INT. J. CONTROL, 1973, VOL. 18, NO. 6, 1189–1199

Modelling a particular class of stochastic systems†

EARL D. EYMAN and THOMAS KERR‡

Department of Electrical Engineering,
University of Iowa, Iowa City

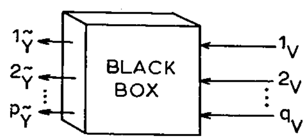
[Received 28 December 1972]

In this paper a method is given for obtaining a mathematical model of a class of black boxes having multiple inputs and multiple outputs in terms of Ito stochastic integral equations. This method is applicable to the class of black boxes having ergodic correlation functions when there is zero applied input. The point of view adopted in this paper is phenomenological in that it is desired that calculations made using the mathematical model should be ' close ' to what is actually observed at the output of the black box.

1. Statement of the problem

Given a black box, as shown in fig. 1 :

Fig. 1



The black box has outputs $1_{\tilde{Y}}$ to $p_{\tilde{Y}}$ and deterministic inputs 1_V to q_V. If, when the input $V=0$, the output Y has an ergodic correlation function matrix, then it is desired to use a mathematical model of the form

$$X_t = C + \int_0^t FX_u \, du + (I) \int_0^t G \, d\beta_u + \int_0^t MV(u) \, du, \tag{1}$$

$$Y_t = HX_t + m, \tag{2}$$

$$E[C]=0, \quad E[CC^{\mathrm{T}}] \triangleq P,$$

$$E[\beta_u]=0 \forall u, \quad E[\beta_s \beta_t^{\mathrm{T}}] = Q \min(t, s), \quad Q>0,$$

as the model for the black box, where Y_t represents the p-dimensional random vector output of the black box, $V(t)$ is the q-dimensional deterministic input of the black box, C is a Gaussian random vector initial condition, $\{\beta_u\}_{u \in \mathfrak{R}}$ is a vector Wiener process independent of C, and $(I) \int_0^t G \, d\beta_u$ is an Ito stochastic integral. Equation (1) is a linear Ito stochastic integral equation. The

† Communicated by Professor E. D. Eyman. This work was partially supported by a NASA Grant NGR-001-090.

‡ Presently at General Electric, Corporate Research and Development Laboratories, Schenectady, New York.

Gaussian random vector initial condition C is completely characterized by its mean, $E[C]=0$, and its variance, $E[CC^{\mathrm{T}}]=P$. The Wiener process $\{\beta_u\}_{u\in\Re}$ is completely characterized by its mean vector, $E[\beta_u]=0\forall u$, and its correlation function matrix, $E[\beta_s\beta_t^{\mathrm{T}}]=Q\min(t, s)$. Essentially, the problem is to find the matrix parameter constants F, G, M, H, m, P, Q (in Appendix 4, it is shown how to reduce the number of unknown matrices by one) so that the solution of eqns. (1) and (2) satisfy the conditions that :

(i) the mathematical model has the same output mean vector as the black box,

(ii) the mathematical model has the same output correlation function matrix as the black box,

(iii) the output of the mathematical model has the same type of sample functions (either continuous or piecewise continuous) as the output of the black box. [When the black box has piecewise continuous sample functions, the Wiener process $\{\beta_u\}_{u\in\Re}$ should be replaced by a centred Poisson process and the integral will still have meaning as a stochastic integral with all its desired properties (Anderson 1966)].

The above conditions (i), (ii), (iii) are the criteria of ' closeness ' that were alluded to in the Abstract.

Overview of what is done

A procedure is given for testing the actual black box under consideration to determine whether the assumed form of the mathematical model, eqns. (1) and (2), is acceptable. This test is based on manipulating the problem into a hypothesis testing situation, where the hypothesis is : ' Does the black box behave in a manner corresponding to the mathematical model of the particular form of eqns. (1) and (2) ?' A test

$$Y(T^*)=H\Phi(T^*, 0)C+(I)\int_0^{T^*} H\Phi(T^*, \tau)G_0\,d\beta_\tau+\int_0^{T^*} H\Phi(T^*, \tau)MV(\tau)\,d\tau,$$

which is, again, a Gaussian random p-vector.

Since M is an $(n\times q)$ constant matrix, let $M=[m_1, m_2, \ldots, m_q]$, where each m_j is a column vector. For an input

$$V^{\delta j}(t)\Delta[\delta_{1j}, \delta_{2j}, \ldots, \delta_{qj}]^{\mathrm{T}}\delta(t-T^*/2),$$

where δ_{ik} is the Kronecker delta, let $[Y^\delta(T^*)]^j$ represent the corresponding p-vector output. An expression for $[Y^\delta(T^*)]^j$ using the sifting property of the Dirac delta function is

$$\begin{aligned}[Y^\delta(T^*)]^j &= H\Phi(T^*, 0)C+(I)\int_0^{T^*} H\Phi(T^*, \tau)G_0\,d\beta_\tau+\int_0^{T^*} H\Phi(T^*, \tau)MV^{\delta j}(\tau)\,d\tau \\ &= H\Phi(T^*, 0)C+(I)\int_0^{T^*} H\Phi(T^*, \tau)G_0\,d\beta_\tau \\ &\qquad +\left\{\int_0^{T^*} H\Phi(T^*, \tau)\delta\left(\tau-\frac{T^*}{2}\right)d\tau\right\}m_j \\ &= H\Phi(T^*, 0)C+(I)\int_0^{T^*} H\Phi(T^*, \tau)G_0\,d\beta_\tau+H\Phi\left(T^*, \frac{T^*}{2}\right)m_j.\end{aligned}$$

(The use of the 'impulse function' in this analysis is just for convenience; later it will be replaced by any arbitrary, easily generated function, without affecting the conclusions of this section.)

The mean of the impulse-excited output is

$$E[Y^{\delta}(T^*)]^j = H\Phi(T^*, 0)E[C] + E\left\{(I)\int_0^{T^*} H\Phi(T^*, \tau)G_0\, d\beta_\tau\right\} + H\Phi\left(T^*, \frac{T^*}{2}\right) m_j.$$

Since $E[C]=0$ (see Appendices 1 and 2) and the expectation of the Ito integral is 0 (Varadhan 1968, p. 129), we have that

$$E[Y^{\delta}(T)^*]^j = H\Phi\left(T^*, \frac{T^*}{2}\right) m_j.$$

The covariance matrix is

$$\text{cov}\,[Y^{\delta}(T^*)]^j = \int_0^{T^*} H\Phi(T^*, u)G_0G_0^{\mathrm{T}}\Phi^{\mathrm{T}}(T^*, u)H^{\mathrm{T}}\, du + H\Phi(T^*, 0)P\Phi^{\mathrm{T}}(T^*, 0)H^{\mathrm{T}};$$

this result is obtained by using the fact that C and β_t are independent for $t \geqslant 0$, and other properties of the expectation procedure is formulated and a method is derived for determining, *a priori*, the number of trials, N, required for a certain confidence, α, in the conclusion of whether to accept or reject the hypothesis. If the hypothesis is accepted, that is, if it is found that eqn. (1) and (2) do adequately describe the behaviour of the black box, then the methods of the pseudo-inverse are employed to determine M. Prior to the determination of M, the other matrix parameters are determined by the method of Bucy and Joseph (see Appendices 4 and 5) with the input $V(t)=0$.

2. Procedure and derivations

The solutions of a system of equations that are equivalent to (1) and (2), which bear the same relationship that (1″) has to (1′) in Appendix 4, are

$$X_t = \Phi(t, 0)C + (I)\int_0^t \Phi(t, \tau)G_0\, d\beta_\tau + \int_0^t \Phi(t, \tau)MV(\tau)\, d\tau,$$

$$Y_t = H\Phi(t, 0)C + (I)\int_0^t H\Phi(t, \tau)G_0\, d\beta_\tau + \int_0^t H\Phi(t, \tau)MV(\tau)\, d\tau,$$

where $\Phi(t, \tau) = \exp[F(t-\tau)]$ (see Appendix 1 for a pertinent discussion). For $t=T^*$, we have that

$$Y(T^*) = H\Phi(T^*, 0)C + (I)\int_0^{T^*} H\Phi(T^*, \tau)G_0\, d\beta_\tau + \int_0^{T^*} H\Phi(T^*, \tau)MV(\tau)\, d\tau.$$

Since

$$\left\{\Phi(t, 0)C + (I)\int_0^t \Phi(t, \tau)G_0\, d\beta_\tau\right\}_{t\epsilon}$$

is a Gauss–Markov process (Jazwinski 1970, p. 111) because it is the solution of the stochastic integral equation

$$X_t = C + \int_0^t FX_\tau \, d\tau + (I) \int_0^t G_0 \, d\beta_\tau,$$

we have that, for fixed $t = T^*$, $\Phi(T^*, 0)C + (I) \int_0^{T^*} \Phi(T^*, \tau)G_0 \, d\beta_\tau$ is a Gaussian random n-vector (recall that a Gaussian process is completely characterized by the first- and second-order distributions which are Gaussian and jointly Gaussian, respectively). Since premultiplying the above by a constant H to obtain $H\Phi(T^*, 0)C + (I) \int_0^{T^*} H(T^*, \tau)G_0 \, d\beta_\tau$ just represents a linear transformation from n into p, the result is still a Gaussian random p-vector. Adding the constant term $\int_0^{T^*} H\Phi(T^*, \tau) V(\tau) \, d\tau$ to the above, for a specific $V(\,.\,)$, to yield

$$Y(T^*) = H\Phi(T^*, 0)C + (I) \int_0^{T^*} H\Phi(T^*, \tau)G_0 \, d\beta_\tau + \int_0^{T^*} H\Phi(T^*, \tau)MV(\tau) \, d\tau,$$

which is, again, a Gaussian random p-vector.

Since M is an $(n \times q)$ constant matrix, let $M = [m_1 \vdots \ldots \vdots m_q]$, where each m_j is a column vector. For an input

$$V^{\delta j}(t) \triangleq [\delta_{1j}, \delta_{2j}, \ldots, \delta_{qj}]^{\mathrm{T}} \quad \delta(t - T^*/2),$$

where δ_{ik} is the Kronecker delta, let $[Y^\delta(T^*)]^j$ represent the corresponding p-vector output. An expression for $[Y^\delta(T^*)]^j$ using the sifting property of the Dirac delta function is

$$\begin{aligned}
[Y^\delta(T^*)]^j &= H\Phi(T^*, 0)C + (I) \int_0^{T^*} H\Phi(T^*, \tau)G_0 \, d\beta_\tau \\
&\qquad + \int_0^{T^*} H\Phi(T^*, \tau)MV_j^\delta(\tau) \, d\tau \\
&= H\Phi(T^*, 0)C + (I) \int_0^{T^*} H\Phi(T^*, \tau)G_0 \, d\beta_\tau \\
&\qquad + \left\{ \int_0^{T^*} H\Phi(T^*, \tau)\delta\left(\tau - \frac{T^*}{2}\right) d\tau \right\} m_j \\
&= H\Phi(T^*, 0)C + (I) \int_0^{T^*} H\Phi(T^*, \tau)G_0 \, d\beta_\tau + H\Phi\left(T^*, \frac{T^*}{2}\right) m_j.
\end{aligned}$$

(The use of the 'impulse function' in this analysis is just for convenience; later it will be replaced by any arbitrary, easily generated function, without affecting the conclusions of this section.)

The mean of the impulse-excited output is

$$\begin{aligned}
E[Y^\delta(T^*)]^j = H\Phi(T^*, 0)E[C] + E\left\{(I) \int_0^{T^*} H\Phi(T^*, \tau)G_0 \, d\beta_\tau\right\} \\
+ H\Phi\left(T^*, \frac{T^*}{2}\right) m_j.
\end{aligned}$$

Since $E[C]=0$ (see Appendices 1 and 2) and the expectation of the Ito integral is zero (Varadhan 1968, p. 129), we have that

$$E[Y^{\delta}(T^*)]^j = H\Phi\left(T^*, \frac{T^*}{2}\right) m_j.$$

The covariance matrix is

$$\text{Cov}\,[Y^{\delta}(T^*)]^j = \int_0^{T^*} H\Phi(T^*, u) G_0 G_0^{\mathrm{T}} \Phi^{\mathrm{T}}(T^*, u) H^{\mathrm{T}}\, du + H\Phi(T^*, 0) P \Phi^{\mathrm{T}}(T^*, 0) H^{\mathrm{T}};$$

this result is obtained by using the fact that C and β_t are independent for $t \geqslant 0$ and other properties of the expectation of the Ito integral and of the Ito integral squared (Varadhan 1968, p. 129). Note that the covariance, $\text{cov}\,[Y(T^*)]^j$, is independent of j and is the same for any input.

With the aid of the above-established results and the material of the Appendices, the procedure for modelling a black box having outputs and inputs will now be given in detail.

First, clamp the inputs $V(t)=0$. If the output of the actual black box behaves in such a manner that the correlation function is ergodic, evaluate the unknown matrices H, F, G_0, P, m as described in Appendix 5. Now everything is known in the mathematical model, eqns. (1) and (2), except M. We must first determine if the added term $\int_0^t MV(\tau)\, d\tau$ validly represents the manner in which the input affects the actual black box under discussion (i.e. is linearity valid ?). A test procedure will now be given and a criterion set to determine if this added input term is valid for the specific black box. If the criterion is satisfied, a method is given for determining M from the same data used in the test.

The test is essentially a test to see if the actual black box behaves linearly and in a time-invariant manner. If it does, then superposition should apply. For fixed j, apply an input $V^{Aj}(t)=[\delta_{1j}, \delta_{2j}, \ldots, \delta_{qj}]^{\mathrm{T}} u^A(t)$, where $u^A(t)$ is an arbitrary scalar function of time. Apply this input serveral times, say N times (it is this N, the number of trials, that will be determined *a priori* as will be shown below), and record the corresponding N outputs ${}^i[Y^A(t)]^j$, $i=1, \ldots, p$; the record should extend from $t=0$ to $t=T^*$ where T^* is chosen for convenience. Divide the time interval $[0, T^*]$ into equispaced points, so that $\Delta = T^*/m$. For $t=k\Delta$, for every fixed $k \in [1, \ldots, m]$, average the N outputs ${}^i[Y^A(k\Delta)]^j$ to yield

$${}^i[\bar{Y}^A(k\Delta)]^j = \frac{1}{N}\sum_{n=1}^{N} {}^i[Y^A(k\Delta)]_n{}^j \quad (i=1, 2, \ldots, p).$$

Still for fixed j, apply an input

$$V^{Bj}(t)=[\delta_{1j}, \delta_{2j}, \ldots, \delta_{qj}]^{\mathrm{T}} u^B(t),$$

where $u^B(t)$ is an arbitrary scalar function of time different from $u^A(t)$. Similarly obtain

$${}^i[\bar{Y}^B(k\Delta)]^j = \frac{1}{N}\sum_{n=1}^{N} {}^i[Y^A(k\Delta)]_n{}^j \quad (i=1, 2, \ldots, p).$$

Finally, for the same fixed j, apply an input $V^{(A+B)j}(t)=[\delta_{1j}, \delta_{2j}, \ldots, \delta_{qj}]^{\mathrm{T}}[u^A(t)+u^B(t)]$. Then, again for every fixed k, average the outputs to obtain

$$i_{[\bar{Y}^{A+B}(k\Delta)]^j}=\frac{1}{N}\sum_{n=1}^{N} i_{[Y^{A+B}(k\Delta)]_n{}^j} \quad (i=1, \ldots, p).$$

This same test procedure is followed for each of the q components of the input.

If the black box were linear and the inputs entered in a time-invariant manner, then $\{[Y^A(k\Delta)]_n{}^j\}_{n=1}{}^N$, for a fixed k, is a sample of size N from a Gaussian population where the population has a known variance (eqn. (5)) of

$$\operatorname{cov}[Y^A(k\Delta)]^j=\int_0^{k\Delta} H\Phi(k\Delta, u)G_0G_0{}^{\mathrm{T}}\Phi^{\mathrm{T}}(k\Delta, u)H^{\mathrm{T}}\,du + H\Phi(k\Delta, 0)P\Phi^{\mathrm{T}}(k\Delta, 0)H^{\mathrm{T}} ;$$

and unknown mean

$$E[Y^A(k\Delta)]^j=\left[\int_0^{k\Delta} H\Phi(k\Delta, \tau)u^A(\tau)\,d\tau\right] m_j,$$

where m_j is an unknown vector.

For $t=k\Delta$, for any fixed k, the probability that the sample mean $[\bar{Y}^A(k\Delta)]^j$ of a sample size N is within ϵ of the population mean μ^{Aj} is given by

$$P\left[\frac{1}{N}\|[\bar{Y}^A(k\Delta)]^j-\mu^{Aj}\|^2(\operatorname{cov}[\bar{Y}^A(k\Delta)]^j)^{-1}\leqslant \epsilon^2\right] = P\left[\frac{1}{N}\|\bar{Z}\|_I{}^2\leqslant\epsilon^2\right]=P[\|\bar{Z}\|_I{}^2\leqslant N\epsilon^2]=\alpha,$$

where use has been made of the transformation

$$Z=\{[Y^A(k\Delta)]^j-\mu^{Aj}\}\{\operatorname{cov}[Y^A(k\Delta)]\}^{-1/2}$$

where Z is a p-dimensional Gaussian vector having

$$p_Z(\beta)=(1/(2\pi)^{p/2}\exp\{-(1/2)\|\beta\|_I{}^2\}$$

as a probability density function. Note that for $t=k\Delta$, for any fixed k, the problem transforms into the same problem in Z with the same sample size N. Since $[\bar{Y}^A(k\Delta)]^j$ was Gaussian, $\bar{Z}$ is Gaussian since only the above linear transformation was used.

Since the i_Z's are independent and have a Gaussian distribution, their squares have a χ^2 distribution and are also independent. The sum of p independent random variables having a χ^2 distribution is also χ^2 distributed with p degrees of freedom, $\chi_p{}^2$. This distribution is well tabulated. From the $\chi_p{}^2$ table it is possible to calculate, *a priori*, the sample size N required so that we have α confidence that the sample mean (a maximum likelihood, sufficient, unbiased, 'efficient' and 'consistent' statistic for the population mean (Hogg and Craig 1970, p. 255)) is within ϵ of the true population mean, where α and ϵ are set in advance. The number of degrees of freedom p is the number of outputs of the black box.

Returning to test the linearity hypothesis, if the actual black box were linear, then

$$\|[\bar{Y}^A(k\Delta)]^j-\mu^{Aj}\|_R{}^2-1=\|\bar{Z}-\mu_Z\|_I{}^2\leqslant\epsilon^2,$$

and we would have :

$$\|[\bar{Y}^A(k\Delta)]^j+[\bar{Y}^B(k\Delta)]^j-[\bar{Y}^{A+B}(k\Delta)]^j\|_{R-1}$$

$$\leqslant\|[\bar{Y}^A(k\Delta)]^j-\mu^{Aj}\|_{R-1}+\|[_Y{}^B(k\Delta)]^j-\mu^{Bj}\|_{R-1}+\|[\bar{Y}^{A+B}(k\Delta)]^j$$

$$-\mu^{(A+B)j}\|_{R-1}+\|\mu^{(A+B)j}-\mu^{Aj}-\mu^{Bj}\|_{R-1};$$

$$\times[\bar{Y}^A(k\Delta)]^j-\mu^{Aj}\|_{R-1}{}^2\leqslant\epsilon^2N;\quad\text{etc.,}$$

$$\times\|\mu^{Aj}+\mu^{Bj}-\mu^{(A+B)j}\|_{R-1}{}^2=0$$

$$\times\|[\bar{Y}^A(k\Delta)]^j+[\bar{Y}^B(k\Delta)]^j-[\bar{Y}^{A+B}(k\Delta)]^j\|^2{}_{R-1}\leqslant 9\epsilon^2N.$$

(R was used in the above to represent the appropriate covariance). Define

$$\gamma=\sum_{k=1}^{m}\|[\bar{Y}^A(k\Delta)]^j+[\bar{Y}^B(k\Delta)]^j-[\bar{Y}^{A+B}(k\Delta)]^j\|^2{}_{R-1}.$$

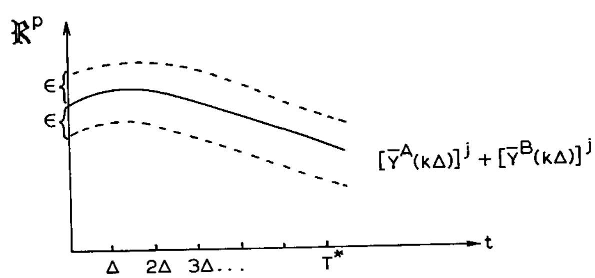
Therefore, to accept the hypothesis that the actual black box is linear and time-invariant it must be that

$$\gamma\leqslant 9m\epsilon^2N\quad\text{for}\quad j=1,2,\ldots,q.$$

If $\gamma>9m\epsilon^2N$, the conclusion is that the black box under consideration cannot be modelled by the methods presented.

The criterion can be interpreted pictorially in $(p+1)$—dimensional Euclidean space in fig. 2. Let the solid line represent $[\bar{Y}^A(k\Delta)]^j+[\bar{Y}^B(k\Delta)]^j$ in p-dimensional Euclidean space. When ϵ, α, and consequently N, have been specified, an ϵ-sheath is defined around the solid line in p-dimensional Euclidean space as represented by the dashed lines in fig. 2. If $[\bar{Y}^{A+B}(k\Delta)]^j$ is within the ϵ-sheath to a degree that, at the m-time points at which it is checked, the sum of the excursions outside the ϵ-sheath are compensated for by its proximity to $[\bar{Y}^A(k\Delta)]^j+[\bar{Y}^B(k\Delta)]^j$ at other times such that $\gamma\leqslant 9m\epsilon^2N$, the hypothesis that the actual black box is linear is accepted.

Fig. 2



Now returning to the problem of identifying M, we have that

$$[\bar{Y}^A(T^*)]^j = \frac{1}{N} \sum_{n=1}^{N} [Y^A(T^*)]_n^j$$

$$\doteqdot E[Y^A(T^*)]^j = \left[\int_0^{T^*} H\Phi(T^*, \tau) u^A(\tau)\, d\tau \right] m_j.$$

Now, $[\bar{Y}^A(T^*)]^j$ is a known p-vector, H is a known $(p \times n)$ matrix, and $\Phi(T^*, T^*/2) = \exp(FT^*/2)$ is a known $(n \times n)$ matrix, and $u^A(t)$ is a known deterministic scalar control ; therefore,

$$[\bar{Y}^A(T^*)]^j = \left[\int_0^{T^*} H\Phi(T^*, \tau) u^A(\tau)\, d\tau \right] m_j$$

is of the form of the algebraic equation $Y = Ax$, where A is a known $(p \times n)$ matrix, x is an unknown q-vector, and Y is a known p-vector. We wish to solve the above equation of m_j, which corresponds to solving $Y = Ax$ for x. If A were square and non-singular, the solution would be $x = A^{-1}Y$. Even when A^{-1} does not exist, it is desirable to solve $Y = Ax$ in some approximate sense ; the theory of the pseudo-inverse and how to find it is given in Appendix II of Aoki (1967, p. 318–324). Let A^+ represent the pseudo-inverse of the $(p \times n)$ matrix A ; then the solution x is $x = A^+Y$. The above analysis can be applied for each j $(j = 1, \ldots, q)$ so that each m_j $(j = 1, \ldots, q)$ is determined. The $(n \times q)$ matrix $M = [m_1, m_2, \ldots, m_q]$ has been identified. The problem of modelling a black box by a linear, constant coefficient, stochastic integral equation has now been solved.

Appendix 1

The argument for the zero-mean restriction follows. The equations of the mathematical model

$$X_t = C + \int_0^t FX_u\, du + (I) \int_0^t G\, d\beta_u,$$

$$Y_t = HX_t,$$

have the solutions

$$X_t = \Phi(t, 0)C + (I) \int_0^t \Phi(t, u)G\, d\beta_u,$$

$$Y_t = H\Phi(t, 0)C + (I) \int_0^t H\Phi(t, u)G\, d\beta_u,$$

where $\Phi(t, u) = \exp F(t-u)$, as can be verified by applying Ito's lemma to each scalar component of the vector solution for X to obtain the original stochastic integral equations (Bucy and Joseph 1968, p. 24). To satisfy the condition that the output of the black box have the same mean as the output of the model requires that

$$m = E[\tilde{Y}(t)] = E[Y(t)] = H\Phi(t, 0)E[C], \quad t.$$

Since m is a constant and $\Phi(t, 0)$ is varying with time, in general, this equation is satisfied if and only if $m = 0 = E[C]$.

Appendix 2

The zero-mean restriction can be removed by assuming a mathematical model of the same form except that $Y_t = HX_t + d$. Now $m = E[\tilde{Y}(t)] = E[Y(t)] = H\Phi(t, 0)E[C] + d$, $\forall\, t$ is satisfied if and only if $m = d$ and $E[C] = 0$. This causes no added difficulty since the covariance can be transformed, factored and manipulated in the same way that the correlation function is transformed, factored and manipulated in Bucy and Joseph (1968, pp. 25–26, 29–42). Using this approach with $Y_t = HX_t + d$, we have that the mathematical model and the black box have the same mean vector and covariance function matrix ; therefore, they have the same correlation function.

Appendix 3

From the solution of the stochastic integral equation mentioned in Appendix 1, from the unique properties of the Ito integral (Varadhan 1968, p. 129, or Jazwinski 1970, p. 99), and the fact that the black box is wide-sense stationary, the following equations are derived which hold true for the problem of modelling the black box with no inputs (Bucy and Joseph 1968, p. 39) :

$$S_{\tilde{y}\tilde{y}}(p) = H(pI - F)^{-1} G_0 G_0^{\mathrm{T}} (-pI - F^{\mathrm{T}})^{-1} H^{\mathrm{T}},$$

$$E[CC^{\mathrm{T}}] = E[X(t)X^{\mathrm{T}}(t)] \triangleq P, \quad \text{where } FP + PF^{\mathrm{T}} + G_0 G_0^{\mathrm{T}} = 0,$$

$$W^{\mathrm{T}}(p) = H(pI - F)^{-1} G_0, \quad \text{where } S_{\tilde{y}\tilde{y}}^{\mathrm{T}}(p) \text{ factors}$$

into $S_{\tilde{y}\tilde{y}}^{\mathrm{T}}(p) = W^{\mathrm{T}}(-p)W(p)$. (Please see Appendix 5 (*d*).)

Appendix 4

If $GQG^{\mathrm{T}} = G_0 G_0^{\mathrm{T}}$, then the system of equations used as a mathematical model in this paper,

$$(1') \qquad X_t = C \int_0^t FX_u \, du + (I) \int_0^t G \, d\beta_u,$$

$$(2') \qquad Y_t = HX_t,$$

$$E[C] = 0 \; ; \quad E[CC^{\mathrm{T}}] = P \; ; \quad E[\beta_t] = 0, \quad t,$$

$$E[\beta_t \beta_s^{\mathrm{T}}] = Q \min(t, s), \quad Q \geqslant 0,$$

where F, G, H, P, Q are the five unknown matrices, can be replaced by the equivalent system of equations

$$(1'') \qquad X_t = C + \int_0^t FX_u \, du + (I) \int_0^t G_0 \, d\beta_u,$$

$$(2'') \qquad X_t = HX_t,$$

$$E[C] = 0 \; ; \quad E[CC^{\mathrm{T}}] = P \; ; \quad E[\beta_t] = 0, \ \forall\, t,$$

$$E[\beta_t \beta_s^{\mathrm{T}}] = I \min(t, s),$$

where F, G_0, H, P are only four unknown matrices. This replacement can be done since the solution of the two X_t stochastic integral equations in both (1′) and (1″) are Gauss–Markov processes (Jazwinski 1970, p. 79) and as Markov processes do not require specification of the entire family of all finite dimensional distributions for a complete characterization ; knowledge of the transition probability densities of the form $p(X, t|y, s)$ suffices. The solutions of the X_t stochastic integral equations in (1′) and (1″) have transition probability density functions which satisfy the forward Kolmogorov or Fokker–Planck equations

$$\frac{\partial p(X, t|X_0, t_0)}{\partial t} = -\left(\frac{\partial}{\partial X}\right)^{\mathrm{T}}[FXp(X, t|X_0, t_0)] + \tfrac{1}{2}\,\mathrm{tr}\left\{\left(\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial X}\right)^{\mathrm{T}}[GQG^{\mathrm{T}}p(X, t|X_0, t_0)]\right\}$$

and

$$\frac{\partial p(X, t|X_0, t_0)}{\partial t} = -\left(\frac{\partial}{\partial X}\right)^{\mathrm{T}}[FX_p(X, t|X_0, t_0)] + \tfrac{1}{2}\,\mathrm{tr}\left\{\left(\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial X}\right)^{\mathrm{T}}[G_0G_0^{\mathrm{T}}p(X, t|X_0, t_0)]\right\}$$

respectively (Jazwinski 1970, p. 130). If $G_0G_0^{\mathrm{T}} = GQG$, these two Kolmogorov equations are the same, so their solutions are the same ; therefore, the X_t processes in (1′) and (1″) are the same process.

Appendix 5

The procedure for identifying the unknown matrices of (1″), (2″) from measurements made at the output of a black box follows below.

(*a*) Obtain an extensive time record of $\tilde{y}(t)$, the actual output of the black box.

(*b*) Process this date by time averaging to obtain the correlation function matrix, $R_{\tilde{y}\tilde{y}}(\tau)$, and the mean, $E[y(t)]$. The only assumption on the whole procedure is that the correlation function matrix is ergodic. (However, this one assumption implies ergodicity of the mean and wide-sense stationarity [Papoulis, 1965, p. 329].)

(*c*–i) Approximate this correlation function matrix in the τ-domain by an exponential series (Laning and Battin, 1956, p. 381). Then take the bilateral Laplace transform of the approximating correlation function to obtain the power spectral density matrix, $S_{\tilde{y}\tilde{y}}(p)$. Since the approximating correlation function consists of exponential terms, the power spectral density matrix has elements that are rational functions (i.e. ratios of polynomials).

(*c*–ii) An alternate procedure to (*c*–i). Instead of approximating in the τ-domain, first obtain the power spectral density matrix by taking the bilateral Laplace transform of the correlation function matrix, then obtain an approximation for the elements of the matrix in terms of rational functions by any one of the four methods mentioned in Solodovnikov (1960, Chap. V).

(*d*) Since every power spectral density matrix which has elements that are ratios of polynomials satisfies the sufficient conditions for applying the matrix factorization procedure (Kerr 1971, p. 330–333), factor $S^{--\mathrm{T}}(p)$ into $S^{--\mathrm{T}}(p) = W^{\mathrm{T}}(-p)W(p)$, where $W(p)$ is analytic in p in $\mathrm{Re}\,(p) \geqslant 0$. This factorization can be accomplished by either of the two methods presented by Youla (1961, method I is theorem 2, method II is theorem 3) or by the method of Davis (1963, p. 296–305). Since $S_{\tilde{y}\tilde{y}}{}^{\mathrm{T}}(p)$ is known, $W^{\mathrm{T}}(p)$ is known. Let $W^{\mathrm{T}}(p) = H(pI - F)^{-1}G_0$, where the triple (H, F, G_0) is to be determined.

(*e*) A triple (H, F, G_0) (not necessarily unique) can be found which satisfies $W^{\mathrm{T}}(p) = H(pI - F)^{-1}G_0$, and such that (H, F) is observable, (F, G) is controllable, and F is stable, either by the methods of obtaining a realization from a 'transfer function' as mentioned in Kalman (1963, p. 152) or by an original method in Kerr (1971, p. 255).

(*f*) Once (H, F, G_0) is known, the solution of $0 = FP + PF^{\mathrm{T}} + G_0G_0{}^{\mathrm{T}}$, where (F, G_0) is completely controllable, is known :

$$P = \int_0^\infty \exp(Ft)\, G_0G_0{}^{\mathrm{T}} \exp(F^{\mathrm{T}}t)\, dt$$

(Anderson 1967, p. 173).

(*g*) From Appendix 2, we have that $E[\tilde{y}(t)] = m$; therefore the matrices H, F, G_0, P, m in the mathematical model of the black box have all been determined. The modelling problem for the black box without inputs is solved.

Appendix 6

From the method of Appendix 5 and from the main method of this paper, mathematical models in terms of Ito stochastic integrals were obtained. Eventually, these mathematical models will be used to make computations which represent what actually occurs at the outputs of the black box. Digital computers do not normally perform Ito integrations, but this can be resolved by using methods of Wong and Zakai which relate Ito integrals to ordinary integrals (1965 a, p. 1560, and 1965 b, p. 213).

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