] holds, and the pseudoinverse again degenerates to $T^{\dagger} = T^{T}$, but Assumption 4 is violated in general [since (48) does *not* hold] unless it is possible (and practically desirable from the standpoint of convenience and economic feasibility) to implement control compensation in the particular application, such that (48) holds. Thus, two cases, one conveniently tractable and one less tractable, have been presented to illustrate realistic MVRO implementation considerations for a time-invariant block upper-triangular system truth model, such as is frequently encountered in realistic navigation and guidance (and even passive bearings-only sonobuoy target tracking) filter applications (e.g., [34, 43, 55–62, 72–80, 146, 147]). A block-diagonal example and one- and two-state filter subsets of an inherent third-order system are treated in Ref. [16, pp. 790–791] with all details visible, while the more realistic navigation examples involving 18-state, 15-state, and 11-state MVRO implementations treated in Ref. [16] unfortunately do *not* expose any of the detailed considerations needed to establish that the Assumptions 1–4 hold, as has been demonstrated here to be of fundamental importance.

Several useful surveys of Kalman filter procedures, experiences, and successes have been offered over the years [54, 64, 65], the most recent being in Ref. [54]. In Ref. [54, p. 503], it is stated that "systematic implementation and analysis of reduced-order models were unavailable" prior to the introduction of MVRO (in 1972 [35] to 1973 [36]). The use of MVRO is advocated in Ref. [54] as a systematic way to handle reduced-order filtering applications and is asserted to be much preferred to current covariance and Monte Carlo simulation techniques, which are referred to in Ref. [54] as being "expensive in both time and computer cost" and essentially "*ad hoc*." On the face of it, MVRO offers a type of analytic beauty that appears to continually entice unwary onlookers (in 1973 [35], in 1979 [44], in 1981 [63], and in 1986 [111]). The other worrisome aspect is encountering these exhortations to use MVRO without corresponding explanations, such as that provided here, of what lies beneath the surface (or what likely disappointments each new enthusiast should expect to encounter after devoting time and effort toward implementing this apparent panacea). It is acknowledged in a somewhat astonished tone in Ref. [54] that conventional approaches to reduced-order filtering continue to be routinely used in practice rather than MVRO. This section explains why, in general, the conventional reduced-order filtering practices are apparently the prudent approach to follow, and should be continued until MVRO is refined beyond its current state, such that the open questions raised herein are adequately answered and resolved. Much stronger objections to the use of MVRO (along other lines) appear in [140] than (charitably) are raised here; these are endorsed here as valid criticisms. The scrutiny of [140] was evidently spawned by my earlier critique of [112].

Other objections to standard reduced-order filtering practice (also embracing the MVRO approach) have been raised in the past [31] on the grounds that explicit consideration of the bias introduced in utilizing a reduced-order filter had not yet been explicitly evaluated. However, current practice is to perform a representative off-line bias evaluation tailored to the intended mission scenario to enable rigorous verification of a satisfactorily low bias magnitude such that the

bias does *not* interfere with the goal of filtering: "to tractably provide reasonably close estimates on-line in real time of the true states of interest."

A UDU^T-type square-root factorization of the type associated with Bierman [45] has more recently been performed for the MVRO [44] under somewhat restrictive conditions. These conditions are [44, p. 578] that:

1) The block-partitioned state vector is arranged so that

$$y = [I \mid 0]\left[\begin{array}{c} x_1 \\ \hline x_2 \end{array}\right]; \tag{60}$$

2) The filter error estimate is instantaneously reset to zero through impulsive control;

3) The measurement covariance matrix R is diagonal.

It is further suggested in Ref. [44, p. 578] that the condition of 1) is *not* restrictive since such ordering of the state vector can be either performed[1] by the analyst in setting up the problem or be automatically performed via pointer arrays in the resulting computer code, 2) will depend on the structure and flexibility of the particular application, and 3) can be attained, if not already present, by using well-known "whitening" procedures. It is agreed here that 3) can be attained using data "whitening" procedures since

$$y' = R^{-1/2}y = R^{-1/2}Hx + R^{-1/2}v \tag{61a}$$

$$= H'x + v' \tag{61b}$$

$$E\left[v'_k v'^T_k\right] = I. \tag{62}$$

However, we note here that these whitening procedures are cumbersome if R is time varying. Moreover, it is noted here that such a whitening would violate any state rearrangement that resulted in the condition of 1) as (60) being satisfied. Thus, in general, it appears that requiring an arbitrary system to satisfy both 1) *and* 3) is contradictory, while either 1) *or* 3) is routinely achievable.

Besides the use of MVRO and safer conventional approaches to reduced-order filtering discussed at the beginning of Section IV, many other novel approaches to reduced-order filtering exist [82–85, 102, 103, 107, 108, 117, 118], but the utility of these other results to navigation applications is yet to be demonstrated. Indeed, kneejerk response commentaries/caveats for some of these other approaches are that:

1) Reference [82] requires that a full two-point boundary-value problem (TPBVP) be solved for reduced-order filtering. (While solving backward and forward in time for Kalman smoothing as a Bryson–Frazier two-filter smoother is standard practice [87], having this comparable computational burden for just filtering is somewhat unexpected, i.e., defying prior physical intuition, and would tend to preclude a convenient real-time implementation.)

2) Reference [83] is less computationally burdensome than Ref. [82], but requires that the state size of the reduced-order filter be the same as the dimension of the measurements in (25) [while the identities of the original underlying states x in (24) are lost]. While the approach of Ref. [83] appears to be mathematically correct, it is of almost no interest for most navigation applications where the dimension of the underlying states can be fairly large: 15–100 states, while the dimension of the measurements is usually fairly small (on the order of 2 or 3). If the method of Ref. [83] were applied to these navigation applications, the dimension of the resulting reduced-order filter would correspondingly be only 2 or 3, respectively, depending on the actual measurement dimension. This is a severely confining restriction on the dimension of the tolerable reduced-order filter that is to capture the essence of the system's underlying dynamics.

3) While Ref. [84] offers a wonderful historical survey and insightful revelations into the various alternative approaches developed to handle filtering situations with some noise-free measurements present, the technique that is developed and advocated in Ref. [84] invokes a similarity transformation where, in general, underlying physical state identities are lost. Practical navigation applications routinely utilize reasonableness tests based on anticipated behavior of the physically enumerated states. In order to employ reasonableness tests (e.g., Ref. [89, p. 288]) for the reduced-order filter of Ref. [84], such tests would have to also be converted to the newly established coordinate system and later backed out for problem isolation (thus constituting a nontrivial computational burden). The advanced algebraic techniques of Ref. [84] are applicable only for time-invariant systems, while many navigation applications are inherently time varying due to the way specific forces and/or gravity anomalies/vertical deflections are handled [114].

4) Reference [85] is in the same vein as Ref. [83], but remedies many of the above-mentioned concerns. However, Ref. [85] is applicable only to time-invariant systems (see caveats above for Ref. [84]) and (as acknowledged in Ref. [85]) does not offer a numerical method for solving the simultaneous matrix equations that arise, nor does it offer a way of avoiding spurious solutions associated with likely multiple local minima that would satisfy the same optimal projection equations. More significant is that the problem formulation is only concerned with asymptotically good estimation and tracking as time gets large rather than being concerned with good tracking for finite horizon mission epochs of usual concern in navigation.

Perhaps newly emerging approaches to reduced-order modeling, such as in Refs. [102, 103, 107, 108, 138, 139, 141–144], will be more fruitful. References [102, 103] require further optimization operations and by so doing depart from a standard Kalman filter formulation, with an additional computational burden incurred. Reasonable complaints have already been raised in Ref. [120] concerning the approach of Refs. [102, 103]. The author also has strong reservations regarding the practical applicability of Refs. [117, 118], but constraints on space

[1] However, an explicit algorithm for achieving this prescribed reordering was not provided in Ref. [44], nor referenced as being easily available.

prevent further elaboration here. References [108], [139], and [141] appear to be particularly promising for future filtering applications.

VI. SUMMARY

Simple nonpathological counterexamples have been provided to demonstrate that two recently proposed algorithms for matrix pseudoinverse calculation are unsatisfactory by not giving the correct answer in the general case. The elements of a widely endorsed computational approach for calculating the matrix pseudoinverse of Penrose, based upon an SVD algorithm, as available in either an EISPACK or IMSL software package, were reviewed as a correct approach to solving this problem. The reader was also availed with insights into related issues and less-well-known open questions and the corresponding followups pertaining to currently used approximate estimates of "condition numbers" employed as a gauge of numerical ill conditioning actually encountered in pseudoinverse calculation of specific matrices. Pointers were supplied to constructive impacts of pseudoinverse calculation across a fairly broad spectrum of application areas as the reason the applications-oriented engineer should be concerned about the proper calculation of the pseudoinverse.

Focusing attention on a likely beneficial impact of pseudoinverse calculation in estimation theory for navigation applications, the so-called minimum variance reduced-order design methodology for selecting reduced-order filters was explored in detail here. The outputs of this investigation are:

1) Analytical statements of inherent assumptions and conditions being offered here as clarifications of requirements that must be met in order to validly use MVRO (but were not previously made explicit).

2) A revelation of the heavy reliance on a *correct* matrix pseudoinverse computation within the MVRO filtering mechanization equations, thus evidence exists of:

   a) Sensitivity of MVRO to computation time expended in forming the pseudoinverse;

   b) Sensitivity of MVRO performance to accuracy of pseudoinverse calculations as a required intermediate computation.

3) A balanced treatment of *both* benefits and drawbacks that should be considered in pursuing an MVRO implementation for a particular application.

Throughout this investigation, constructive remedies were offered whenever possible: 1) to strengthen observed weakness in previously recommended approaches for pseudoinverse calculations and, 2) to bolster the MVRO design methodology.

By alerting Kalman filter practitioners to these weaknesses associated with MVRO, the previous pitfalls can be circumvented. It is hoped that the cautions extended here are viewed constructively along with the following other warnings in the Kalman filter analysis and application area:

1) On limitations of a structural reformation for solving an algebraic Riccati equation [68] associated with filtering;

2) On some pitfalls in seeking to use age-weighted filters [66];

3) On problems in particular formulations of extended Kalman filtering [69] (with additional remedies offered in Ref. [86]);

4) On problems in the early analytic proofs of the stability of the Kalman filter [67; 70, Appendix C] and even in later proofs (see Ref. [25, Section 4.2 and Appendix A.1] for additional occurrences and ramifications);

5) On tradeoffs existing between degree of accuracy achievable versus computational time delay incurred (as gauged in terms of operations counts) for several popular alternative square-root filter formulations [18, Chapter 7].

6) Some problems relating to the lack of numerical stability when using the widely hailed "Schur Approach" for calculating Matrix Riccati equation solution [135];

7) Some problems associated with certain approaches to Matrix Spectral Factorization (as discussed in [137]) in putting a problem involving serially time-correlated additive noise into standard Kalman filter form (which expects only uncorrelated white additive noises) via "state augmentation" [11, pp. 133–135].

VII. APPENDIX A: THEORETICAL BASIS FOR THE PSEUDOINVERSE OF A CONTINUOUS LINEAR TRANSFORMATION

The significance of working with Hilbert spaces as done here is simply explained in Ref. [12, Chapter 3] as being motivated by the capability to carry over geometric intuition and experience honed in two- and three-dimensional Euclidean space to other situations [133], such as in achieving a better understanding of the underlying geometric structure of spaces of functions given that the *domain space* G is a *Hilbert space* (i.e., it is a linear vector space that has an *inner product* and is "complete" in the sense that all Cauchy sequences converge to a point within the space without "gaps").

H, the *range space*, is a Hilbert space.

θ_x and θ_y are the *null elements* or *additive identities* in the linear vector spaces G and H, respectively.

B(G, H) is notation for the class of *bounded* (*continuous*) *linear functions* from domain G *into* range H (notation: $f:G \rightarrow H$ or the function f *maps* G into H and f is continuous on G).

f is a function from B(G, H).

The main idea associated with the above preliminary definitions is portrayed geometrically in Fig. 2.

In Fig. 2, $\mathcal{R}(f)$ is the notation for the *range of f*. The fact that f is merely *into* H means that there may exist y in H such that there is no $\overset{\circ}{x}$ in G with

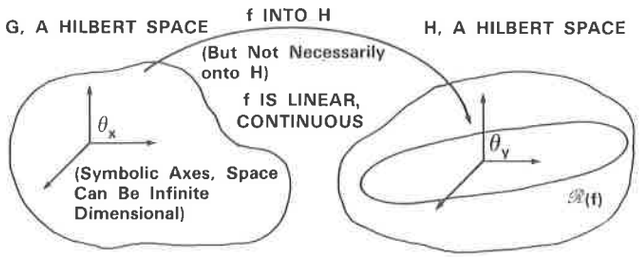

Fig. 2. Function f maps the domain space G into the range space H.

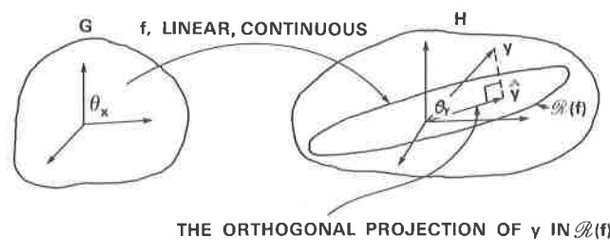

Fig. 3. Projecting y into $\mathscr{R}(f)$ by the Hilbert space projection theorem.

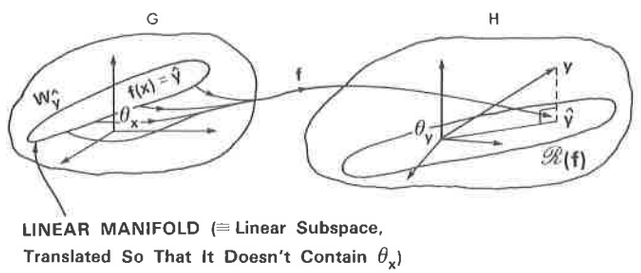

Fig. 4. Obtaining the manifold of x values such that $f(x) = \hat{y}$.

$$f(\overset{\circ}{x}) = \overset{\circ}{y}. \tag{63}$$

If f were *onto* H, then for every y in H, there would exist an $\overset{\circ}{x}$ in G such that (63) holds. If, in addition, f were *one-to-one* (notation: 1–1) then, for every $\overset{\circ}{y}$ in H there would exist only *one* x in G, such that (63) holds (i.e., two or more points in G could not map into the same y value in H). If f were 1–1 *and* onto, then an ordinary inverse function would exist and there would be no need to consider a pseudoinverse since it would reduce to this *unique* ordinary inverse. The situations of interest for potential use of the pseudoinverse occur when f is *not* 1–1 and onto. (For finite-dimensional matrices, these cases of interest correspond to either the situation of a square matrix being singular or the situation of having a nonsquare matrix where the finite-dimensional Euclidean domain space has a dimension different from that of the Euclidean range space.

The range of f, $\mathscr{R}(f)$ is a *linear subspace* (in analogy to a plane through the origin in Euclidean 3-space, E^3). If G or H is finite dimensional (or if it can be otherwise shown), then $\mathscr{R}(f)$ is *closed* [125]. Consequently, $\mathscr{R}(f)$ is a *closed linear subspace*. This is just what is needed in the hypothesis to apply the Hilbert space projection theorem [126, p. 76, Theorem 4.11] in specifying what rigorously constitutes a pseudoinverse of the transformation f.

A. CONSTRUCTION OF THE GENERAL PSEUDOINVERSE

Given a particular arbitrary y in H (notation: $y \in H$), consider all the x_1 in G (notation: $x_1 \in G$) such that

$$\|f(x_1) - y\| = \min_{x \in G} \|f(x) - y\|. \tag{64}$$

The norm $\|\cdot\|$ here is defined naturally in terms of the existing inner product associated with the Hilbert space as $\|\cdot\| = \sqrt{(\cdot \mid \cdot)}$.

Approximation of y in H. As seen in Fig. 3,

$$\min_{x \in G} \|f(x) - y\|$$

exists since, by the Hilbert space projection theorem, any arbitrary y in H may be approximated in the closed linear subspace $\mathscr{R}(f)$ by its orthogonal projection, $\hat{y}$. Since $\mathscr{R}(f)$ is the range space of f, there exists an x_1 in G [not necessarily unique, as shown in Fig. 4 (since there may be more than one x_1 value in G)], such that

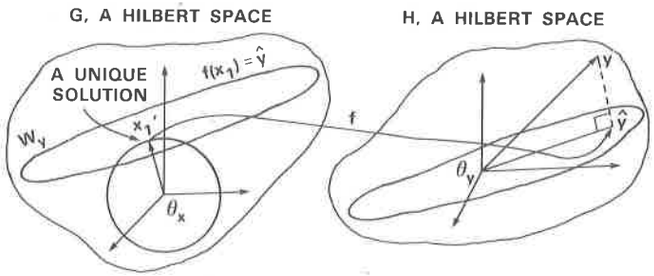

Fig. 5. x_0' is the unique point of $W_{\hat{y}}$ which has minimum norm.

$$f(x_1) = \hat{y} \tag{65}$$

[this occurs since f is onto $\mathscr{R}(f)$].

Therefore, the following equalities hold:

$$\|f(x_1) - y\| = \min_{x \in G} \|f(x) - y\| = \min_{p \in \mathscr{R}(f)} \|p - y\| = \|\hat{y} - y\|. \tag{66}$$

The following device is used to obtain a unique x_1 that satisfies (65).

Uniqueness by Minimizing Norm in G. Represent the linear *manifold* of Fig. 4 by

$$W_{\hat{y}} = \{x_1 \text{ such that } f(x_1) = \hat{y}\}. \tag{67}$$

That (67) describes a linear manifold (also known as a linear *variety* or linear *flat*) can be demonstrated by showing that $W_{\hat{y}}$ satisfies the requirements to be classified as a linear manifold as presented in Ref. [12, Chapter 3]. Now

$$W_{\theta_y} = \{x_1 \text{ such that } f(x_1) = \theta_y\} \triangleq \text{null space of } f = \eta(f) \tag{68}$$

so $W_{\hat{y}}$ is just a translation of the null space of (68). While $W_{\hat{y}}$ is not a subspace, since it does not contain θ_x, it is a complete convex subset of Hilbert space; hence, by Ref. [126, p. 78, Theorem 4.10], there is a unique point x_1' which is

closest to the origin (the null element) θ_x of G. Figure 5 amply portrarys this situation.

The point of tangency of the smallest ball, centered at the origin of θ_x which interesects $W_{\hat{y}}$, is x_1'. The point x_1', defined in this way, is now unique. The above-described mechanism of *associating a unique x_1' with an arbitrary fixed y such that x_1' satisfies (66)* defines a function from H *into* G which is the pseudoinverse, $f^\dagger$. A nice analytic proof of the minimum norm property of the pseudoinverse is in Ref. [124, pp. 89–90, Theorem 3.7].

By an alternative approach, since $\mathscr{R}(f)$ is a closed linear subspace in a Hilbert space H, and $\eta(f)$ is a closed linear subspace in a Hilbert space G, invocation of the Hilbert space decomposition theorem [126, p. 79, Theorem 4.11] yields that

$$G = \eta(f) \oplus [\eta(f)]^\perp \tag{69}$$

$$H = \mathscr{R}(f) \oplus [\mathscr{R}(f)]^\perp \tag{70}$$

where (69) and (70) have the following interpretation: for arbitrary fixed x in G, there exists a *unique* $\overline{x}$ and $\tilde{x}$ with

$$\overline{x} \text{ in } \eta(f) \tag{71}$$

$$\tilde{x} \text{ in } [\eta(f)]^\perp \tag{72}$$

{i.e., each element of $[n(f)]^\perp$ results in zero when an inner product is formed with it and every element of $\eta(f)$}, such that

$$x = \overline{x} + \tilde{x} \tag{73}$$

(i.e., this is an *orthogonal decomposition*, with one element in the linear subspace and the other element orthogonal to it!). Similarly, for arbitrary fixed y in H, there exists a *unique* $\hat{y}$ and $\tilde{y}$ with

$$\hat{y} \text{ in } R(f) \tag{74}$$

and

$$\tilde{y} \text{ in } [R(f)]^\perp \tag{75}$$

i.e., each element of $[R(f)]^\perp$ results in zero when an inner product is formed with it and every element of R(f), such that

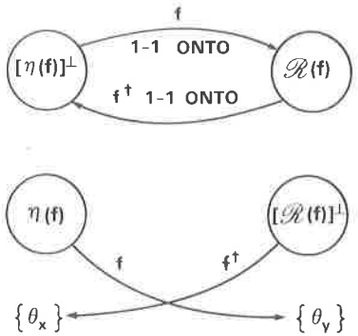

Fig. 6. The pseudoinverse as an extension of the ordinary inverse function associated with an original function that is 1–1 and onto [12, p. 164; 113, p. 578, Fig. C.17.1].

$$y = \hat{y} + \tilde{y} \tag{76}$$

As discussed in Ref. [12, Chapter 3], the continuous function, when restricted to have only domain $[\eta(f)]^{\perp}$, can be regarded as a function from the Hilbert space $[\eta(f)]^{\perp}$ onto the Hilbert space $\mathscr{R}(f)$. This closed linear subset of a Hilbert space is itself again a Hilbert space [125].

Between $[\eta(f)]^{\perp}$ and $\mathscr{R}(f)$, f is *one-to-one* and *onto*, and it has an inverse that is continuous and linear. The inverse of f defines $f^{\dagger}$ on $\mathscr{R}(f)$. Its domain is extended to all of H by defining the function's extension as

$$f^{\dagger}(y) = \theta_x \text{ for all y in } [R(f)]^{\perp}. \tag{77}$$

This alternate but equivalent interpretation of the pseudoinverse afforded by (69)–(77) is summarized in Fig. 6.

B. PROPERTIES OF THE GENERAL PSEUDOINVERSE

For a continuous linear function f, having its range R(f) closed and pseudoinverse being $f^{\dagger}$, then [12, p. 165]:

(a) $f^{\dagger}$ is linear;

(b) $f^{\dagger}$ is continuous;

(c) $(f^{\dagger})^{\dagger} = f$;

(d) $(f^{*})^{\dagger} = (f^{\dagger})^{*}$;

(e) $f^{\dagger} \circ f \circ f^{\dagger} = f^{\dagger}$;

(f) $f \circ f^{\dagger} \circ f = f$;

(g) $(f^{\dagger} \circ f)^{*} = f^{\dagger} \circ f$;

(h) $f^{\dagger} = (f^{*} \circ f)^{\dagger} \circ f^{*}$;

(i) $f^{\dagger} = f^{*} \circ (f \circ f^{*})^{\dagger}$; (78)

where, in the above, $f \circ g \triangleq f[g(\bullet)]$ is the composite function, and * is the adjoint operator, and represents the adjoint of f, which is defined using the following definitions of *inner product* on the domain $(\bullet \mid \bullet)_G$ and on the range $(\bullet \mid \bullet)_H$. The adjoint operator f* is defined so that $(x \mid f^{*}(y))_G = (f(x) \mid y)_H$ for each fixed y in H and all x in G. Therefore, f* is unique, linear, continuous, and identical in norm to f as $\|f\| = \|f^{*}\|$.

In *certain limiting cases* it is possible to give a *simple explicit formula for* $f^{\dagger}$ as follows: If $f \circ f^{*}$ is invertible, then

$$f^{\dagger} = [f^{*} \circ f]^{-1} \circ f^{*}. \tag{79}$$

If $f \circ f^{*}$ is invertible, then

$$f^{\dagger} = f^{*} \circ [f \circ f^{*}]^{-1}. \tag{80}$$

In general, however, a simple explicit closed-form formula for $f^{\dagger}$ does not exist [12, Chapter 3].

C. MANIPULATING IDENTITIES

As an example of how the identities of Section VII,B can be manipulated to analytically establish certain desired relationships, consider the following exercise of seeking to establish that

$$(I + f)^{-1} \stackrel{?}{=} (I - f^{\dagger} \circ f) + (I + f^{\dagger})^{-1} \circ f^{\dagger}, \tag{81}$$

where I is the identity transformation.

To establish the above identity as the goal, proceed by performing a sequence of completely reversible operations that eventually reduce to an identity that is obvious; then by retracing the steps in reverse order, the desired identity is

obtained. To this end, first perform a composite "postmultiplication" by (I + f) to yield

$$I = (I - f^{\dagger} \circ f) \circ (I + f) + (I + f^{\dagger})^{-1} \circ f^{\dagger} \circ (I + f) \tag{82a}$$

$$= I - f^{\dagger} \circ f + [f - f^{\dagger} \circ f \circ f] + (I + f^{\dagger})^{-1} \circ f^{\dagger} + (I + f^{\dagger})^{-1} \circ f^{\dagger} \circ f \tag{82b}$$

$$= I - f^{\dagger} \circ f + [f - f \circ f^{\dagger} \circ f] + (I + f^{\dagger})^{-1} \circ f^{\dagger} + (I + f^{\dagger})^{-1} \circ f \circ f^{\dagger} \tag{82c}$$

$$= I + (I + f^{\dagger})^{-1} \circ [-(I + f^{\dagger}) \circ f \circ f^{\dagger} + f^{\dagger} + f \circ f^{\dagger}] \tag{82d}$$

$$= I + (I + f^{\dagger})^{-1} \circ [-f \circ f^{\dagger} + [-f^{\dagger} \circ f \circ f^{\dagger} + f^{\dagger}] + f \circ f^{\dagger}] \tag{82e}$$

$$= I + (I + f^{\dagger})^{-1} \circ [-f \circ f^{\dagger} + f \circ f^{\dagger}] \tag{82f}$$

$$= I, \tag{82g}$$

where in going from (82a) to (82b) the indicated expansions were performed. In going from (82b) to (82c), the expressions within brackets in both are equivalent. In going from (82c) to (82d), the expression within brackets is zero via the identity of (78f). In going from (82d) to (82e), the expression $(I + f^{\dagger})$ within the brackets is expanded out with its composite "postmultipliers." In going from (82e) to (82f), the expression within the brackets is zero via the identity of (78e). In going from (82f) to (82g), the expression within the brackets is the zero transformation which, when composed with $(I + f^{\dagger})$, is also zero. Thus, retracing steps in reverse order, and finally using a composite function "postmultiplication" by $(I + f)^{-1}$, yields the desired identity of (81). Thus, this exercise demonstrates standard manipulations that can be performed with pseudoinverse transformations in accordance with the established "rules" or identities of (78). The exercise of Eq. (82), as used to verify Eq. (81), was actually solicited from the author within an application by James Taylor in seeking a more general result as an extension of those in [145], as evident from his acknowledgment of me.

VIII. APPENDIX B: LONGHAND CALCULATION OF A MATRIX PSEUDOINVERSE

A. MATRIX PSEUDOINVERSE AS A SPECIAL CASE OF THE GENERAL PSEUDOINVERSE OF PENROSE

Please consider the diagram of Fig. 5 with the Hilbert domain space $G \equiv E^n$ and the range space $H \equiv E^m$, where E represents Euclidean space (i.e., E^1 is the real line). All linear continuous functions f from E^n to E^m can be represented as

$$\underline{y}^{(m\times 1)} = f(\underline{x}) = C^{(m\times n)}\underline{x}^{(n\times 1)} \tag{83}$$

where C is an (m × n) matrix.

For matrices, properties of (78d)–(78g) completely specify a unique $C^{\dagger}$ and are sometimes used as the definition of $C^{\dagger}$. Alternatively,

$$C^{\dagger}C\underline{y} = \underline{y} \text{ for all } \underline{y} \text{ in } \mathscr{R}(C^{T}) \tag{84}$$

$$C^{\dagger}\underline{z} = 0 \text{ for all } \underline{z} \text{ in } \eta(C^{T}) \tag{85}$$

$$C^{\dagger}(\underline{y} + \underline{z}) = C^{\dagger}\underline{y} + C^{\dagger}\underline{z} \text{ for all } \underline{y} \text{ in } \mathscr{R}(C) \text{ and all } \underline{z} \text{ in } \eta(C^{T}) \tag{86}$$

and (84)–(86) may be used as definitions of $C^{\dagger}$, where in the above the adjoint of C is merely the matrix transpose.

By Ref. [51, Theorem 4.22], if any solution to

$$C\underline{x} = \underline{y} \tag{87}$$

exists, it can be expressed as

$$\underline{x} = C^{\dagger}\underline{y} + (I - C^{\dagger}C)\underline{z}, \tag{88}$$

where $\underline{z}$ is any arbitrary conformable vector.

{As an aside, the expression within parentheses in the second term in (88) is idempotent in that it is its own square [123, p. 41]. Hence, the pseudoinverse can be routinely used to create examples of idempotent matrices.}

B. EXPLICIT PROCEDURE FOR CALCULATING THE PSEUDOINVERSE

The following procedure is from Ref. [51].

Situation 1. Given a diagonal matrix

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{89}$$

where some λ_i may be zero, then the corresponding pseudoinverse is

$$\Lambda^\dagger \triangleq \text{diag}\,(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}), \tag{90}$$

where

$$\lambda_i^{-1} = \begin{cases} \dfrac{1}{\lambda_i}, & \text{if } \lambda_i \neq 0 \\ 0, & \text{if } \lambda_i = 0 \end{cases} \tag{91}$$

[cf. (22), (23)].

Situation 2. Given a Hermitian matrix, i.e.,

$$H = \overline{H}^T, \tag{92}$$

where the vinculum represents the complex conjugate, let

$$H = U\Lambda\overline{U}^T, \tag{93}$$

where U is a unitary matrix

$$\overline{U}^T = U^{-1}. \tag{94}$$

(U is simply the normalized eigenvector matrix associated with H, and the eigenvectors of an Hermitian matrix are always distinct and H can always be diagonalized [127, Sections 7.12 and 7.13].) The corresponding pseudoinverse is

$$H^\dagger = U\Lambda^\dagger \overline{U}^T, \tag{95}$$

where $\Lambda^\dagger$ can be found from the procedure of Situation 1 above. Naturally,

$$\overline{U}^T = U^T \tag{96}$$

for H and U real.

Situation 3. Given an arbitrary m × n matrix C, let

$$H \triangleq C^T C; \tag{97}$$

then

$$C^\dagger \triangleq H^\dagger C^T, \tag{98}$$

where $H^\dagger$ can be computed by the procedure of Situations 1 and 2 above.

The above procedure can be put in perspective by considering the following direct quote from Ref. [19, p. 136]: "Several proposals for computing $C^\dagger$ have been made in the literature. These algorithms are often very inefficient from the point of view of numerical computation (although they may be useful for getting exact answers in simple cases)." For instance, in establishing the existence of the Penrose inverse for a symmetric matrix Λ, we used the existence of a diagonalizing transformation, but this involved finding all the eigenvalues of C, which is a more difficult mathematical problem than the computation of $\Lambda^\dagger$. Moreover, other algorithms proposed thus far do not simplify when the actual inverse C exists, since then it should be that

$$C^\dagger = C^{-1}. \tag{99}$$

C. A NUMERICAL EXAMPLE OF PSEUDOINVERSE CALCULATION

The following direct quote from Ref. [19, p. 145] shows what order of difficulty to anticipate by way of computational burden incurred in this calculation: "The exact computation of the generalized inverse involves a very large amount of work." This may be an understatement. In seeking to find the pseudoinverse of the matrix C_1 in (11) of Section II,D, first form

$$H \triangleq C_1^* C_1 = C_1^T C_1 \tag{100}$$

as

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 4 & 6 & 2 \\ 1 & 2 & 1 \end{bmatrix}. \tag{101}$$

As observed in Section II,D, both $C_1^T C_1$ and $C_1 C_1^T$ are singular; so there can be no recourse to the simple expressions of (9) or (10) [cf. (80) and (79), respectively,

for specifying the pseudoinverse]. The pseudoinverse must instead be determined from the general expression of (98); therefore, as an intermediate step $H^{\dagger}$ must be determined via the procedure of Situation 2 of Section VIII,B.

Finding the Eigenvalues of H.

$$0 = \det[H - \lambda I_3] = \det\begin{bmatrix} 3-\lambda & 4 & 1 \\ 4 & 6-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{bmatrix}$$

$$= \lambda(-6 + 10\lambda - \lambda^2). \tag{102}$$

Therefore, the eigenvalues of H correspond to the solutions of

$$\lambda = 0 \tag{103}$$

$$\lambda^2 - 10\lambda + 6 = 0, \tag{104}$$

where further use of the quadratic formula for (104) yields the following three eigenvalues of H:

$$\lambda = 0, 5 \pm \sqrt{19}. \tag{105}$$

The corresponding eigenvectors are now calculated. For $\lambda = 0$:

$$H - (0)I_3 = \begin{bmatrix} 3 & 4 & 1 \\ 4 & 6 & 2 \\ 1 & 2 & 1 \end{bmatrix}. \tag{106}$$

It may be verified that an unnormalized eigenvector of H may be obtained by finding the cofactors associated with the first row of (106) as

$$\underline{e}_1^T = [2, -2, 2]^T. \tag{107}$$

Normalizing (107) so that

$$e_{1j}^2 + e_{2j}^2 + e_{3j}^2 = 1, \text{ for } j = 1 \tag{108}$$

yields:

$$\hat{\underline{e}}_1^T = \left[\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T. \tag{109}$$

For $\lambda = 5 + \sqrt{19}$:

$$H - (5 + \sqrt{19})I_3 = \begin{bmatrix} -2-\sqrt{19} & 4 & 1 \\ 4 & 1-\sqrt{19} & 2 \\ 1 & 2 & -4-\sqrt{19} \end{bmatrix}. \tag{110}$$

Similarly, an unnormalized eigenvector may be obtained by finding the cofactors associated with the first row of (110) as

$$\underline{e}_2^T = \left[11 + 3\sqrt{19}, 18 + 4\sqrt{19}, 7 + \sqrt{19}\right]^T. \tag{111}$$

Normalizing (111) so that (108) holds with j = 2 yields

$$\hat{\underline{e}}_2^T = \left[\frac{11 + 3\sqrt{19}}{\left(988 + 224\sqrt{19}\right)^{1/2}}, \frac{18 + 4\sqrt{19}}{\left(988 + 224\sqrt{19}\right)^{1/2}}, \frac{7 + \sqrt{19}}{\left(988 + 244\sqrt{19}\right)^{1/2}}\right]. \tag{112}$$

Similarly, for $\lambda = 5 - \sqrt{19}$:

$$H - (5 - \sqrt{19})I_3 = \begin{bmatrix} -2+\sqrt{19} & 4 & 1 \\ 4 & 1+\sqrt{19} & 2 \\ 1 & 2 & -4+\sqrt{19} \end{bmatrix}. \tag{113}$$

An unnormalized eigenvector may be obtained by finding the cofactors associated with the first row of (110) as

$$\underline{e}_3^T = \left[11 - 3\sqrt{19}, 18 - 4\sqrt{19}, 7 - \sqrt{19}\right]^T. \tag{114}$$

Similarly, normalizing (114) so that (108) holds with j = 3, yields

$$\hat{\underline{e}}_3^T = \left[\frac{11 - 3\sqrt{19}}{\left(988 - 244\sqrt{19}\right)^{1/2}}, \frac{18 - 4\sqrt{19}}{\left(988 - 224\sqrt{19}\right)^{1/2}}, \frac{7 - \sqrt{19}}{\left(988 - 244\sqrt{19}\right)^{1/2}} \right]. \quad (115)$$

A check on these calculations occurs in noting that: $\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3$ are mutually orthogonal, as theoretically predicted and, consequently, for the normalized eigenvector matrix

$$U \triangleq \left[\hat{\underline{e}}_1 \mid \hat{\underline{e}}_2 \mid \hat{\underline{e}}_3 \right] \quad (116)$$

it checks that

$$UU^T = I_{(3\times 3)}. \quad (117)$$

Now the normalized eigenvector matrix U so obtained diagonalizes the symmetric matrix H as

$$U^T H U = \text{diag}\left\{0,\ 5 + \sqrt{19}\ ,\ 5 - \sqrt{19}\right\}. \quad (118)$$

As discussed in Situation 1 of Section VIII,B, the pseudoinverse of the diagonal matrix is:

$$\Lambda^\dagger \triangleq \text{diag}\left\{0,\ 1/(5 + \sqrt{19}\ ),\ 1/(5 - \sqrt{19}\right\} \quad (119)$$

$$= \text{diag}\left\{0,\ 0.1069,\ 1.560\right\}. \quad (120)$$

By (95) of Situation 2 of Section VIII,B, the pseudoinverse of the symmetric matrix is:

$$H^\dagger \triangleq \begin{bmatrix} 0.5774 & 0.5432 & -0.6096 \\ -0.5774 & 0.7995 & 0.1657 \\ 0.5774 & 0.2563 & 0.7752 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1069 & 0 \\ 0 & 0 & 1.560 \end{bmatrix} U^T$$

$$= \begin{bmatrix} 0 & 0.0580 & -0.9508 \\ 0 & 0.0854 & 0.2584 \\ 0 & 0.0274 & 1.209 \end{bmatrix} \begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \\ 0.5432 & 0.7995 & 0.2563 \\ -0.6096 & 0.1657 & 0.7752 \end{bmatrix}$$

$$= \begin{bmatrix} 0.61\bar{1} & -0.11\bar{1} & -0.72\bar{2} \\ -0.11\bar{1} & 0.11\bar{1} & 0.22\bar{2} \\ -0.72\bar{2} & 0.22\bar{2} & 0.94\bar{4} \end{bmatrix} \quad (121)$$

By (96) of Situation 3 of Section VIII,B, the pseudoinverse is:

$$C^\dagger \triangleq \begin{bmatrix} 0.61\bar{1} & -0.11\bar{1} & -0.72\bar{2} \\ -0.11\bar{1} & 0.11\bar{1} & 0.22\bar{2} \\ -0.72\bar{2} & 0.22\bar{2} & 0.94\bar{4} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.33\bar{3} & 0.5 & 0.5 \\ 0.33\bar{3} & 0.0 & 0.0 \\ 0.66\bar{6} & -0.5 & -0.5 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}, \quad (122)$$

which corresponds to (12) in Section II, D.

IX. APPENDIX C: AN APPLICATION OF THE PSEUDOINVERSE IN MINIMUM ENERGY OPTIMAL CONTROL

Given a linear dynamic system described by a set of differential equations of the form

$$\dot{\underline{x}}(t) = F\underline{x}(t) + \underline{b}u(t), \quad (123)$$

where $\underline{x}$ is an n vector, F is an n × n matrix, $\underline{b}$ is an n vector, and u is a scalar control function. Assume that initial condition $x(0) = \theta_y$ (the null element of the range space) and that the goal is to transfer the system to the final state $\underline{x}(T) = \underline{x}_1$ by application of suitable control u(t). Of the class of controls which accomplish the desired state transfer, our objective is to determine the one that minimizes the energy:

$$\int_0^T u^2(t)\,dt. \tag{124}$$

The explicit solution to the differential equation of state (123) is

$$x(T) = \int_0^T e^{F(T-t)}\underline{b}u(t)\,dt.$$

Thus, defining the function f mapping $L_2[0, T]$ into Euclidean n space E^n by

$$f(u) = \int_0^T e^{F(T-t)}\underline{b}u(t)\,dt \tag{125}$$

yields the following associations in terms of the notation of Appendix A:

$$G = L^2[0, T];\quad H = E^n,$$

and the corresponding appropriate inner products are

$$(x_1 \mid x_2)_G = (x_1 \mid x_2)_{L^2} \triangleq \int_0^T x_1(t)\cdot x_2(t)\,dt \tag{126}$$

and

$$(y_1 \mid y_2)_H = (\underline{y}_1 \mid \underline{y}_2)_{E^n} \triangleq \underline{y}_1^T \cdot \underline{y}_2, \tag{127}$$

and the corresponding norm for the control (in terms of the specified inner product) is

$$\|u\|_{L^2} \triangleq \sqrt{(u \mid u)_{L^2}} = \left(\int_0^T u^2(t)\,dt\right)^{1/2}. \tag{128}$$

The problem of minimizing (124) is equivalent to that of determining the *u of minimum norm* in (128) satisfying

$$f(u) = \underline{x}_1'. \tag{129}$$

Since the range of f, $\mathscr{R}(f)$, is finite dimensional, it is therefore closed. Thus, the results of the following theorem apply.

Theorem 1 [12, p. 161]. Let G and H be Hilbert spaces and let arbitrary $f \in B(G, H)$ with range closed in H. The vector x of minimum norm satisfying

$$f(x) = \underline{y} \tag{130}$$

is given by

$$x = f^*(z), \tag{131}$$

where z is any solution of the composite function

$$f \circ f^*(\underline{z}) = \underline{y}. \tag{132}$$

By virtue of the above Theorem 1, the optimal solution of (129) is

$$u = f^*(\underline{z}), \tag{133}$$

where

$$f \circ f^*(z) = \underline{x}_1'. \tag{134}$$

It remains to compute the above functions f* and f ∘ f* for this specific application. For any $u \in L_2$, $\underline{y} \in E^n$

$$(\underline{y} \mid f(u))_{E^n} = \underline{y}^T \int_0^T e^{F(T-t)}\underline{b}u(t)\,dt = \int_0^T \left[\underline{y}^T e^{F(T-t)}\underline{b}\right]u(t)\,dt$$

$$= (f^*(y) \mid u)_{L^2}; \tag{135}$$

hence, by the property of adjoint transformations

$$f^*(y) = \underline{b}^T e^{F^T(T-t)} \underline{y}. \tag{136}$$

It turns out that f ∘ f* is the following n × n matrix

$$f \circ f^* = \int_0^T e^{F(T-t)} \underline{b}\underline{b}^T e^{F^T(T-t)}\, dt. \tag{137}$$

If the matrix [f ∘ f*] is invertible [i.e., if (F, b) is a controllable pair], then the optimal control u(t) can be found directly to be of the form

$$u(t) = f^* \circ [f \circ f^*]^{-1} \underline{x}_1', \tag{138}$$

or, more explicitly, as

$$u(t) = \underline{b}^T e^{F^T(T-t)} \left(\int_0^T e^{F(T-s)} \underline{b}\underline{b}^T e^{F^T(T-s)}\, ds \right)^{-1} \underline{x}_1'. \tag{139}$$

Another unrelated pseudoinverse application in control and estimation theory relates to computationally determining a basis for the null space of an arbitrary matrix [109].

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REFERENCES

1. T. H. KERR, An invalid norm appearing in control and estimation. *IEEE Trans. Autom. Control*, Vol. AC-23, pp. 73, 74 (1978).
2. T. H. KERR, Correction to 'An invalid norm appearing in control and estimation.' *IEEE Trans. Autom. Control (Tech. Notes Corresp.)*, Vol. AC-23, pp. 1117, 1118 (1978).

3. C.-H. J. HSU, Comments on 'An invalid norm appearing in control and estimation.' *IEEE Trans. Autom. Control (Tech. Notes Corresp.)* **AC-25**(5), 1012 (1980).
4. T. H. KERR, Three important matrix inequalities currently impacting control and estimation applications. *IEEE Trans. Autom. Control*, Vol. AC-23, pp. 1110, 1111 (1978).
5. T. H. KERR, Rectifying several pervasive errors appearing in estimation and control theory. (internal memo available from author upon request).
6. A. S. WILLSKY and H. L. JONES, "A Generalized Likelihood Ratio Approach to State Estimation in Linear Systems Subject to Abrupt Changes," Rep. No. ESL-P-538. Electron. Syst. Lab., Massachusetts Institute of Technology, Cambridge, 1974.
7. A. S. WILLSKY and H. L. JONES, A generalized likelihood ratio approach to state estimation in linear systems subject to abrupt changes. *Proc. Conf. Decision Control*, pp. 846–853 Phoenix, AZ (1974).
8. A. S. WILLSKY and H. L. JONES, A generalized likelihood ratoi approach to detection and estimation of jumps in linear systems. *IEEE Trans. Autom. Control* **AC-21**(1), 108–112 (1976).
9. A. BEN-ISRAEL and T. N. E. GREVILLE, "Generalized Inverses: Theory and Applications." Wiley, New York, 1974.
10. C. R. RAO and S. K. MITR , "Generalized Inverses of Matrices and Its Applications." Wiley, New York, 1971.
11. A. GELB, ed., "Applied Optimal Estimation." MIT Press, Cambridge, Massachusetts, 1974.
12. D. G. LUENBERGER, "Optimization by Vector Space Methods." Wiley, New York, 1969.
13. E. K. BLUM, "Numerical Analysis and Computation Theory and Practice." Addison-Wesley, Reading, Massachusetts, 1972.
14. V. C. KLEMA and A. J. LAUB, The singular value decomposition: It's computation and some applications. *IEEE Trans. Autom. Control* **AC-25**(2), 164–176 (1980).
15. B. S. GARBOW J. M. BOYLE, J. J. DONGARRA, and C. B. MOLER, Matrix eigensystem routines EISPACK guide extension. *Lect. Notes Comput. Sci.* **51** (1977).
16. C. E. HUTCHINSON, J. A. D'APOLITO, and K. J. ROY, Applications of minimum variance reduced-state estimators. *IEEE Trans. Aerosp. Electron. Syst.* **AES-11**(5), pp. 785–794 (1975).
17. E. M. DUIVEN, Suboptimal linear filtering. *J. Spacecr. Rockets* **11**(3), 196–198 (1974).
18. P. S. MAYBECK, "Stochastic Models, Estimation, and Control," Vol. 1. Academic Press, New York, 1979.
19. R. E. KALMAN and T. S. ENGLAR, A user's manual for the automatic synthesis program (Program C). *NASA [Contract. Rep.], CR* **NASA CR-475** (1966).
20. G. J. BIERMAN, Power series evaluation of transition and covariance matrices. *IEEE Trans. Autom. Control* **AC-17**(2), 228–232 (1972).
21. P. R. BELANGER and T. P. McGILLIVRAY, Computational experience with the solution of the matrix Lyapunov equations. *IEEE Trans. Autom. Control (Tech. Notes Corresp.)* **AC-21**(5), 799–800 (1976).
22. E. FOGEL, A comment on the use of spectral radius of a matrix. *IEEE Trans. Autom. Control* **AC-26**(3), 798–799 (1981).
23. T. H. KERR, A conservative view on the GLR failure and event detection approach, March, 1982 (internal memo available from author on request).
24. C. W. SANDERS, E. C. TACKER, T. D. LINTON, and R.Y-S. LING, Specific structures for large-scale state estimation algorithms having information exchange. *IEEE Trans. Autom. Control* **AC-23**(2), pp. 255–261 (1978).
25. T. H. KERR and L. CHIN, The theory and techniques of discrete-time decentralized filters. *Inl* "Advances in the Theory and Technology of Nonlinear Filters and Kalman Filters" (C. T. Leondes, ed.). NATO Advis. Group Aerosp. Res. Dev., AGARDograph Noordhoff International Publishing, Leiden, 1981.

26. J. J. DONGARRA C. B. MOLER, J. R. BUNCH, and G. W. STEWART, "LINPACK User's Guide." SIAM, Philadelphia, Pennsylvania, 1979.
27. A. K. CLINE, C. B. MOLER, G. W. STEWART, and J. H. WILKINSON, An estimate for the condition number of a matrix. *SIAM J. Numer. Anal.* **16**(2), 368–375 (1979).
28. S. MAXUMDAR, C. C. LI, and G. R. BRYCE, Correspondence between a linear restriction and a generalized inverse in linear model analysis.. *Am. Stat.* **34**(2), 103–105 (1980).
29. S. L. CAMPBELL and C. D. MEYER, "Generalized Inverse of Linear Transformations." Pitman, San Francisco, California, 1979.
30. E. D. SONTAG, On generalized inverses of polynmomial and other matrices. *IEEE Trans. Autom. Control* **AC-25**(3), 514–517 (1980).
31. R. B. ASHER, K. D. HERRING, and J. C. RYLES, Bias, variance, and estimation error in reduced-order filters. *Automatica* **12**, 589–600 (1976).
32. C. W. SANDERS, E. C. TACKER, and T. D. LINTON, "Decentralized Estimation Via Constrained Filters," Tech. Rep. ECE-73-1. University of Wisconsin, Madison, 1973.
33. C. B. MOLER, Three research problems in numerical linear algebra. *Numer. Anal. Proc. Symp. Appl. Math.* **22** (1978).
34. T. H. KERR, Modeling and evaluating and empirical INS difference monitoring procedure used to sequence SSBN navaid fixes. *Navigation, J. Inst. Navig.* **28**(4), pp. 263–285 (1981).
35. C. E. HUTCHINSON and J. A. D'APPOLITO, Minimum variance reduced state filters. *Proc. Conf. Decision Control, 1972*, pp. 670–671 (1973).
36. J. A. D'APPOLITO and K. J. ROY, Reduced order filtering with applications in hybrid navigation. *Proc. Electron. Aerosp. Syst. Conv. (EASCON), 1973*, pp. 1–7 (1973).
37. M. ATHANS and F. C. SCHWEPPE, "Gradient Matrices and Matrix Calculations," TN 1965-63. Lincoln Lab., Massachusetts Institute of Technology, Lexington, 1965.
38. G. W. STEWART, On the perturbation of pseudo-inverses, projections, and least squares problems. *SIAM Rev.* **19**, 634–662 (1977).
39. R. K. MEHRA, Digital simulation of multidimensional Gauss–Markov random processes. *IEEE Trans. Autom. Control* **AC-14**(2), 112–113 (1969).
40. T. H. KERR, Real-time failure detection: A static nonlinear optimization problem that yields a two-ellipsoid overlap test. *J. Optim. Theory Appl.* **22**, pp. 509–536 (1977).
41. T. H. KERR, Statistical analysis of a two-ellipsoid overlap test for real-time failure detection. *IEEE Trans. Autom. Control* **AC-25**(4), 762–772 (1980).
42. T. H. KERR, False alarm and correct detection probabilities over a time interval for restricted classes of failure detection algorithms. *IEEE Trans. Inf. Theory* **IT-28**(4), pp. 619–631 (1982).
43. P. SINHA, "Covariance Analysis Program: User's Guide," GPS IV&V Memo No. 21-80. Intermetrics, 1980 (program under configuration control No. B13704A-00.00A).
44. J. VAGNERS and J.. WITTE, Design of numerically stable flight filters from minimum variance reduced-order estimators. *Proc. Nat. Aerosp. Electron. Conf. (NAECON), 1979* (1979).
45. G. J. BIERMAN, "Factorization Methods for Discrete Sequential Estimation." Academic Press, New York, 1977.
46. T. H. KERR, Examining the controversy over the acceptability of SPRT and GLR techniques in some applications. *Proc. Am. Control Conf.* **3**, 966–977 (1983).
47. T. F. CHAN, Algorithm 581: An improved algorithm for computing the singular value decomposition F1. *ACM Trans. Math. Software* **5**(1), 84–88 (1982).
48. T. F. CHAN, An improved algorithm for computing the singular value decomposition. *ACM Trans. Math. Software* **8**(1), 72–83 (1982).
49. B. SHEFAI, R. L. CARROLL *et al.*, Comments on "On the application of matrix generalized inverses to the design of observers for time-varying and time-invariant linear systems." *IEEE Trans. Autom. Control* **AC-29**(12), 1125–1126 (1984).

50. D. G. LUENBERGER, An introduction to observers. *IEEE Trans. Autom. Control* **AC-16**(6), pp. 596–602 (1971).
51. D. M. WIBERG, "Schaum's Outline Series: State Space and Linear Systems." McGraw-Hill, New York, 1971.
52. D. E. CATLIN and R. L. GEDDES, State estimation and divergence analysis. *IEEE Trans. Aerosp. Electron.* **AES-20**(5), 594–602 (1984).
53. R. BYERS, A LINPACK-style condition estimator for the equation $AX - XB^T = C$. *IEEE Trans. Autom. Control* **AC-29**(10), 926–928 (1984).
54. C. E. HUTCHINSON, The Kalman filter applied to aerospace and electronic systems. *IEEE Trans. Aerosp. Electron. Syst.* **AES-20**(4), 500–504 (1984).
55. T. H. KERR, "Poseidon Improvement Studies: Real-Time Failure Detection in the SINS/ESGM," TASC Rep. TR-418-20, Reading, MA, 1974 (confidential).
56. T. H. KERR, "Failure Detection in the SINS/ESGM System," TASC Rep. TR-528-3-1, Reading, MA, 1975 (confidential).
57. T. H. KERR, "Improving ESGM Failure Detection in the SINS/ESGM System (U)," TASC Rep. TR-678-3-1, Reading, MA, 1976 (confidential).
58. T. H. KERR, Preliminary Quantitative Evaluation of Accuracy/Observables Trade-off in Selecting Loran/NAVSAT Fix Strategies," TASC Tech. Inf. Memo. TIM-889-3-1, Reading, MA, 1977 (confidential).
59. T. H. KERR, "Improving C-3 SSBN Navaid Utilization," TASC Tech. Inf. Memo. TIM-1390-3-1, Reading, MA, 1979 (secret).
60. T. H. KERR, "Stability Conditions for the RelNav Community as a De-Centralized Estimator – Final Report," Rep. No. IR-480, Intermetrics, Cambridge, MA, 1980.
61. J. E. SACKS, N. A. CARLSON, and T. H. KERR, "Integrated Navigation Concept Study," Rep. IR-MA-321, Intermetrics, 1984.
62. T. H. KERR, Decentralized filtering and redundancy management/failure detection for multisensor integrated navigation systems. *Proc. Inst. Navig. Nat. Tech. Meet., Avion. Sess., pp. 191–208, 1985* (1985).
63. P. S. MAYBECK, Reduced-order Kalman filter design and performance analysis. *In* "Advances in the Theory and Technology of Applications of Nonlinear Filters and Kalman Filters (C. T. Leondes, ed.). NATO Advis. Group Aerosp. Res. Deve., AGARDograph Noordhoff International Publishing, Leiden, Chapter 10, 1981.
64. C. T. LEONDES, ed., "Practical Aspects of Kalman Filtering Implementation, AGARD Lec. Ser. No. 82. NATO Advis. Group Aerosp. Res. Dev., AGARDograph Noordhoff International Publishing, Leiden, 1976.
65. S. F. SCHMIDT, The Kalman filter: Its recognition and development for aerospace applications. *J. Guidance Control* **4**(1), 4–7 (1981).
66. G. W. GAWRYS and V. D. VANDELINDE, On the steady-state error of the fading memory filter. *IEEE Trans. Autom. Control* **AC-21**(4), 624–625 (1976).
67. K. L. HITZ, T. E. FORMANN, and B. D. O. ANDERSON, A note on bounds on solutions of Riccati equations. *IEEE Trans. Autom. Control* **AC-20**(1), 178 (1972).
68. G. J. BIERMAN, Commens on an analytic solution for an algebraic matrix Riccati equation. *IEEE Trans. Autom. Control* **AC-20**(2), 300–301 (1975).
69. H. E. EMARA-SHABAIK and C. T. LEONDES, A note on the extended Kalman filter. *Automatica* **17**(2), 411–412 (1981).
70. T. P. McGARTY, "Stochastic Systems and State Estimation." Wiley (Interscience), New York, 1974.
71. I. C. F. IPSEN, "Singular Value Decomposition with Systolic Arrays," NASA Tech. Rep. NASA-CR-172396. Langley Research Center, Hampton, Virginia, 1984.
72. T. H. KERR, Failure detection aids for human operator decisions in a precision inertial navigation system complex. *Proc. Symp. Appl. Decision Theory Probl. Diagn. Repair*. Sponsored by the American Statistical Association, Fairborn, Ohio, 1976.
73. T. H. KERR and L. CHIN, A stable decentralized filtering implementation for JTIDS RelNav. *Proc. IEEE Positin, Location, Navig. Symp. (PLANS), pp. 318–329, 1980* (1981).
74. T. H. KERR, "Phase III GPS Integration," Vol. 1, Rep. No. IR-MA-177. Intermetrics, Cambridge, MA, 1983.

75. T. H. KERR, "Navy GPS/SSN Phase II User Equipment DT&E: Magnavox Modification Center (Mod Center) Test Report," Rep. No. IR-MA-391 (NADC Rep. No. GPS-85-19432-032). Intermetrics, Cambridge, MA, 1984.
76. T. H. KERR, "Navy GPS/SSN Phase II User Equipment DT&E: Rockwell–Collins Developmental Test & Evaluation (Operational Readiness) [DT&E(OR)] Test Report)," Rep. No. IR-MA-484 (NADC No. GPS-85-19600-061). Intermetrics, Cambridge, MA, 1985.
77. T. H. KERR, "Functional and Mathematical Structural Analysis of the Passive Tracking Algorithm (PTA)," Rep. IR-MA-208 for Naval Air Development Center. Intermetrics (1983).
78. T. H. KERR, "Assessment of the Status of the Current Post-Coherent Localization Algorithm," Rep. IR-MA-319 for NADC. Intermetrics, Cambridge, MA (1984).
79. C. S. HWANG and R. R. MOHLER, "Nonlinear Observability and Mixed-Coordinate Bearings-Only Signal Processing," Rep. No. OSU-OWR TR 84-4 for ONR. Naval Postgraduate School, Monterey, CA, 1984.
80. S. E. HAMMEL and V. J. AIDALA, Observability requirements for three-dimensional tracking via angle measurements. *IEEE Trans. Aerosp. Electron. Syst.* **AES-21**(2) (1985).
81. B. D. O. ANDERSON and J. B. MOORE, "Optimal Filtering." Prentice-Hall, Englewood Cliffs, New Jersey, 1979.
82. C. S. SIMS and R. B. ASHER, Optimal and suboptimal results in full- and reduced-order linear filtering. *IEEE Trans. Autom. Control* **AC-23**(3) (1978).
83. C. S. SIMS, Reduced-order modeling and filtering. *Control Dyn. Syst.* **18** (1982).
84. V. B. HAAS, Reduced-order state estimation for linear systems with exact measurements. *Automatica* **21**(2) (1984).
85. D. S. BERSTEIN and D. C. HYLAND, The optimal projection equations for reduced-order state estimation. *IEEE Trans. Autom. Control* **AC-30**(6) , pp. 1201–1211 (1985).
86. G. B. DiMASI, An approximation for the nonlinear filtering problem, with error bound. *Stochastics* **14**, 247–271 (1985).
87. J. E. WALL, A. S. WILLSKY, and N. R. SANDELL, On the fixed-interval smoothing problem. *Stochastics* **5**, 1–41 (1981).
88. M. AOKI and J. R. HUDDLE, Estimation of the state vector of a linear stochastic system with a constrained estimator. *IEEE Trans. Autom. Control* **AC-12**(4) (1967).
89. D. F. LIANG, J. C. McMILLAN *et al.*, Low cost integrated marine navigation system. *Navigation, J. Inst. Navig.* **30**(4) , pp. 281–300, (1983–1984).
90. R. ISERMAN, Process fault detection based on modeling and estimation methods – A survey. *Automatica* **20**(4), 387–404 (1984).
91. B. WALKER, Recent developments in fault diagnosis and accommodation. Presented at *AIAA Guidance Control Conf.*, but too late for inclusion in proceedings (available from author), August, 1983.
92. T. H. KERR, Decentralized filtering and redundancy management for multisensor navigation. *IEEE Trans. Aerosp. Electron. Syst.* **AES-23**(2) (1987).
93. T. H. KERR, Testing matrices for positive definiteness and semidefiniteness and application examples that spawn the need. *AIAA J. Guidance, Control, Dyn.* **10**(5) (1987).
94. M. VERHAEGEN and P. VAN DOOREN, Numerical aspects of different Kalman filter implementations. *IEEE Trans. Autom. Control* **AC-31**(10), 907–917 (1986).
95. C. L. LAWSON and R. J. HANSON, "Solving Least-Squares Problems (Appendix C)," Prentice-Hall, Englewood Cliffs, New Jersey, 1974.
96. G. H. GOLUB and C. F. VAN LOAN, "Matrix Computations." Johns Hopkins Univ. Press, Baltimore, Maryland, 1983.
97. J. K. CULLUM and R. A. WILLOUGHBY, "Lanczos Algorithms for Large Symmetric Eigenvalue Computations," Vols. 1 and 2, Birkhaeuser, Boston, Massachusetts, 1985.
98. R. SCHREIBER, Implementation of adaptive array algorithms. *IEEE Trans. Acoust., Speech, Signal Process.* **ASSP-34**(5), 1038–1045 (1986).

99. D. W. TUFTS and C. D. MELLISSINOS, Simple, effective computation of principal eigenvalues and their eigenvectors and application to high resolution estimation of frequencies. *IEEE Trans. Acoust., Speech, Signal Process.* **ASSP-34**(5), 1046–1053 (1986).
100. R. DEUTSCH, "Estimation Theory." Prentice-Hall, Englewood Cliffs, New Jersey, 1965.
101. G. GOLUB and W. KAHAN, Calculating the singular values and pseudo-inverse of a matrix. *SIAM J. Numer. Anal. Ser. B* **2**, 205–224 (1965).
102. F. W. FAIRMAN, Reduced-order state estimation for discrete time stochastic systems. *IEEE Trans. Autom. Control* **AC-22** , pp. 874–876 (1977).
103. E. FOGEL and Y. F. HUANG, Reduced-order optimal state estimator for linear systems with partially noise-corrupted measurement. *IEEE Trans. Autom. Control* **AC-25**(5), 994–996 (1980).
104. D. W. TUFTS and C. D. MELISSINOS, Effective computation of principal eigenvectors and their eigenvalues and application to high-resolution estimation of frequencies. *IEEE Trans. Acoust., Speech, Signal Process.* **ASSP-34**(5), 1046–1053 (1986).
105. R. O. SCHMIDT, Multiple emitter location and signal parameter estimation. *Proc. RADC Spectrum Estim. Workshop, 1979*, pp. 243–258 (1979).
106. E. K. L. HUNG, R. M. TURNER, and R. W. HERRING, A pitfall in using the pseudoinverse of a singular covariance matrix in adaptive signal processing. *Proc. ASSP Spectrum Estim. Workshop, 2nd, 1983*, pp. 253–258 (1984).
107. P. G. COXSON, Structured model reduction. *Proc. Conf. Decision Control, 23rd, 1984*, pp. 139–142 (1985).
108. U. B. DESAI, Reduced-order modeling of stochastic processes with applications in estimation. *In* "Modeling and Application of Stochastic Processes" (U. B. Desai, ed.), pp. 43–74. Kluwer Academic Publ., Boston, Massachusetts, 1986.
109. J. D. APLEVICH, A simple method for finding a basis for the null-space of a matrix. *IEEE Trans. Autom. Control* **AC-21**(3), 402 (1976).
110. M. L. LION, Author's reply to 'On the evaluation of e^{At} by power series.' *IEEE Proc.* **55**, 413 (1967).
111. J. A. D'APPOLITO, Kalman filter theory and application – A short course. A tutorial presented at NAECON 86 and included in *Tutorial Proc. NAECON , 1986* (1986).
112. T. H. KERR, The proper computation of the matrix pseudoinverse and its impact in MVRO filtering. *IEEE Trans. Aerosp. Electron. Syst.* **AES-21**(5), 711–724 (1985).
113. L. A. ZADEH and C. A. DESOER, "Linear System Theory: The State Space Approach." McGraw-Hill, New York, 1963.
114. W. H. FINCKE and A. J. KLEINMAN, Maneuver-dependent component error models and synchronized reset filters for inertial navigation systems. *Proc. Position, Location, Navig. Symp. (PLANS), 1980*, pp. 107–114 (1981).
115. T. H. KERR, Cautions concerning occasional misuse of shaping filters for marginally stable systems. *IEEE Trans. Aerosp. and Electron. Syst* (to be published).
116. A. PAULRAJ, R. ROY, and T. KAILATH, A subspace approach to signal parameter estimation. *Proc. IEEE* **74**(7), 1044–1045 (1986).
117. K. YONEZAWA, Reduced--order Kalman filtering with incomplete observability. *AIAA J. Guidance Control* **3**(3), 280–282 (1980).
118. A. K. MAHALANABIS, On minimizing the divergence in discrete filters. *IEEE Trans. Autom. Control* **AC-17**(2), 239–240 (1972).
119. W. P. M. P. DAYAWANSA, Comments on 'On the application of matrix generalized inverses to the design of observers for time-varying and time-invariant linear systems.' *IEEE Trans. Autom. Control* **AC-27**(5), 1137–1138 (1982).
120. J. O'REILLY, Reply to author's reply to 'Commens on two recent papers on reduced-order optimal state estimation for linear systems with partially noise-corrupted measurement.' *IEEE Trans. Autom. Control* **AC-27**(6), 1266–1267 (1982).
121. G. W. STEWART, "Introduction to Matrix Computations." Academic Press, New York, 1973.

122. T. S. LEE, Numerical aspects of the inverse function theory. *Proc. Conf. Decision Control, 22nd, 1983*, pp, 424–429 (1984).
123. M. J. R. HEALY, "Matrices for Statistics." Oxford Univ. Press (Clarendon), London and New York, 1986.
124. R. A. USMANI, "Applied Linear Algebra." Dekker, New York, 1987.
125. S. K. BERBERIAN, "Introduction to Hilbert Spaces." Oxford Univ. Press, London and New York, 1961.
126. W. RUDIN, "Real and Complex Analysis." McGraw-Hill, New York, 1966.
127. E. KREYSZIG, "Advanced Engineering Mathematics." Wiley, New York, 1963.
128. *Proceedings of the International Workshop on SVD and Signal Processing*. Sponsored by European Association for Signal Processing (EURASIP), Les Houches, Grenoble, France, 1987.
129. C. L. PATTERSON, G. BUECHLER, and H. C. ANDREWS, Pseudoinversion in digital image restoration. *Proc. Conf. Decision Control, 1974*, pp. 449–452 (1975).
130. E. VERRIEST, B. FRIEDLANDER, and M. MORF, Distributed processing in estimation and detection. *Proc. Conf. Decision Control, 18th, 1979*, pp. 153–158 (1980).
131. D. J. TYLAVSKY and G. R. L. SOHIE, Generalization of the matrix inversion lemma. *Proc. IEEE* 74(7), 1050–1052 (1986).
132. C. B. CHANG and K. P. DUNN, On the GLR detection and estimation of unexpected inputs in linear discrete systems. *IEEE Trans. Autom. Control* **AC-24**(3), 499–500 (1979).
133. G. R. L. SOHIE and L. H. SIBUL, Application of Hilbert space theory to optimal and adaptive space–time processing. *Proc. Conf. Inf. Sci. Syst.*, pp. 17–22 (1982).
134. P. BURRIDGE and A. HALL, Convergence of the Kalman filter gain for a class of nondetectable signal extraction problems, *IEEE Trans. Autom. Control*, **AC-32**(11), 1036–1039 (1987).
135. P. H. PETKOV, N. D. CHRISTOV, and M. M. KONSTANTINOV, On the numerical properties of the Schur approach for solving the matrix Riccati equation, *Sys. Control Lett.* 9(3), 197–201 (1987).
136. C. BRUNI, A. ISIDORI, and A. RUBERTI, "On the calculation of pseudo-inverses," *IEEE Trans. Autom. Control*, **AC-11**(2), 204–205 (1966).
137. T. H. KERR, "Multichannel Shaping Filter Formulations for BRVAD Using Matrix Spectral Factorization," Report No. PA-500. MIT Lincoln Laboratory, Lexington (1987).
138. A. L. DORAN, Coordinate Selection Issues in the Order Reduction of Linear Systems. In "Control and Dynamic Systems: Advances in Theory and Applications," Vol. 25: System Identification and Adaptive Control, Part 1 of 3 (C. T. Leondes, ed.). Academic Press, NY (1987).
139. Y. BARAM and G. KALIT, Order reduction in linear state estimation under performance constraints, *IEEE Trans. Autom. Control* **AC-32**(11), 983–989 (1987).
140. R. SETTERLUND, New insights into minimum variance reduced order filters, *AIAA J. Guidance Control, Dyn.* (1988).
141. U. M. AL-SAGGAF and G. F. FRANKLIN, An error bound for a discrete reduced order model of a linear multivariable system, *IEEE Trans. Autom. Control* **AC-32** (9), 815–819 (1987).
142. H.-F. CHEN and L. GUO, Consistent estimation of the order of stochastic control systems, *IEEE Trans. Autom. Control* **AC-32** (6), 531–535 (1987).
143. E. A. JONCKHEEVE, On stochastic model reduction, *IEEE Trans. Autom. Control* **AC-32** (6), 530–531 (1987).
144. H. D. MITTELMANN and J. A. CADZOW, Continuity of closest rank-p approximations to matrices, *IEEE Trans. Acoustics, Speech, and Sig. Proc.* **ASSP-35**(8), 1211–1212 (1987).
145. J. H. TAYLOR, Cramer–Rao estimation error lower bound analysis for nonlinear systems with unknown deterministic variables, *IEEE Trans. Autom. Control* **AC-24** (2), 343–344 (1979).
146. T. H. KERR, An almost completely analytic closed-form implementation example of a Schweppe likelihood ratio detector, submitted.

147. T. H. KERR, Kalman filter tracking in the multitarget case and an associated practical sonobuoy selection procedure, submitted.
148. T. H. KERR, Overview status of Cramer–Rao-like bounds for nonlinear filtering, submitted.