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CORPORATE RESEARCH AND DEVELOPMENT
Schenectady, N.Y.

ADA70 STEADY-STATE INITIAL VALUE CONVERGENCE TECHNIQUES

by

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Information Science and Engineering

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SUMMARY: Three distinct approaches for obtaining the ADA steady-state initial values are discussed:

(1) using the ADA steady-state option, which requires the use of convergence factors;

(2) calculating the inital values directly by hand, by a graphical analysis, or by using a special purpose computer program;

(3) using the transient start option with assumed initial conditions and integrate to a steady-state.

The underlying theory of each of the three approaches is discussed to show when and how they may be used.

A method is presented for determining bounds on the convergence acceleration factors that ensure convergence for linear systems. Extensions of all the techniques to particular classes of nonlinear systems are included.

Numerous concrete examples of how the various techniques are applied to actual computer simulations are included to facilitate reading. Although slanted toward ADA, theory is applicable to the universal steady-state evaluation problem faced by any general purpose continuous simulation program.

KEY WORDS: ADA, differential equations, dynamic systems, simulation, digital computations, continuous systems

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ADA70 Steady-State Initial Value Convergence Techniques

Section I: Introduction

The following approaches may be used to obtain the steady-state initial conditions for ADA simulations:

(1) use the steady-state start option, which requires the use of convergence factors;

(2) calculate the initial values directly by hand, by graphical analysis, or by using a special purpose computer program,

(3) use the transient start option with assumed initial conditions and integrate to a steady-state.

For a stable linear system, all of the above approaches yield the correct steady-state initial conditions. However, further investigation is required for each nonlinear system to determine whether or not the approaches will converge. There is no all inclusive theory for nonlinear systems (please see the remarks in Section III for some particular peculiarities).

The underlying theory of each of the three approaches is discussed to show when and how they may be used. The method of using the steady-state start feature of ADA is discussed in Section II. A method is presented for determining bounds on the convergence acceleration factors that ensures convergence for linear systems. Several general cases are presented so that the user may identify his problem as a special case of one of the general cases and make use of the results of the general analysis. For

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many problems a quick sketch is all that is required to determine the proper bounds.

Methods of calculating the steady-state initial values directly by solving the associated system of simultaneous algebraic equations is discussed in Section III.

The method of making a transient start and computing until all of the variables balance to obtain the steady-state initial conditions is discussed in Section IV. The limitations of this method are also emphasized.

In Section V, examples are given of linear system models where both the steady-state iteration method and the integration method fail. However, based on proofs of necessity and sufficiency, it is concluded that this double failure only occurs when the system being simulated is an inherent type n system, where $n > 0$ (D'Azzo and Houpis, 1960, p. 172). For this type of system the initial values must be specified to assure a zero input to the integrator. When both the iteration and integration methods fail to converge, a more detailed analysis of the model is required. A system may not appear to be of type n ($n > 0$) but actually is (please see Example 10). A method is given, using signal flow-graph techniques, for detecting when a linear system is of type n ($n > 0$).

A technique is presented in the Appendix which may be used to solve systems of nonlinear algebraic equations (or to determine that a solution does not exist).

(Note: The version of ADA implemented on the computers at some G.E. installations reverses the sign of the convergence acceleration factor, so that if one wants $R_2 = -.9$, he must enter $+.9$)

Section II: Computation of Convergence (Acceleration) Factors for Linear Systems

The steady state initial value computations of ADA normally use a Gauss Sidel Iteration technique. The iteration equations are in actuality difference equations, so knowing whether the iterations converge is equivalent to knowing whether the difference equations which describe the iterations converge. Associated with each differential equation in an ADA simulation is a constant known as a convergence acceleration factor to be specified by the user. This Section will deal with how to determine bounds on the convergence factors to ensure that the steady-state iterations converge when convergence is possible.

The iterations are described by difference equations and difference equations converge as the number of iterations become "large" if and only if the characteristic equations for the system of difference equations has all its roots within the unit circle. In Section II:A and Section II:B, the roots of the characteristic equation will be examined directly. Jury's stability test (B.C. Kuo, 1963, p. 156) enables one to determine whether a system of difference equations has all its roots within the unit circle simply by observing whether certain inequalities derived from the coefficients of the characteristic equation are satisfied. In this report, for systems involving two differential equations, Sections II:C-D, these inequalities are replaced by equivalent functions of two real variables, which are investigated to determine where these functions change from positive to negative. Once these boundary regions are established, the results of each individual graph can be superimposed to yield the regions in which all the inequalities

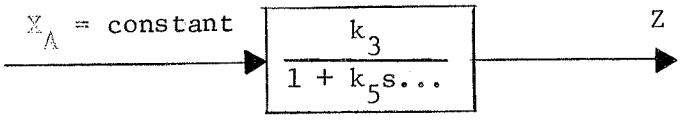
associated with Jury's stability test are satisfied. Essentially, this is a graphical application of Jury's stability test to determine for what values of the convergence acceleration factors the iterations converge. If there are no values of the convergence acceleration factors for which the iterations converge, then this fact will be evident in the graphical technique.

The purpose of this report is to derive the bounds on the convergence acceleration factor for the single differential equation, no loop and single loop case, and to do all the derivations for the graphical technique to be applied to the two differential equation configurations. To use the results of this report the user will only have to identify his problem as a particular special case of the general case treated, and quickly use the results that are derived in this report.

It is possible that some systems will not converge for any choice of convergence factors (please see Example 10). In case this situation should arise, the conclusion that it is impossible is readily apparent from the graphical technique.

Finally, a method of approach will be given (Section II:D) for extending the results of Section I to more complicated systems both linear and nonlinear. The overall purpose of Section II is to present a rationale in the selection of convergence acceleration factors.

A. Single D.E. with no Feedback



i.c. = Guess?

The iteration equation for calculating the steady-state initial condition is:

$$Z_N = k_3 X_A + R_2 [k_3 X_A - Z_{N-1}] \tag{1}$$

$$Z_N + R_2 Z_{N-1} = k_3 (1 + R_2) X_A$$

The associated homogeneous equation is:

$$Z_N + R_2 Z_{N-1} = 0 \tag{2}$$

Assuming a solution of the form $Z_k = e^{rk}$, where r is to be determined, and substituting into (2) yields

$$e^{rN} + R_2 e^{r(N-1)} = 0$$

or

$$e^{r} + R_2 = 0 \qquad \text{since } e^{r(N-1)} \neq 0.$$

Therefore $r = \ln[- R_2]$, and

$$Z_k = C_1 e^{(\ln[- R_2])k} = C_1 [e^{\ln[- R_2]}]^k = C_1 (- R_2)^k \tag{3}$$

is the transient solution of (1). Stability of (1) (i.e., whether the transients die out as $k \to \infty$) can be determined by investigating (3); it is from this equation that bounds are determined for the convergence acceleration factor R_2. Consider

$$\lim_{k\to\infty} Z_k = \lim_{k\to\infty} C_1 (- R_2)^k . \tag{4}$$

The limit in (4) exists only for

(5) $$-1 \leq R_2 < 1 \quad .$$

Let us now investigate the particular solution of equation (1). By the method of undetermined coefficients (DeRusso, Roy, Close, 1965, p. 86), we can find a particular solution of equation 1 making the assumption that X_A is a constant. Assume $Z_{P.S.}(k) = C_2$ and substitute into equation (1),

$$C_2 + R_2 C_2 = k_3 (1 + R_2) X_A$$

$$C_2 (1 + R_2) = k_3 (1 + R_2) X_A$$

(6) $$C_2 = k_3 X_A \quad \text{for} \quad R_2 \neq -1 \ ;$$

hence $$Z_{P.S.}(k) = k_3 X_A$$

The general solution is $Z_{G.S.}(k) = C_1 (-R_2)^k + k_3 X_A$. Now $\lim_{k\to\infty} Z_{G.S.}(k)$ $= C_1 \cdot 0 + k_3 X_A = k_3 X_A$ for $-1 < R_2 < 1$.

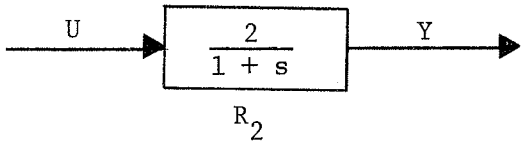
In summary, the iteration equation converges for $-1 < R_2 < 1$, and this iteration equation converges to $k_3 X_A$ as shown in (17), where it has been assumed that X_A is a constant.

In actual practice for a D.E. with no loops, if X_A is a constant, it is best to let $R_2 = 0$, in which case $Z_N = k_3 X_A$ for all N. Here the steady-state is reached on the first pass of ADA.

If X_A is not a constant but varies as a result of various inputs or feedback loops we cannot determine the required R_2 unless we know explicitly what X_A is; in general, we must expand our field of view to encompass all

the feedback loops to determine what acceleration factors are needed to ensure convergence.

Example 1: Consider the following block diagram:



By the theory for D.E.'s with no loops, for convergence, it must be that $-1 < R_2 < +1$.

```
$LIST ITER2
01/03/72    10:56

00010 STRP,/,1,  ,8,16,:,7,13,19,25,31,37,43,:,14,26,38,50,62,',0
00020 /$ SNUMB THKAA
00030 /$ IDENT T.KERR,KERR  37  578D  5-4171
00040 /$ PRØGRAM RLHS
00050 /$ LIMITS 1,16000,,1000
00060 /$ PRMFL H*,Z,R,WATSØN/ADA70A
00070 /$ DISC Q2,X2R,20L
00080 /$ PRMFL P*,R/A/S,L,*/P
00090 /$ NØTIFY */S
00100 0;40;2;1
00110 'T.KERR 9/21/71
00120 'R&DC 37 578D 8*235-4171
00130 'CØNVERGENCE ØF ITERATIØNS FØR S.S. FØR NØ LØØP
00140 1;1;50;1;5;1
00150 2
00160 1-5:1-5:1-6:1-2:10
00170 0:1:1:0:0
00180 ;2;U;W
00190 U;6
00200 3
00210 W;1;U;1;1
00220 10:-.25:2:1:1
00230 0;40;1
00240 /$ ENDJØB

READY
```

$R_2 = +.25$

ADA70 01/03/72 10.517

CASE 1 STEADY-STATE INITIAL VALUES ITERATIØN 10

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U	3.0000E 00				

DIFFERENTIAL EQUATIØNS

NAME	Y	DY1 ...
W	6.0000E 00	

CASE 1 FINAL VALUES AT TIME 1.0000E 00

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U	3.0000E 00				

DIFFERENTIAL EQUATIØNS

NAME	Y	DY1 ...
W	6.0000E 00	

CASE 1 CØNVERGENCE ØF ITERATIØNS FØR S.S. FØR NØ LØØP

TIME	U	W
0.	3.0000E 00	6.0000E 00
0.	3.0000E 00	6.0000E 00
E1.0000E 00	3.0000E 00	6.0000E 00

CPU TIME - TØTAL = 0.410 SECØNDS.

READY

$R_2 = - .25$

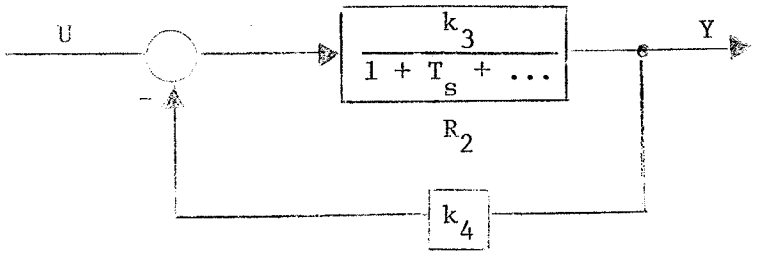
```
ADA70  01/03/72  11.414

 CASE    1      STEADY-STATE INITIAL VALUES  ITERATIØN     10

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U        3.0000E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y           DY1 ...
 W        6.0000E 00

 CASE    1      FINAL VALUES AT TIME  1.0000E 00

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U        3.0000E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y           DY1 ...
 W        6.0000E 00

 CASE    1    CØNVERGENCE ØF ITERATIØNS FØR S.S. FØR NØ LØØP

   TIME         U            W

 0.           3.0000E 00   6.0000E 00
 0.           3.0000E 00   6.0000E 00
E1.0000E 00   3.0000E 00   6.0000E 00


CPU TIME - TØTAL =    0.417 SECØNDS.


READY
```

$R_2 = 0$

```
$LIST ØUT7
01/Ø3/72   11:32

 ADA7Ø  Ø1/Ø3/72  11.454

  CASE    1      STEADY-STATE INITIAL VALUES  ITERATIØN      2

 CØNSTANT, STEP, AND TABULAR FUNCTIØNS

  NAME      VALUE       NAME      VALUE       NAME      VALUE
  U        3.0000E 00

 DIFFERENTIAL EQUATIØNS

  NAME       Y           DY1  ...
  W        6.0000E 00

  CASE    1      FINAL VALUES AT TIME  1.0000E 00

 CØNSTANT, STEP, AND TABULAR FUNCTIØNS

  NAME      VALUE       NAME      VALUE       NAME      VALUE
  U        3.0000E 00

 DIFFERENTIAL EQUATIØNS

  NAME       Y           DY1  ...
  W        6.0000E 00

  CASE    1    CØNVERGENCE ØF ITERATIØNS FØR S.S. FØR NØ LØØP

    TIME         U            W

  0.          3.0000E 00   6.0000E 00
  0.          3.0000E 00   6.0000E 00
E1.0000E 00   3.0000E 00   6.0000E 00


 CPU TIME - TØTAL =    0.466 SECØNDS.

READY
```

The problem is programmed on file ITER2. File OUT5 is the output when $R_2 = + 0.25$. File OUT6 is the output when $R_2 = - 0.25$. Both the positive and negative value of R_2 yield convergence at the same rate (within 10 iterations) for the no loop case. As mentioned in the previous theoretical discussion, for no loops, it is best to have $R_2 = 0$. The file OUT7 is the output when $R_2 = 0$. Convergence is achieved within 2 iterations.

B. Single D.E. with Arbitrary Feedback



Algebraic solution

$$k_3 (U - k_4 Y) = Y -$$

$$k_3 U = [1 + k_3 k_4] Y$$

$$Y = \frac{k_3 U}{[1 + k_3 k_4]}$$

The above block diagram can be used for analyzing the convergence when there is positive feedback simply by making the necessary changes in k_4.

The difference equation which describes the iteration in the steady-state iteration procedure is

$$Y_{N+1} = k_3 (U - k_4 Y_N) + R_2 [k_3 (U - k_4 Y_N) - Y_N]$$

or

$$Y_{N+1} + [k_3\, k_4 + R_2\, (k_3\, k_4 + 1)]\, Y_N = k_3\, (1 + R_2)\, U.$$

The solution of the associated homogeneous equation satisfies

$$\beta^{N+1} + [k_3\, k_4 + R_2\, (k_3\, k_4 + 1)]\, \beta^N = 0 \quad ,$$

which is the characteristic equation; the solution is

$$\beta = - [k_3\, k_4 + R_2\, (k_3\, k_4 + 1)].$$

We have convergence of the iteration equation if and only if

$$- 1 < - [k_3\, k_4 + R_2\, (k_3\, k_4 + 1) < 1.$$

This means that using the above inequality we can obtain bounds on R_2, such that the above inequality is satisfied. Rearranging the above inequality we obtain

$$k_3\, k_4 - 1 < - R_2\, (k_3\, k_4 + 1) < k_3\, k_4 + 1 \ .$$

Assume $k_3 > 0$, we have five cases to investigate.

Case 1: $\boxed{1 \geq k_3\, k_4 > 0}$

$$- \frac{(k_3\, k_4 - 1)}{(k_3\, k_4 + 1)} > R_2 > - 1$$

Now $1 \geq k_3 k_4$

$0 \geq k_3 k_4 - 1$

$-(k_3 k_4 - 1) \geq 0$

$-\dfrac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)} \geq 0$

$k_3 k_4 > 0 \Longrightarrow 0 > -k_3 k_4$

$k_3 k_4 > 0 > -k_3 k_4$

$k_3 k_4 > -k_3 k_4$

$1 + k_3 k_4 > 1 - k_3 k_4 = -(k_3 k_4 - 1)$

$1 = \dfrac{1 + k_3 k_4}{1 + k_3 k_4} > -\dfrac{(k_3 k_4 + 1)}{(k_3 k_4 + 1)}$

For $k_3 k_4 > 0$, we have that $1 > \dfrac{-(k_3 k_4 - 1)}{(k_3 k_4 + 1)} > R_2 > -1$, for convergence.

Case 2: $k_3 k_4 > 1$

Now $k_3 k_4 - 1 > 0$

$\dfrac{k_3 k_4 - 1}{k_3 k_4 + 1} > 0$

$-\dfrac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)} < 0$

For $k_3 k_4 > 1$, we have that $0 > -\dfrac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)} > R_2 > -1$, for convergence.

Case 3: (Positive feedback) $k_3 k_4 < -1$

$k_3 k_4 < -1$

$k_3 k_4 + 1 < 0$

$1 > -1$

$$k_3 k_4 + 1 > k_3 k_4 - 1$$

$$1 < \frac{k_3 k_4 - 1}{k_3 k_4 + 1}$$

$$-1 > -\frac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)}$$

For $k_3 k_4 < -1$, we have that $-\frac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)} < R_2 < -1$, where

$-\frac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)} < -1$, for convergence.

Case 4: (Positive feedback) $\boxed{-1 < k_3 k_4 < 0}$

$$k_3 k_4 + 1 > 0$$

$$k_3 k_4 < 0$$

$$-k_3 k_4 > k_3 k_4$$

$$-(k_3 k_4 + 1) = -k_3 k_4 - 1 > k_3 k_4 - 1$$

$$-1 > \frac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)}$$

$$1 < -\frac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)}$$

For $-1 < k_3 k_4 < 0$, we have that $-\frac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)} > R_2 > -1$, where

$-\frac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)} > 1$, for convergence.

Case 5: (Positive feedback) $\boxed{k_3\, k_4 = -1}$

The difference equation which describes the iteration is:

$$Y_{N+1} = k_3\, U - k_3\, k_4\, Y_N + R_2\, [k_3\, U - k_3\, k_4\, Y_N - Y_N]$$

$$= k_3\, U + Y_N + R_2\, k_3\, U$$

or

$$Y_{N+1} - Y_N = k_3\, (1 + R_2)\, U.$$

The characteristic equation is

$Y_{N+1} - Y_N = 0$ or $(E - 1)\, Y_N = 0$; the iterations will not converge no matter what the acceleration factor is.

(However, this is good since the closed loop transfer function is

$$\frac{Y}{U} = \frac{\dfrac{k_3}{1 + Ts + \ldots}}{1 + \dfrac{-1}{1 + Ts + \ldots}}$$

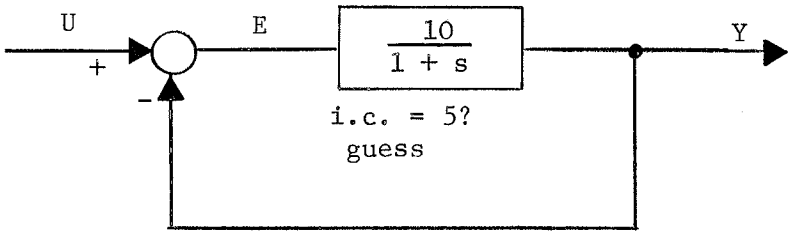
$$= \frac{k_3}{1 + Ts + \ldots - 1} = \frac{k_3}{s\,(T + \ldots)} \quad ;$$

which is called a type one system and which should have the initial condition specified exactly.)

Example 2: Negative Feedback

A numerical example will now be worked to illustrate how to apply the general results of the above analysis to a specific problem with negative

feedback. Consider the following block diagram:



$$k_3 = 10$$

$$k_4 = 1$$

$$k_3 k_4 = 10 > 1$$

By Case 2 of the analysis of a single differential equation with a feedback loop, the required convergence factor has the following bounds:

$$-\frac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)} > R_2 > -1 \qquad \text{or}$$

$$-.8\overline{181} = -\frac{9}{11} > R_2 > 1 \text{ .}$$

These results are verified in the following computer simulation:

```
$LIST ITER1
01/03/72   11:46
00010 STRP,/,1, ,8,16,:,7,13,19,25,31,37,43,:,14,26,38,50,62,',0
00020 /$ SNUMB THKAA
00030 /$ IDENT T.KERR,KERR  37  578D  5-4171
00040 /$ PRØGRAM RLHS
00050 /$ LIMITS 1,16000,,1000
00060 /$ PRMFL H*,Z,R,WATSØN/ADA70A
00070 /$ DISC 02,X2R,20L
00080 /$ PRMFL P*,R/A/S,L,*/P
00090 /$ NØTIFY */S
00100 0;40;2;1
00110 'T.KERR 9/21/71
00120 'R&DC 37 578D 8*235-4171
00130 'CØNVERGENCE ØF ITERATIØNS FØR S.S. FØR 1 LØØP
00140 1;1;50;1;5;1
00141 2
00150 1-5:1-5:1-6:1-2:10
00160 0:1:1:0:0
00170 ;2;Y;U;E
00180 U;6
00190 3
00200 E;11;2;U;Y
00210 0:1:-1
00220 Y;1;E;1;1
00230 5:-.91:10:1:1
00240 0;40;1
00250 /$ ENDJØB

READY
```

Necessary for Convergence

$$-1 < R_2 < -\frac{9}{11} = -.8\overline{181}$$

$R_2 = - .91$

$- 1 < - .91 < - .8181$

ADA7Q 01/03/72 11.688

CASE 1 STEADY-STATE INITIAL VALUES ITERATIØN 5

ALGEBRAIC FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
E	2.7273E-01				

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U	3.0000E 00				

DIFFERENTIAL EQUATIØNS

NAME	Y	DY1 ...
Y	2.7273E 00	

CASE 1 FINAL VALUES AT TIME 1.0000E 00

ALGEBRAIC FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
E	2.7270E-01				

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U	3.0000E 00				

DIFFERENTIAL EQUATIØNS

NAME	Y	DY1 ...
Y	2.7273E 00	

CASE 1 CØNVERGENCE ØF ITERATIØNS FØR S.S. FØR 1 LØØP

TIME	Y	U	E
0.	2.7273E 00	3.0000E 00	2.7273E-01
0.	2.7273E 00	3.0000E 00	2.7273E-01
E1.0000E 00	2.7273E 00	3.0000E 00	2.7270E-01

CPU TIME - TØTAL = 0.446 SECØNDS.

$LIST ØUT9
01/03/72 12:43

$R_2 = -.79$

$.8\overline{181} = -\frac{9}{11} < -.79$

ADA70 01/03/72 12.527

ADA CASE 1

STEADY-STATE INITIAL CØNDITIØNS HAVE NØT BEEN FØUND
WITHIN 50 ITERARIØNS.
RESULTS ØF LAST ITERATIØN FØLLØW-

ALGEBRAIC FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
E	-9.6732E 05				

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U	3.0000E 00				

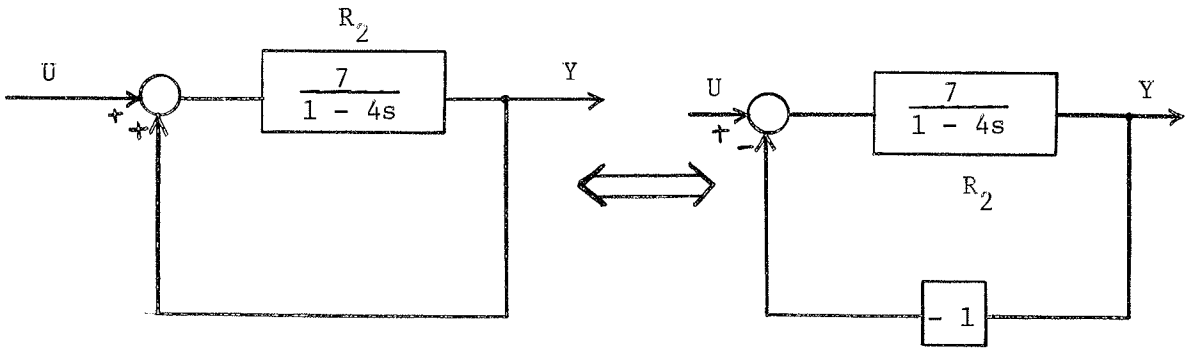
DIFFERENTIAL EQUATIØNS

NAME	VALUE	ERRØR	NAME	VALUE	ERRØR
Y	-1.2672E 06	1.0641E 07			

READY

Example 3: Positive Feedback

A numerical example will now be worked to illustrate how to apply the general results of the above analysis to a specific problem with positive feedback. Consider the following block diagram:



$$k_3 = 7 > 0$$

$$k_4 = -1$$

$$k_3 k_4 = 7(-1) = -7 < -1 < 0$$

By Case 3 of the analysis of a single differential equation with a feedback loop, the required convergence factor has the following bounds:

$$-\frac{(k_3 k_4 - 1)}{(k_3 k_4 + 1)} < R_2 < -1 \quad \text{or}$$

$$-1.\overline{333} = -\frac{-8}{-6} < R_2 < -1 \ .$$

These results are verified in the following computer simulation.

```
$LIST ITER5

01/03/72    14:31                    -22-


00010 STRP,/,1,  ,8,16,;,7,13,19,25,31,37,43,:,14,26,38,50,62,',0
00020 /$ SNUMB THKAA
00030 /$ IDENT T.KERR,KERR  37  578D  5-4171
00040 /$ PRØGRAM RLHS
00050 /$ LIMITS 1,16000,,1000
00060 /$ PRMFL H*,Z,R,WATSØN/ADA70A
00070 /$ DISC 02,X2R,20L
00080 /$ PRMFL P*,R/A/S,L,*/P
00090 /$ NØTIFY */S
00100 0;40;2;1
00110 'T.KERR 9/21/71
00120 'R&DC 37 578D 8*235-4171
00130 'CØNVERGENCE ØF ITERATIØNS FØR S.S. FØR PØSITIVE FDBK,1L00P
00140 I;1;50;1;5;1
00150 2
00160 1-5:1-5:1-6:1-2:10
00170 0:5:1:0:0
00180 32;Y;U;E
00190 U;6
00200 6
00210 E;19;2;U;Y
00220 Y;1;E;1;1
00230 1:-1.1666:7:1:-4
00240 0;40;1
00250 /$ ENDJØB

READY
```

Necessary for Convergence
$-1.33\overline{3} < R_7 < -1$

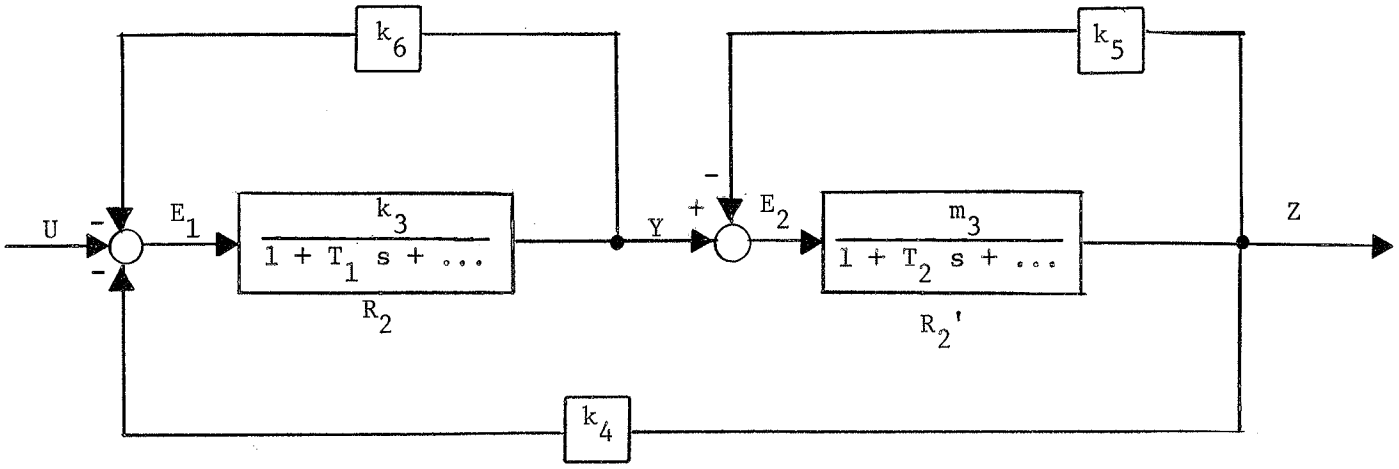
```
$LIST ØUT10
01/03/72   14:35
```

$R_2 = -1.1666$

```
ADA70  01/03/72  14.510


 CASE   1      STEADY-STATE INITIAL VALUES  ITERATIØN      3

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E       -1.0000E 00

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U        6.0000E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Y       -7.0000E 00


 CASE   1      FINAL VALUES AT TIME  5.0000E 00

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E       -1.0000E 00

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U        6.0000E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Y       -7.0000E 00


 CASE   1   CØNVERGENCE ØF ITERATIØNS FØR S.S. FØR PØSITIVE FDBK,1L00P

   TIME         Y             U             E

 0.          -7.0000E 00  6.0000E 00 -1.0000E 00
 0.          -7.0000E 00  6.0000E 00 -9.9999E-01
 1.0000E 00  -7.0000E 00  6.0000E 00 -1.0000E 00
 2.0000E 00  -7.0000E 00  6.0000E 00 -1.0000E 00
 3.0000E 00  -7.0000E 00  6.0000E 00 -1.0000E 00
 4.0000E 00  -7.0000E 00  6.0000E 00 -1.0000E 00
E5.0000E 00  -7.0000E 00  6.0000E 00 -1.0000E 00


CPU TIME - TØTAL =    0.509 SECØNDSN
```

C. Two D.E. with Arbitrary Feedback

Case 1: Series



The above block diagram can be used for positive feedback simply by making the necessary gains (k_4, k_5, or k_6) negative. The absence of feedback can be obtained by making the appropriate gains zero.

The difference equations which describe the iteration in the initial value calculation for the above configuration are:

$$Y_{N+1} = k_3 (U - k_6 Y_N - k_4 Z_N) + R_2 [k_3 (U - k_6 Y_N - k_4 Z_N) - Y_N]$$

$$Z_{N+1} = m_3 (Y_N - k_5 Z_N) + R_2' [m_3 (Y_N - k_5 Z_N) - Z_N]$$

or

$$Y_{N+1} = k_3 (1 + R_2) U - k_3 k_6 Y_N - R_2 k_3 k_6 Y_N - R_2 Y_N - k_3 k_4 Z_N - R_2 k_3 k_4 Z_N$$

$$Z_{N+1} = m_3 Y_N + R_2' m_3 Y_N - m_3 k_5 Z_N - R_2' m_3 k_5 Z_N - R_2' Z_N$$

or

$$Y_{N+1} + [k_3 k_6 (1 + R_2) + R_2] Y_N + k_3 k_4 (1 + R_2) Z_N = k_3 (1 + R_2) U$$

$$Z_{N+1} + [m_3 k_5 (1 + R_2') + R_2'] Z_N - m_3 (1 + R_2') Y_N = 0$$

or, equivalently,

$$\{E + [k_3 k_6 (1 + R_2) + R_2]\} Y_N + k_3 k_4 (1 + R_2) Z_N = k_3 (1 + R_2) U$$

$$- m_3 (1 + R_2') Y_N + \{E + [m_3 k_5 (1 + R_2') + R_2']\} Z_N = 0.$$

Now eliminate the variable Y_N to obtain a second degree characteristic equation for the system of iterations. The associated homogenous difference equation is:

$$E^2 + [m_3 k_5 (1 + R_2') + R_2' + k_3 k_6 (1 + R_2) + R_2] E +$$

$$\{[m_3 k_5 (1 + R_2') + R_2'] [k_3 k_6 (1 + R_2) + R_2] + m_3 k_3 k_4 (1 + R_2) (1 + R_2')\} = 0$$

The characteristic equation for the system of case 1 is:

$$E^2 + [m_3 k_5 (1 + R_2') + k_3 k_6 (1 + R_2) + R_2 + R_2'] E$$

$$+ R_2 R_2' + m_3 k_3 (k_5 k_6 + k_4) (1 + R_2) (1 + R_2') + R_2 m_3 k_5 (1 + R_2')$$

$$+ R_2' k_3 k_6 (1 + R_2) = 0 \quad .$$

The iteration equations converge if and only if the roots of the characteristic equation are within the unit circle. Jury's stability test gives a convenient criteria for determining when the roots of the characteristic equation are within the unit circle by looking at relationships between the coefficients of the characteristic equation. Jury's stability test (B.C. Kuo, 1963, p. 156) indicates that for the system of case 1, we have convergence if and

only if all of the following four inequalities are satisfied:

Condition I:

$$1 + [m_3 k_5 (1 + R_2') + k_3 k_6 (1 + R_2) + R_2 + R_2'] +$$

$$+ R_2 R_2' + m_3 k_3 (k_5 k_6 + k_4) (1 + R_2) (1 + R_2') + R_2 m_3 k_5 (1 + R_2')$$

$$+ R_2' k_3 k_6 (1 + R_2) > 0$$

Condition II:

$$1 - [m_3 k_5 (1 + R_2') + k_3 k_6 (1 + R_2) + R_2 + R_2']$$

$$+ R_2 R_2' + m_3 k_3 (k_5 k_6 + k_4) (1 + R_2) (1 + R_2') + R_2 m_3 k_5 (1 + R_2')$$

$$+ R_2' k_3 k_6 (1 + R_2) > 0$$

Condition III:

$$R_2 R_2' + m_3 k_3 (k_5 k_6 + k_4) (1 + R_2) (1 + R_2') + R_2 m_3 k_5 (1 + R_2')$$

$$+ R_2' k_3 k_6 (1 + R_2) + 1 > 0$$

Condition IV:

$$0 > R_2 R_2' + m_3 k_3 (k_5 k_6 + k_4) (1 + R_2) (1 + R_2') + R_2 m_3 k_5 (1 + R_2')$$

$$+ R_2' k_3 k_6 (1 + R_2) - 1$$

In order to assure convergence of the iterations, we must pick R_2 and R_2' so that all four of the inequalities of the preceding paragraph are satisfied. It is frequently useful to make the following definitions:

let $H_1 = 1 + R_2$

$H_2 = 1 + R_2'$

Now conditions I-IV become

Condition I-a:

$$(k_3 k_6 m_3 k_5 + m_3 k_3 k_4 + m_3 k_5 + k_3 k_6 + 1) H_1 H_2 > 0 \quad ,$$

Condition II-a:

$$4 - 2 (m_3 k_5 + 1) H_2 - 2 (k_3 k_6 + 1) H_1$$

$$+ (k_3 k_6 m_3 k_5 + m_3 k_3 k_4 + m_3 k_5 + k_3 k_6 + 1) H_1 H_2 > 0 \quad ,$$

Condition III-a:

$$2 + (k_3 k_6 m_3 k_5 + m_3 k_3 k_4 + m_3 k_5 + k_3 k_6 + 1) H_1 H_2$$

$$- (m_3 k_5 + 1) H_2 - (k_3 k_6 + 1) H_1 > 0 \quad ,$$

Condition IV-a:

$$(k_3 k_6 m_3 k_5 + m_3 k_3 k_4 + m_3 k_5 + k_3 k_6 + 1) H_1 H_2$$

$$- (m_3 k_5 + 1) H_2 - (k_3 k_6 + 1) H_1 < 0 \quad ,$$

Once we know in what region of the $H_1 - H_2$ plane these inequalities are satisfied, we know (by a translation) in what region of the $R_2 - R_2'$ plane the inequalities are all satisfied. Now let the following real valued functions of the two real variables H_1, H_2 be

$$h (H_1, H_2) \triangleq (k_3 k_6 m_3 k_5 + m_3 k_3 k_4 + m_3 k_5 + k_3 k_6 + 1) H_1 H_2 \quad ,$$

$$g (H_1, H_2) \triangleq 4 - 2 (m_3 k_5 + 1) H_2 - 2 (k_3 k_6 + 1) H_1$$

$$+ (k_3 k_6 m_3 k_5 + m_3 k_3 k_4 + m_3 k_5 + k_3 k_6 + 1) H_1 H_2 \quad ,$$

$$p(H_1, H_2) \triangleq 2 - (m_3 k_5 + 1) H_2 - (k_3 k_6 + 1) H_1$$

$$+ (k_3 k_6 m_3 k_5 + m_3 k_3 k_4 + m_3 k_5 + k_3 k_6 + 1) H_1 H_2 \quad ,$$

and

$$q(H_1, H_2) \triangleq - (m_3 k_5 + 1) H_2 - (k_3 k_6 + 1) H_1$$

$$+ (k_3 k_6 m_3 k_5 + m_3 k_3 k_4 + m_3 k_5 + k_3 k_6 + 1) H_1 H_2 \quad .$$

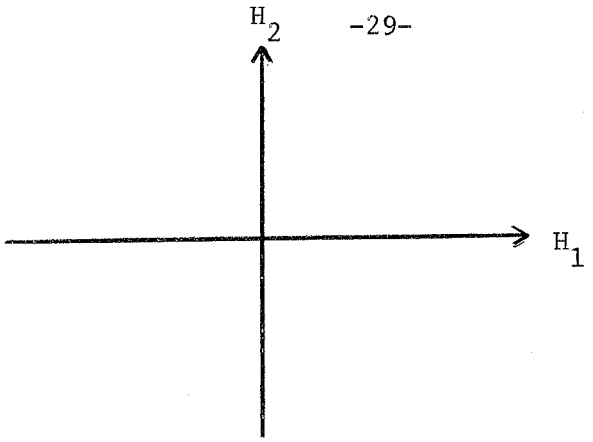
We want to find the (H_1, H_2) a member of Euclidean 2-space, $\mathbb{R}^2$, such that $h(H_1, H_2) > 0$, $g(H_1, H_2) > 0$, $p(H_1, H_2) > 0$, and $q(H_1, H_2) < 0$, that is, we want the set

$\{(H_1, H_2) \mid h(H_1, H_2) > 0\} \bigcap \{(H_1, H_2) \mid g(H_1, H_2) > 0\} \bigcap \{(H_1, H_2) \mid p(H_1, H_2) > 0\}$

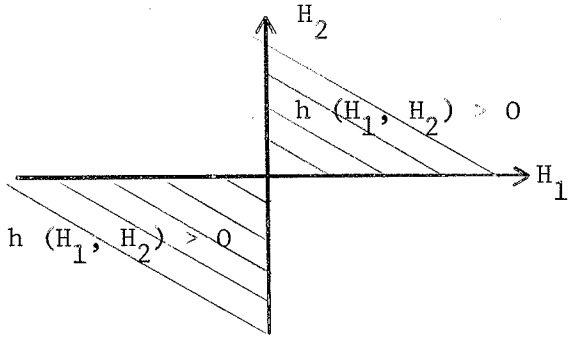
$\bigcap \{(H_1, H_2) \mid q(H_1, H_2) < 0\}$, the region of the $H_1 - H_2$ plane in which all of the inequalities are satisfied.

We can find the regions in which, simultaneously, $h(H_1, H_2) > 0$, $g(H_1, H_2) > 0$, $p(H_1, H_2) > 0$ and $q(H_1, H_2) < 0$ by first finding the boundaries of the region for each individual function, determining on which side of the boundary the correct inequality is satisfied for each function, and then taking the intersection of the allowable region of the $H_1 - H_2$ plane for all four functions.

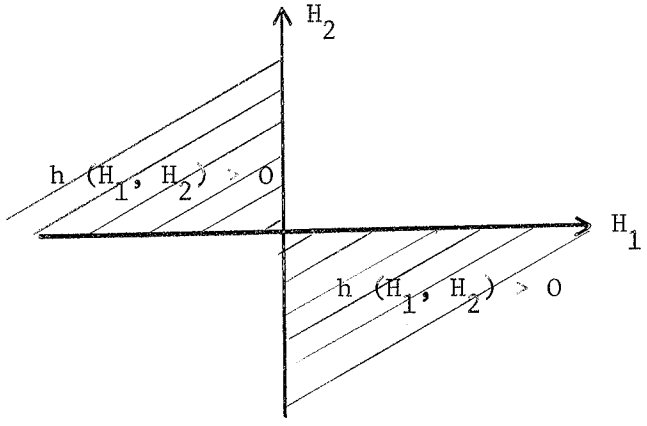
For $h(H_1, H_2)$, the boundary occurs where $h(H_1, H_2) = 0$, which is $H_1 H_2 = 0$ (i.e., $H_1 = 0$ and $H_2 = 0$ are the boundaries).



If $(k_3\ k_6\ m_3\ k_5 + m_3\ k_3\ k_4 + m_3\ k_5 + k_3\ k_6 + 1) > 0$, then the allowable region is



If $(k_3\ k_6\ m_3\ k_5 + m_3\ k_3\ k_4 + m_3\ k_5 + k_3\ k_6 + 1) < 0$, then the allowable region is



For $g\ (H_1, H_2)$, the boundary occurs where $g\ (H_1, H_2) = 0$, which is

$$H_2 = \frac{[-\ 4 + 2\ (k_3\ k_6 + 1)\ H_1]}{[-\ 2\ (m_3\ k_5 + 1) + (k_3\ k_6\ m_3\ k_5 + m_3\ k_3\ k_4 + m_3\ k_5 + k_3\ k_6 + 1)\ H_1]}$$

a hyperbola having a vertical asymptote at

$$H_1 = \frac{2\ (m_3\ k_5 + 1)}{(k_3\ k_6\ m_3\ k_5 + m_3\ k_3\ k_4 + m_3\ k_5 + k_3\ k_6 + 1)} \quad ,$$

a horizontal asymptote at

$$H_2 = \frac{2\ (K_3\ k_6 + 1)}{k_3\ k_6\ m_3\ k_5 + m_3\ k_3\ k_4 + m_3\ k_5 + k_3\ k_6 + 1} \quad ,$$

an H_1 intercept at

$$H_1 = \frac{2}{(k_3\ k_6 + 1)} \quad ,$$

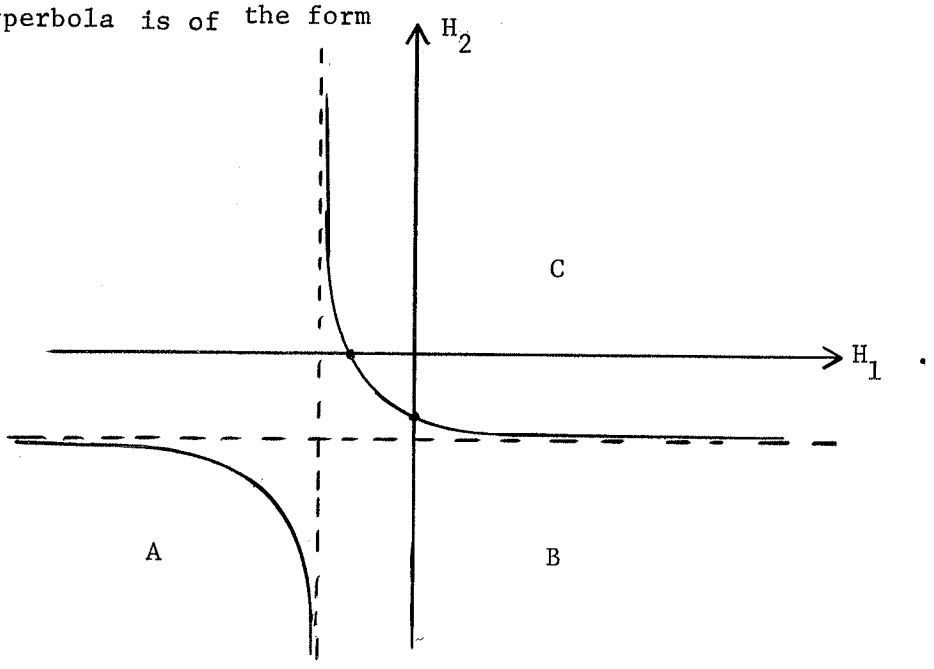
and an H_2 intercept at

$$H_2 = \frac{2}{(m_3\ k_5 + 1)}$$

If the H_1 - intercept is larger than the vertical asymptote, i.e., if

$$\frac{2\ (m_3\ k_5 + 1)}{(k_3\ k_5\ m_3\ k_5 + m_3\ k_3\ k_4 + m_3\ k_5 + k_3\ k_6 + 1)} < \frac{2}{k_3\ k_6 + 1} \quad ,$$

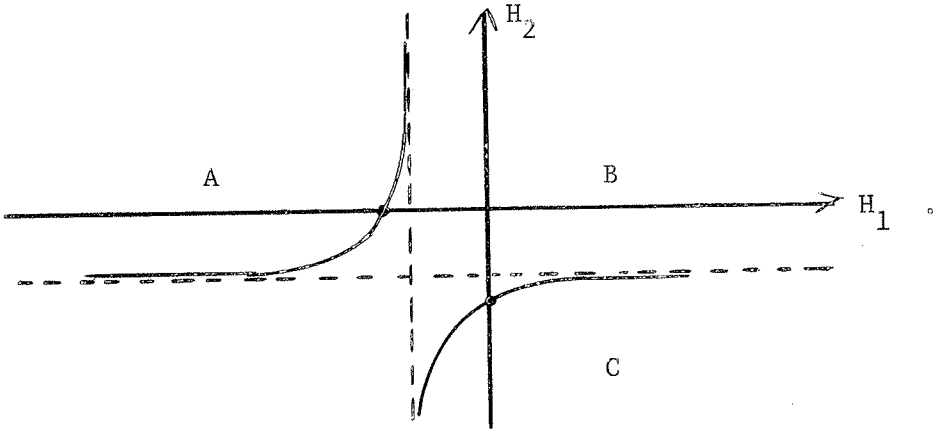
the hyperbola is of the form



However, if the H_1 - intercept is smaller than the vertical asymptote, i.e., if

$$\frac{2\ (m_3\ k_5 + 1)}{(k_3\ k_6\ m_3\ k_5 + m_3\ k_3\ k_4 + m_3\ k_5 + k_3\ k_6 + 1)} > \frac{2}{k_3\ k_6 + 1}\ ,$$

the hyperbola is of the form



The region partitioned by the graph of the hyperbola must be further examined to determine in which regions, A, B, and C, the inequality, $g\ (H_1, H_2) > 0$, is satisfied. This can be determined by picking one convenient point (H_1^*, H_2^*) and computing $g\ (H_1^*, H_2^*)$. The direction of the inequality between $g\ (H_1, H_2)$ and 0 is the same in region A as it is in region C, while the inequality is reversed in region B.

Similarly, the boundaries for $p\ (H_1, H_2)$ and $q\ (H_1, H_2)$ are the hyperbolas

$$H_2 = \frac{[-\ 2 + (k_3\ k_6 + 1)\ H_1]}{[-\ (m_3\ k_5 + 1) + (k_3\ k_6\ m_3\ k_5 + m_3\ k_3\ k_4 + m_3\ k_5 + k_3\ k_6 + 1)\ H_1]}$$

and

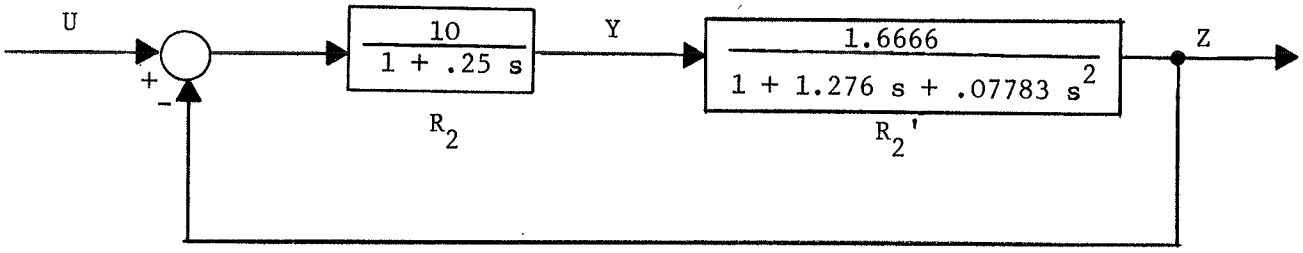
$$H_2 = \frac{[+\ (k_3\ k_6 + 1)\ H_1]}{[-\ (m_3\ k_5 + 1) + (k_3\ k_6\ m_3\ k_5 + m_3\ k_3\ k_4 + m_3\ k_5 + k_3\ k_6 + 1)\ H_1]}\ ,$$

respectively. The orientation of the hyperbolas and the regions in which the inequalities $p(H_1, H_2) > 0$, $q(H_1, H_2) < 0$ are determined in a manner analogous to that described for $g(H_1, H_2)$.

Once the regions in which each inequality is satisfied is obtained, one may obtain the region in which all of the inequalities are satisfied by superimposing the various individual graphs. This can be visualized as overlaying several transparancies of the four individual graphs to obtain the appropriate region.

Example 4: Series with Negative Feedback

A numerical example will now be worked to illustrate how to apply the general results of Section II:Case 1 to a specific problem. Consider the following block diagram which is to be simulated using ADA:



In the simulation, it is desired that the steady-state initial values be calculated by ADA; to do this, it is necessary to specify R_2 and R_2', the acceleration factors. To obtain correct bounds on R_2 and R_2' we can use the results of Case 1 with

$$k_5 = 0 = k_6$$

$$k_4 = 1 \; ; \; k_3 = 10 \; ; \; m_3 = 1.6666.$$

Now $k_3 k_6 m_3 k_5 = 0$; $m_3 k_3 k_4 = 16.666$; $m_3 k_5 = 0 = k_3 k_6$. The four functions are

$$h(H_1, H_2) = (17.666) H_1 H_2 ,$$

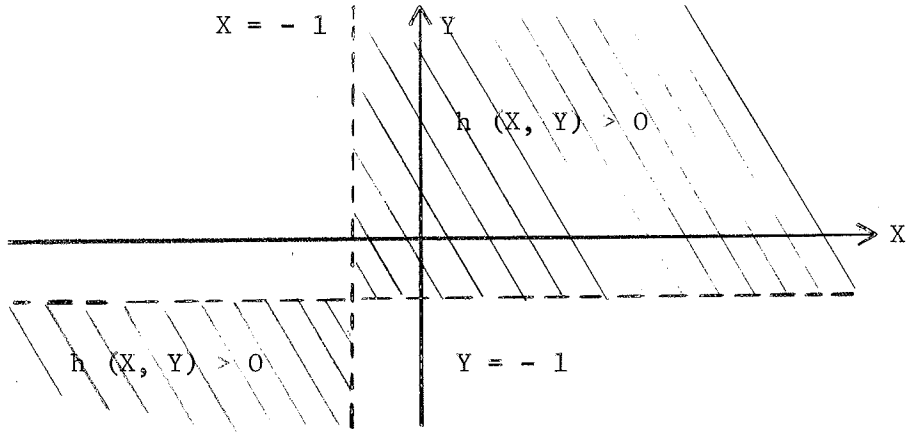
$$g(H_1, H_2) = 4 - 2 H_2 - 2 H_1 + (17.666) H_1 H_2 ,$$

$$p(H_1, H_2) = 2 - H_2 - H_1 + (17.666) H_1 H_2 ,$$

$$q(H_1, H_2) = - H_2 - H_1 + (17.666) H_1 H_2 ;$$

we seek the region in which $h(H_1, H_2) > 0$, $g(H_1, H_2) > 0$, $p(H_1, H_2) > 0$, and $q(H_1, H_2) < 0$ simultaneously. Recall from Case 1 that $H_1 \triangleq 1 + R_2$, $H_2 \triangleq 1 + R_2'$; for convenience let $X \triangleq R_2$, $Y = R_2'$.

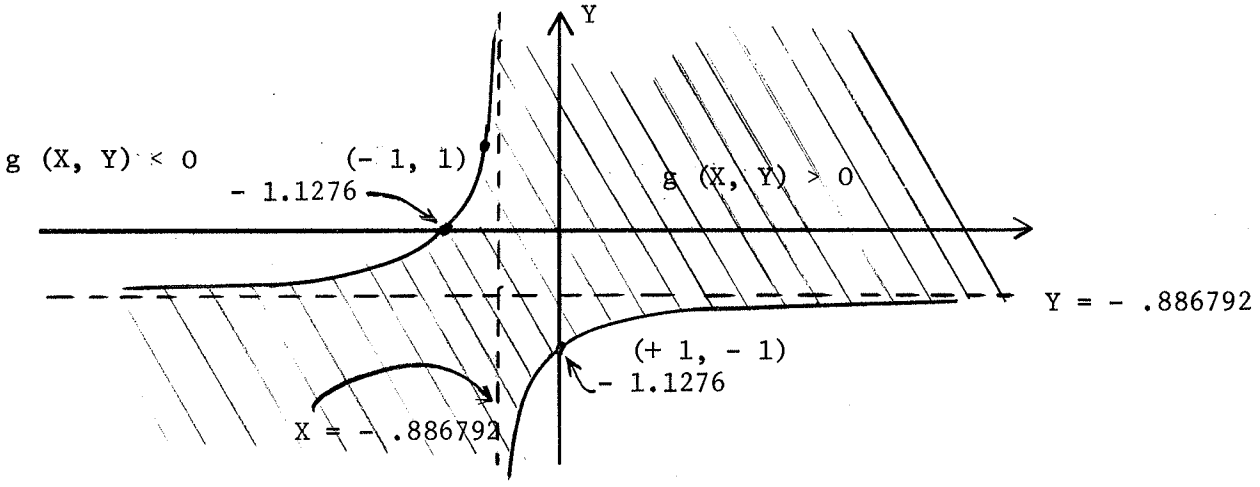
Considering the first equation, $h(X, Y) = 17.666 (1 + Y) (1 + X)$; the boundary of the desired region is where $h(X, Y) = (17.666) (1 + Y) (1 + X) = 0$; this means that $Y = -1$ and $X = -1$ are boundaries. The region in which $h(X, Y) > 0$ is shaded below.



Considering the second equation, $g(X,Y,) = 4 - 2 (1 + X) - 2 (1 + Y) + 17.\overline{6} (1 + X) (1 + Y)$; the boundary of the desired region is where $g(X, Y) = 4 - 2 - 2X - 2 - 2Y + 17.666 + 17.666 X + 17.666 Y + 17.666 XY = 0$, or

$$Y = \frac{-(17.666 + 15.666\ X)}{(15.666 + 17.666\ X)} .$$

The above is the equation of a hyperbola having vertical and horizontal asymptates of - .886792 and X- and Y- intercepts of - 1.1276. The region in which g (X, Y) > 0 is shaded below.

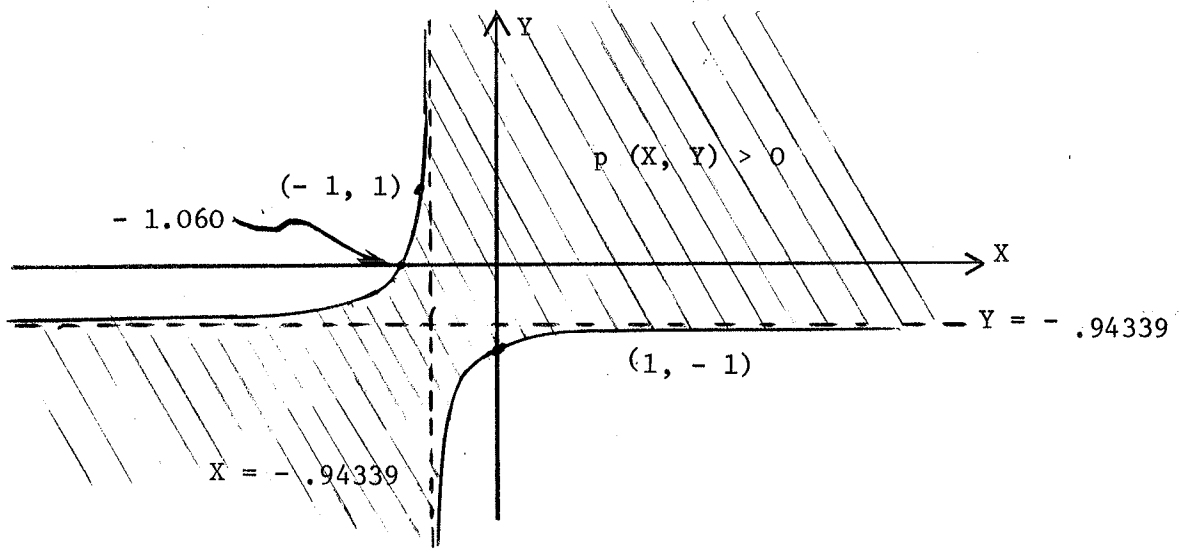


Considering the third equation, p (X, Y) = 2 - (1 + Y) - (1 + X) + 17.666 (1 + X + Y + X Y) ; the boundary of the desired region is where

p (X, Y) = Y (16.666 + 17.666 X) + (17.666 + 16.666 X) = 0

or $Y = \frac{-(17.666 + 16.666\ X)}{(16.666 + 17,666\ X)}$

The region in which p (X, Y) > 0 is shaded below.

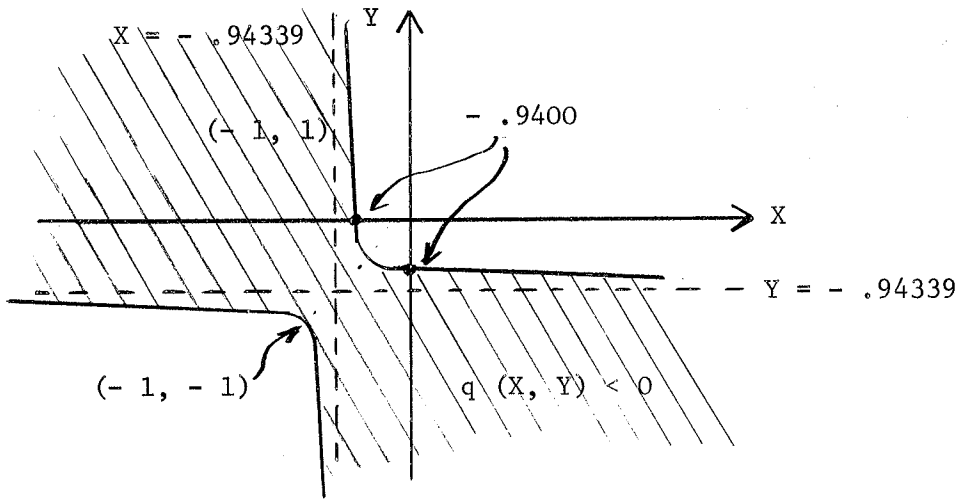


Considering the fourth equation, $q(X, Y) = -(1 + Y) - (1 + X) + 17.666(1 + X)(1 + Y)$; the boundary of the desired region is where

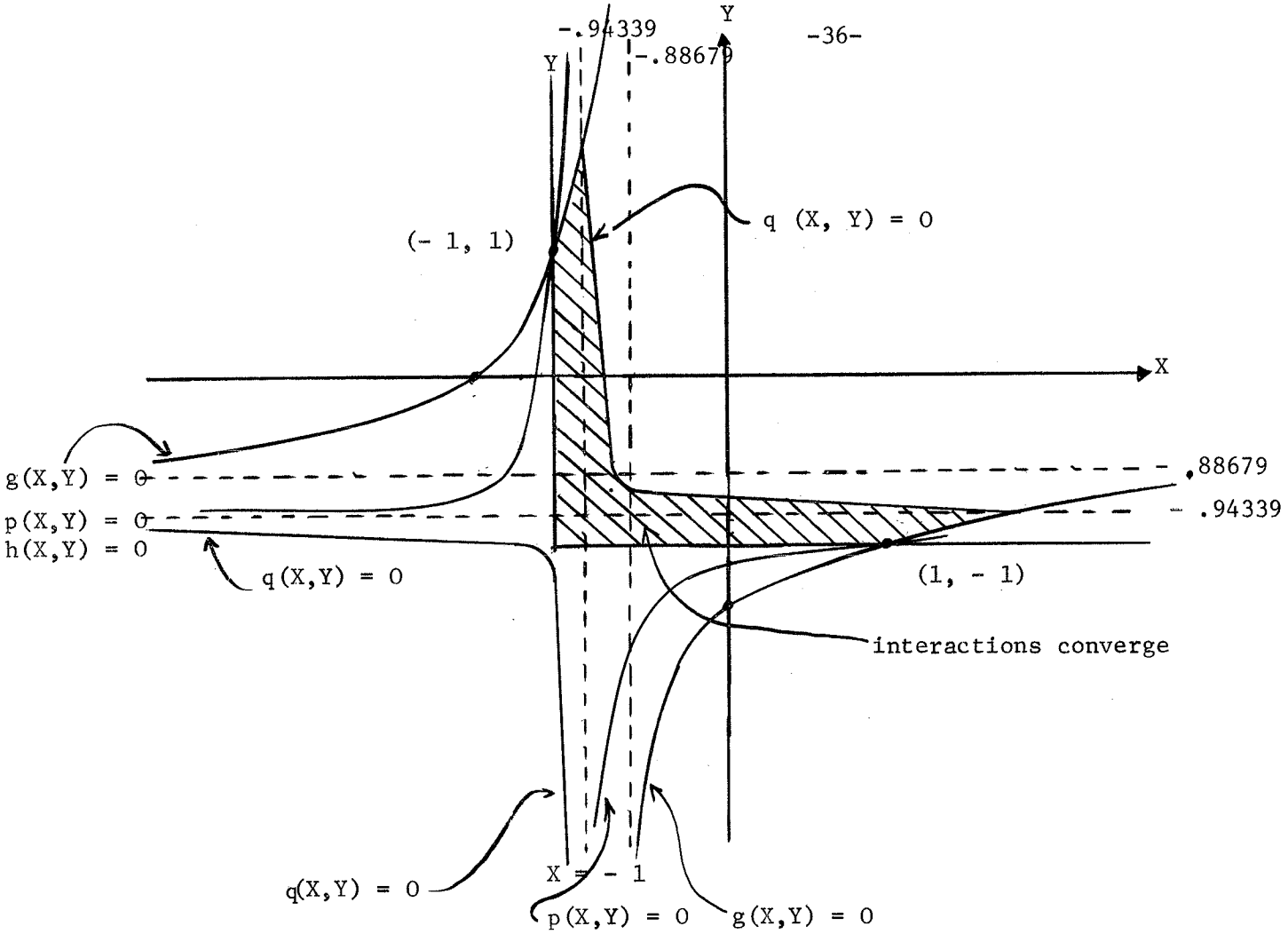
$$q(X, Y) = 15.666 + 16.666\,X + 16.666\,Y + 17.666\,XY = 0$$

or

$$Y = \frac{-(15.666 + 16.666\,X)}{(16.666 + 17.666\,X)}$$; the region in which $q(X, Y) < 0$ is shaded below.

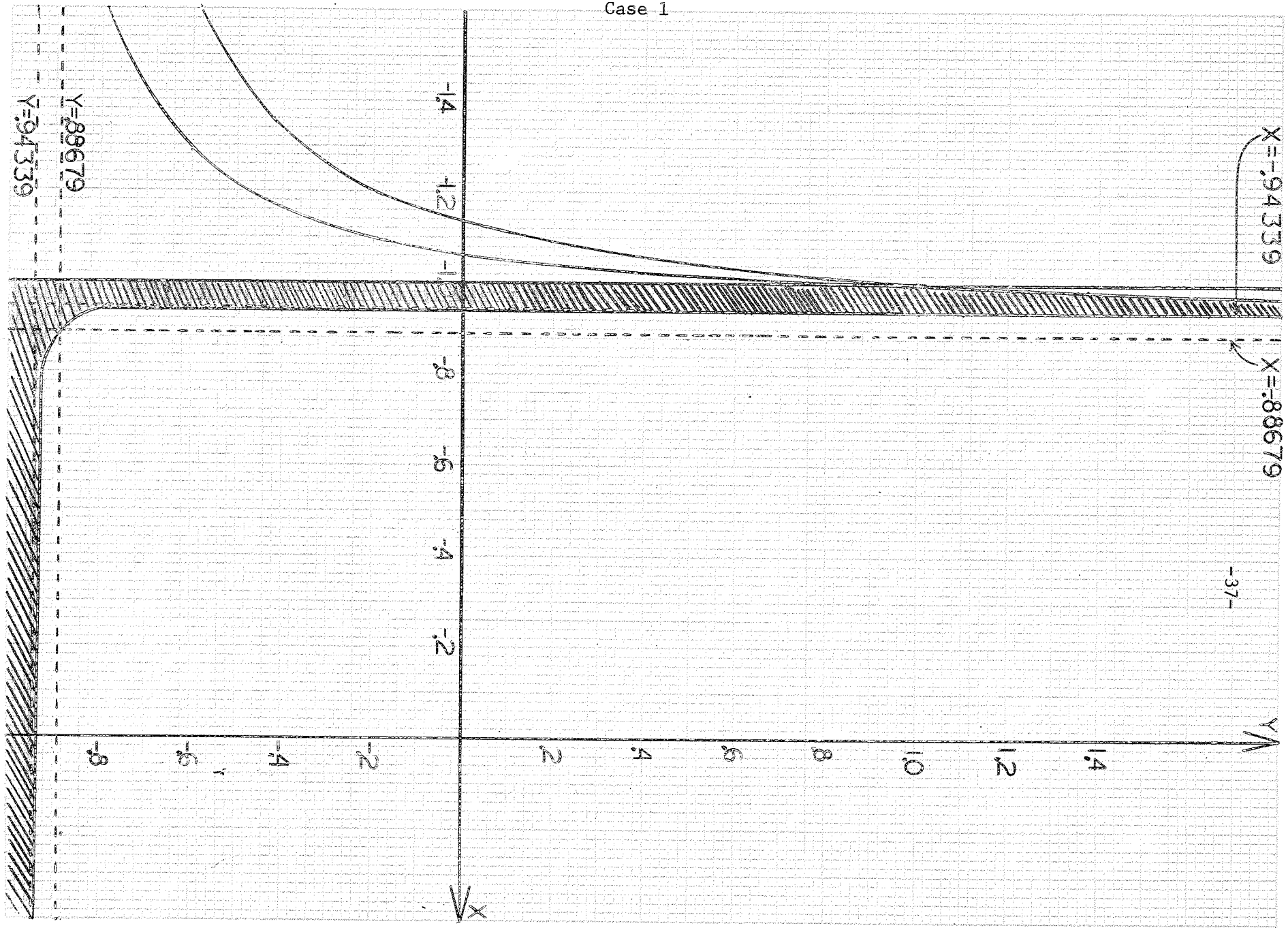


Now that we have the region in which the four inequalities are satisfied, we can plot them all on the same graph to find their common intersection (i.e., that region in which all four inequalities are satisfied). The detailed graph of all four is now given.



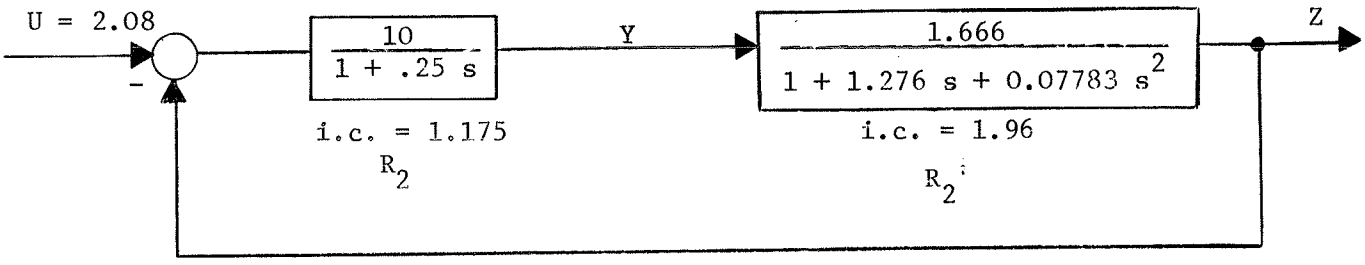
The shaded area in the region of the (X, Y) - plane (i.e., the region of the $(R_2 - R_2')$ - plane) for which the iteration scheme for calculating the initial conditions converge. A more detailed graph which has the proper scale follows.

Region of Convergence for $R_2 + R_2'$ FOR EXAMPLE ILLUSTRATING Case 1



Simulation of Example

A computer simulation can now be performed to verify the results of the theoretical calculations.



Algebraic result:

$Y = 10\ (U - Z)$

$Z = 16.\overline{6}\ [U - Z]$

$17.\overline{6}\ Z = 16.\overline{6}\ U$

$$Z = \frac{16.\overline{6}}{17.\overline{6}}\ U$$

$Y = 10\ (1 - 0.94339)\ U$

For U = 2.08, the answers should be Z = 1.9622

Y = 1.1778

These answers are verified by the computer choosing R_2 and R_2' in the shaded region of the composite graph. For $R_2 = -.94339$, $R_2' = +1.2$, this point is near the boundary of the allowable region for convergence and 225 iterations are required. For $R_2 = -.97$, $R_2' = 1.4$, well within the shaded region, convergence is obtained in 22 iterations. Convergence is obtained for $R_2 = +1.4$ and $R_2' = -.97$. It is not necessary for the initial guess of the initial condition to be close to the actual solution. They can be off by a large factor and still the iterations will converge. Notice that the

```
$LIST ITER4
01/03/72   12:55

00010 STRP,/,1,  ,8,16,;,7,13,19,25,31,37,43,:,14,26,38,50,62,',0
00020 /$ SNUMB THKAA
00030 /$ IDENT T.KERR,KERR  37  578D  5-4171
00040 /$ PRØGRAM RLHS
00050 /$ LIMITS 1,16000,,1000
00060 /$ PRMFL H*,Z,R,WATSØN/ADA70A
00070 /$ DISC 02,X2R,20L
00080 /$ PRMFL P*,R/A/S,L,*/P
00090 /$ NØTIFY */$
00100 0;40;2;1
00110 'T.KERR 9/21/71
00120 'R&DC 37 578D 8*235-4171
00130 'CØNVERGENCE ØF TWØ LØØP S.S. ITERATIØNS FØR I.C.
00140 1;1;300;1;5;1
00150 2
00160 1-5:1-5:1-6:1-2:10
00170 0:1:1:0:0
00180 ;2;U;Y;Z;E
00190 U;6
00200 2.08
00210 E;11;2;U;Z
00220 0:1:-1
00230 Y;1;E;1;1
00240 1.175:-.97:10:1:.25
00250 Z;1;Y;2;1
00260 1.96:1.4:1.666:1:1.276:.07783
00270 0;40;1
00280 /$ ENDJØB

READY
```

```
$LIST ØUT11
Ø1/Ø3/72   13:32
```

$R_2 = 1.4$

$R_2'' = -.97$

```
ADA7Ø  Ø1/Ø3/72  12.900


 CASE   1      STEADY-STATE INITIAL VALUES  ITERATIØN     22

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E        1.1778E-01

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U        2.0800E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Z        1.9622E 00   0.
 Y        1.1778E 00


 CASE   1      FINAL VALUES AT TIME   1.0000E 00

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E        1.1778E-01

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U        2.0800E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Z        1.9622E 00   1.0507E-06
 Y        1.1778E 00


 CASE   1   CØNVERGENCE ØF TWØ LØØP S.S. ITERATIØNS FØR I.C.

   TIME         U            Y            Z            E

 0.          2.0800E 00   1.1778E 00   1.9622E 00   1.1778E-01
 0.          2.0800E 00   1.1778E 00   1.9622E 00   1.1778E-01
E1.0000E 00  2.0800E 00   1.1778E 00   1.9622E 00   1.1778E-01


 CPU TIME - TØTAL =    0.527 SECØNDS.
READY
```

```
$LIST ØUT12
01/03/72   13:42
```

$R_2 = -.97$

$R_2'' = 1.4$

```
ADA7Ø  01/03/72  13.643


 CASE   1      STEADY-STATE INITIAL VALUES  ITERATIØN     24

 ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E       1.1778E-01

 CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U       2.0800E 00

 DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Z       1.9622E 00   0.
 Y       1.1778E 00


 CASE   1      FINAL VALUES AT TIME  1.0000E 00

 ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E       1.1778E-01

 CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U       2.0800E 00

 DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Z       1.9622E 00   1.2341E-05
 Y       1.1778E 00


 CASE   1   CØNVERGENCE ØF TWØ LØØP S.S. ITERATIØNS FØR I.C.

   TIME         U            Y            Z            E

 0.          2.0800E 00  1.1778E 00  1.9622E 00  1.1778E-01
 0.          2.0800E 00  1.1778E 00  1.9622E 00  1.1778E-01
E1.0000E 00  2.0800E 00  1.1778E 00  1.9622E 00  1.1778E-01


 CPU TIME - TØTAL =    0.575 SECØNDS.
READY
```

```
$LIST ØUT13

01/03/72   13:51

ADA7Q  01/03/72  13.818
```

$R_2 = -.94339$

$R_2' = 1.2$

```
 CASE   1      STEADY-STATE INITIAL VALUES  ITERATIØN   225

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E       1.1778E-01

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U       2.0800E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Z       1.9622E 00   0.
 Y       1.1778E 00

 CASE   1      FINAL VALUES AT TIME  1.0000E 00

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E       1.1778E-01

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U       2.0800E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Z       1.9622E 00  -1.2558E-06
 Y       1.1778E 00

 CASE   1   CØNVERGENCE ØF TWØ LØØP S.S. ITERATIØNS FØR I.C.

   TIME         U            Y            Z            E

 0.          2.0800E 00   1.1778E 00   1.9622E 00   1.1778E-01
 0.          2.0800E 00   1.1778E 00   1.9622E 00   1.1778E-01
E1.0000E 00  2.0800E 00   1.1778E 00   1.9622E 00   1.1778E-01


 CPU TIME - TØTAL =    0.735 SECØNDS.
READY
```

```
$LIST ITER4
01/03/72    14:06

00010 STRP,/,1,  ,8,16,:,7,13,19,25,31,37,43,:,14,26,38,50,62,',0
00020 /$ SNUMB THKAA
00030 /$ IDENT T.KERR,KERR  37  578D  5-4171
00040 /$ PRØGRAM RLHS
00050 /$ LIMITS 1,16000,,1000
00060 /$ PRMFL H*,Z,R,WATSØN/ADA70A
00070 /$ DISC 02,X2R,20L
00080 /$ PRMFL P*,R/A/S,L,*/P
00090 /$ NØTIFY */S
00100 0;40;2;1
00110 'T.KERR 9/21/71
00120 'R&DC 37 578D 8*235-4171
00130 'CØNVERGENCE ØF TWØ LØØP S.S. ITERATIØNS FØR I.C.
00140 1;1;300;1;5;1
00150 2
00160 1-5:1-5:1-6:1-2:10
00170 0:1:1:0:0
00180 ;2;U;Y;Z;E
00190 U;6
00200 2.08
00210 E;11;2;U;Z
00220 0:1:-1
00230 Y;1;E;1;1
00240 5:-.97:10:1:.25
00250 Z;1;Y;2;1
00260 6:1.4:1.666:1:1.276:.07783
00270 0;40;1
00280 /$ ENDJØB

READY
```

Initial Guess
Way Off

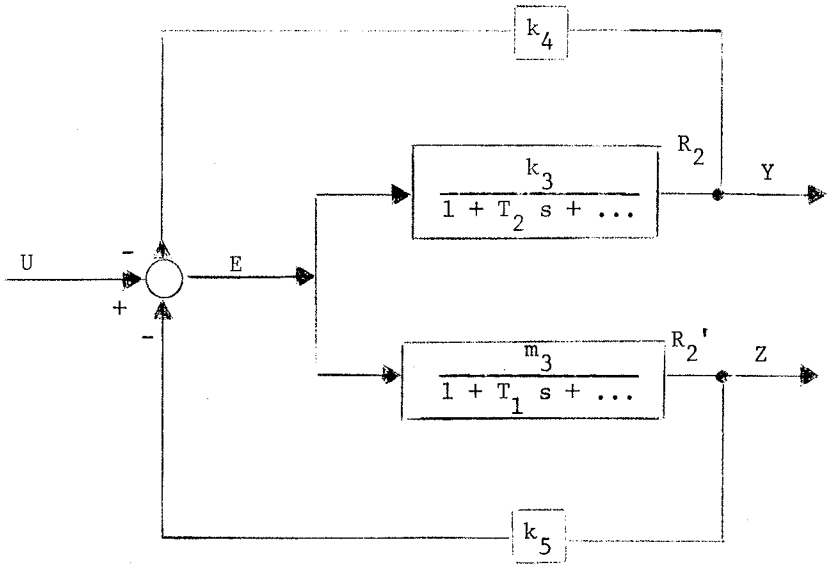
```
$LIST ØUT14
01/03/72   14:09

ADA70  01/03/72  14.099
```

$R_2' = 1.4$
$R_2 = -.97$
Initial Guess Way Off

```
 CASE   1      STEADY-STATE INITIAL VALUES  ITERATIØN    39

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E       1.1778E-01

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U       2.0800E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Z       1.9622E 00   0.
 Y       1.1778E 00


 CASE   1      FINAL VALUES AT TIME  1.0000E 00

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E       1.1778E-01

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U       2.0800E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1 ...
 Z       1.9622E 00   1.4475E-06
 Y       1.1778E 00


 CASE   1   CØNVERGENCE ØF TWØ LØØP S.S. ITERATIØNS FØR I.C.

   TIME         U            Y            Z            E

 0.          2.0800E 00   1.1778E 00   1.9622E 00   1.1778E-01
 0.          2.0800E 00   1.1778E 00   1.9622E 00   1.1778E-01
E1.0000E 00  2.0800E 00   1.1778E 00   1.9622E 00   1.1778E-01


 CPU TIME - TØTAL =    0.615 SECØNDS.
READY
```

choice $R_2 = 1.4$, $R_2'' = -.97$ was nearer to the center of the shaded region in the graph than was the choice $R_2 = -.97339$, $R_2' = 1.2$, hence convergence was faster.

Case 2: Parallel



The above block diagram can be used for a situation in which there is positive feedback simply by making the necessary gains (k_4, k_5) negative. The absence of feedback can be obtained by making the appropriate gain zero.

The difference equations which describe the iteration in the initial value calculation for the above configuration are:

$$Y_{N+1} = k_3 (U - k_4 Y_N - k_5 Z_N) + R_2 [k_3 (U - k_4 Y_N - k_5 Z_N) - Y_N]$$

$$Z_{N+1} = m_3 (U - k_4 Y_N - k_5 Z_N) + R_2' [m_3 (U - k_4 Y_N - k_5 Z_N) - Z_N]$$

or

$$Y_{N+1} = k_3 (1 + R_2) U - [k_3 k_4 (1 + R_2) + R_2] Y_N - [k_3 k_5 (1 + R_2)] Z_N$$

$$Z_{N+1} = m_3 (1 + R_2') U - [m_3 k_4 (1 + R_2')] Y_N - [m_3 k_5 (1 + R_2') + R_2'] Z_N$$

or, equivalently,

$$\{E + [k_3 k_4 (1 + R_2) + R_2]\} Y_N + k_3 k_5 (1 + R_2) Z_N = k_3 (1 + R_2) U$$

$$[m_3 k_4 (1 + R_2')] Y_N + \{E + [m_3 k_5 (1 + R_2') + R_2']\} Z_n = m_3 (1 + R_2') U .$$

Eliminating the variable Y_N to obtain a second degree characteristic equation for the system of iterations yields a characteristic equation for the system of

$$0 = E^2 + [m_3 k_5 (1 + R_2') + R_2' + k_3 k_4 (1 + R_2) + R_2] E$$
$$+ \{[m_3 k_5 (1 + R_2') + R_2'] [k_3 k_4 (1 + R_2) + R_2] - m_3 k_4 k_3 k_5 (1 + R_2) (1 + R_2')\}$$

With the substitution $H_1 \triangleq 1 + R_2$, $H_2 \triangleq 1 + R_2'$, the characteristic equation becomes

$$0 = E^2 + [(m_3 k_5 + 1) H_2 + (k_3 k_4 + 1) H_1 - 2] E$$
$$+ [(m_3 k_5 + k_3 k_4 + 1) H_1 H_2 - (k_3 k_4 + 1) H_1 - (m_3 k_5 + 1) H_2 + 1]$$

The four conditions that must be satisfied in order that the roots of the characteristic equation remain within the unit circle are:

Condition I-a:

$$(m_3 k_5 + k_3 k_4 + 1) H_1 H_2 > 0 ,$$

Condition II-a:

$$4 - 2 (m_3 k_5 + 1) H_2 - 2 (k_3 k_4 + 1) H_1 + (m_3 k_5 + k_3 k_4 + 1) H_1 H_2 > 0 ,$$

Condition III-a:

$$2 - (k_3 k_4 + 1) H_1 - (m_3 k_5 + 1) H_2 + (m_3 k_5 + k_3 k_4 + 1) H_1 H_2 > 0 ,$$

Condition IV-a:

$$- (k_3 k_4 + 1) H_1 - (m_3 k_5 + 1) H_2 + (m_3 k_5 + k_3 k_4 + 1) H_1 H_2 < 0 .$$

Now we proceed in a manner analogous to Case 1 by defining functions of the two variables H_1, H_2, and finding the boundaries upon which the functions are identically zero.

Let

$$h (H_1, H_2) \triangleq (m_3 k_5 + k_3 k_4 + 1) H_1 H_2 ,$$

$$g (H_1, H_2) \triangleq 4 - 2 (m_3 k_5 + 1) H_2 - 2 (k_3 k_4 + 1) H_1 + (m_3 k_5 + k_3 k_4 + 1) H_1 H_2 ,$$

$$p (H_1, H_2) \triangleq 2 - (k_3 k_4 + 1) H_1 - (m_3 k_5 + 1) H_2 + (m_3 k_5 + k_3 k_4 + 1) H_1 H_2 ,$$

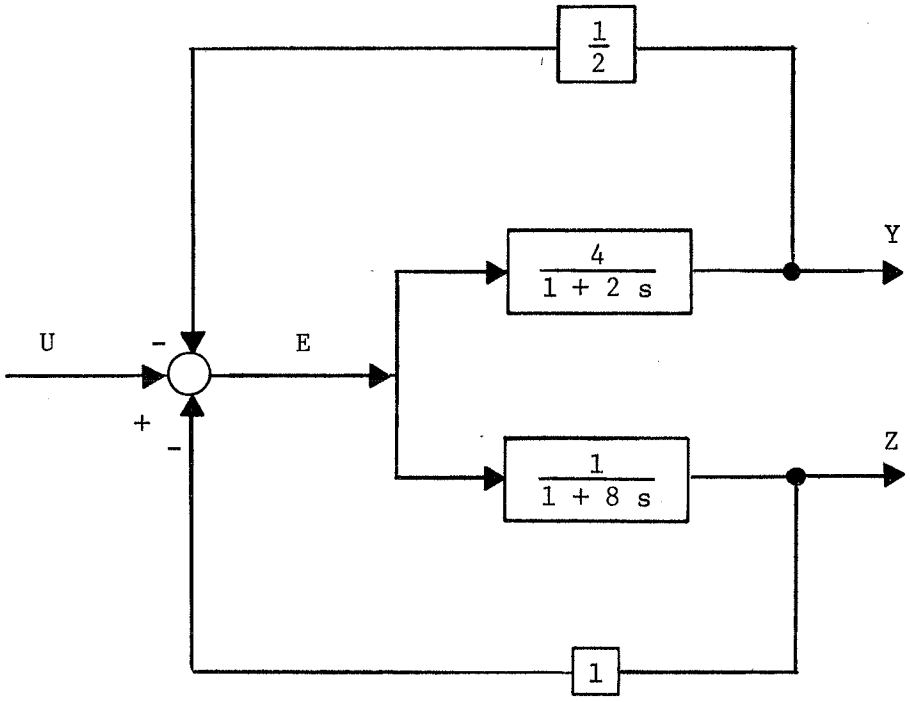
$$q (H_1, H_2) \triangleq - (k_3 k_4 + 1) H_1 - (m_3 k_5 + 1) H_2 + (m_3 k_5 + k_3 k_4 + 1) H_1 H_2 .$$

We want to find the region in which $h (H_1, H_2) > 0$, $g (H_1, H_2) > 0$, $p (H_1, H_2) > 0$, and $q (H_1, H_2) < 0$. First the boundaries are found from $h (H_1, H_2) = 0$, $g (H_1, H_2) = 0$, $p (H_1, H_2) = 0$, and $q (H_1, H_2) = 0$. Then by superimposing the various graphs, we find

$\{(H_1, H_2) \mid h(H_1, H_2) > 0\} \cap \{(H_1, H_2) \mid g(H_1, H_2 > 0\} \cap \{(H_1, H_2) \mid p(H_1, H_2) > 0\}$ $\cap$ $\{(H_1, H_2) \mid q(H_1, H_2) < 0\}$, the region in which all of the inequalities are satisfied.

Example 5: Parallel

An example will now be presented which illustrates how to apply the results of the analysis of how to pick the acceleration factors to ensure the convergence of the i.c. calculations for two parallel differential equations.



$k_4 = \frac{1}{2}$; $k_3 = 4$; $m_3 = 1$; $k_5 = 1$

$m_3 k_5 = 1$; $k_3 k_4 = 2$; $(m_3 k_5 + 1) = 2$

$(m_3 k_5 + k_3 k_4 + 1) = (1 + 2 + 1) = 4$; $(k_3 k_4 + 1) = 3$

The four functions of interest are:

$$h\ (H_1, H_2) \triangleq 4\ H_1\ H_2$$

$$g\ (H_1, H_2) \triangleq 4 - 2 \cdot 2\ H_2 - 2 \cdot 3\ H_1 + 4\ H_1\ H_2\ ,$$

$$= 4 - 4\ H_2 - 6\ H_1 + 4\ H_1\ H_2\ ,$$

$$p\ (H_1, H_2) \triangleq 2 - 3\ H_1 - 2\ H_2 + 4\ H_1\ H_2\ ,$$

$$q\ (H_1, H_2) \triangleq - 3\ H_1 - 2\ H_2 + 4\ H_1\ H_2\ .$$

Next we find the boundaries where these functions of two variables change from negative to positive (and since they are continuous, pass through zero).

For convenience let $X \triangleq R_2 = H_1 - 1$, $Y \triangleq R_2' = H_2 - 1$. From $h\ (X, Y) = 0$, we obtain

$0 = 4\ (1 + X)\ (1 + Y) \implies X = - 1$, $Y = - 1$, a degenerate hyperbola.

From $g\ (X, Y) = 0$, we obtain

$$0 = 4 - 4\ (1 + Y) - 6\ (1 + X) + 4\ (1 + X + Y + XY)$$

$$= 4 - 4 - 6 + 4 - 2\ X + 4\ X\ Y$$

or $Y = \frac{2 + 2\ X}{4\ X} = \frac{1 + X}{2\ X}$, a hyperbola.

From $p\ (X, Y) = 0$, we obtain

$$0 = 2 - 3\ (1 + X) - 2\ (1 + Y) + 4\ (1 + X + Y + XY)$$

$$= 2 - 3 - 2 + 4 + X + 2\ Y + 4\ XY$$

or

$$Y = \frac{-(1 + X)}{(2 + 4X)} = -\frac{1}{4} \cdot \frac{(1 + X)}{(\frac{1}{2} + X)} ,$$

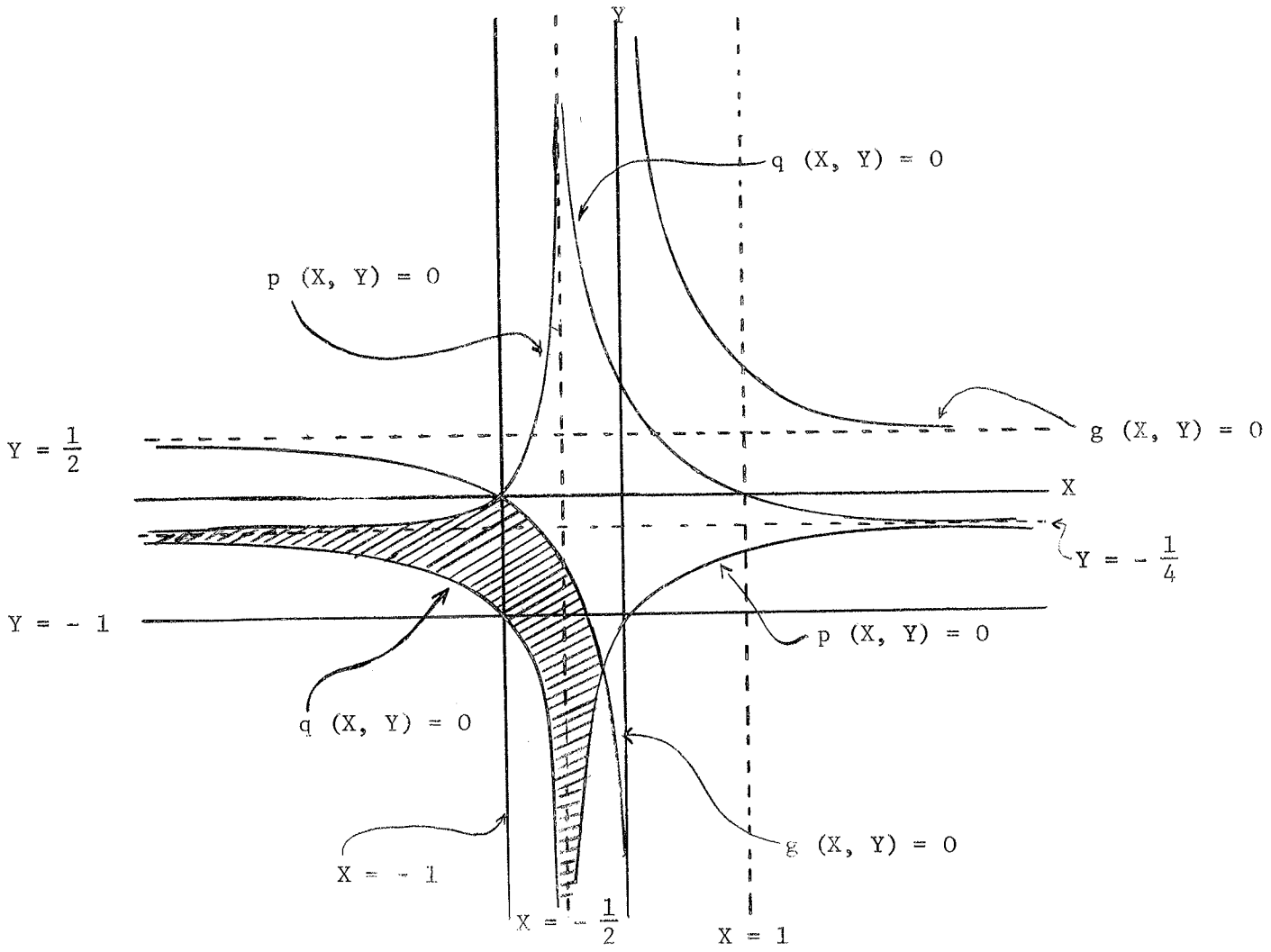
an hyperbola.

From $q(X, Y) = 0$, we obtain

$$0 = -3(1 + X) - 2(1 + Y) + 4(1 + X + Y + XY)$$

$$= -3 - 2 + 4 + X + 2Y + 4 + XY$$

or $Y = \frac{1 - X}{(2 + 4X)} = \frac{1}{4} \cdot \frac{(1 - X)}{(\frac{1}{2} + X)}$, an hyperbola.



```
$LIST ITER6
01/03/72    15:10                    -51-

00010 STRP,/,1, ,8,16,3,7,13,19,25,31,37,43,:,14,26,38,50,62,',0
00020 /$ SNUMB THKAA
00030 /$ IDENT T.KERR,KERR  37  578D  5-4171
00040 /$ PRØGRAM RLHS
00050 /$ LIMITS 1,16000,,1000
00060 /$ PRMFL H*,Z,R,WATSØN/ADA70A
00070 /$ DISC Q2,X2R,20L
00080 /$ PRMFL P*,R/A/S,L,*/P
00090 /$ NØTIFY */S
00100 0;40;2;1
00110 'T.KERR 9/21/71
00120 'R&DC 37 578D 8*235-4171
00130 'CØNVERGENCE ØF TWØ LØØP S.S. ITERATIØNS FØR I.C.
00140 1;1;300;1;5;1
00150 2
00160 1-5:1-5:1-6:1-2:10
00170 0:1:1:0:0
00180 ;2;U;Y;Z;E
00190 U;6
00200 4
00210 E;11;3;U;Y;Z
00220 0:1:-.5:-1
00230 Y;1;E;1;1
00240 4.001:-.99:4:1:2
00250 Z;1;E;1;1
00260 1.001:-.5:1:1:8
00270 0;40;1
00280 /$ ENDJØB

READY
```

$R_2 = - .99$

$R_2' = - .5$

ADA70 01/03/72 15.148

CASE 1 STEADY-STATE INITIAL VALUES ITERATIØN 208

ALGEBRAIC FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
E	1.0000E 00				

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U	4.0000E 00				

DIFFERENTIAL EQUATIØNS

NAME	Y	DY1 ...
Z	1.0000E 00	
Y	4.0000E 00	

CASE 1 FINAL VALUES AT TIME 1.0000E 00

ALGEBRAIC FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
E	1.0000E 00				

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U	4.0000E 00				

DIFFERENTIAL EQUATIØNS

NAME	Y	DY1 ...
Z	1.0000E 00	
Y	4.0000E 00	

CASE 1 CØNVERGENCE ØF TWØ LØØP S.S. ITERATIØNS FØR I.C.

TIME	U	Y	Z	E
0.	4.0000E 00	4.0000E 00	1.0000E 00	1.0000E 00
0.	4.0000E 00	4.0000E 00	1.0000E 00	1.0000E 00
E1.0000E 00	4.0000E 00	4.0000E 00	1.0000E 00	1.0000E 00

```
$LIST ØUT16
01/03/72   15:26
```

$R_2 = -.75$

$R_2' = -.75$

```
ADA70  01/03/72  15.259

 CASE   1      STEADY-STATE INITIAL VALUES  ITERATIØN     17

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E        1.0000E 00

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U        4.0000E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1  ...
 Z        1.0000E 00
 Y        4.0000E 00

 CASE   1      FINAL VALUES AT TIME  1.0000E 00

ALGEBRAIC FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 E        1.0000E 00

CØNSTANT, STEP, AND TABULAR FUNCTIØNS

 NAME      VALUE       NAME      VALUE       NAME      VALUE
 U        4.0000E 00

DIFFERENTIAL EQUATIØNS

 NAME       Y            DY1  ...
 Z        1.0000E 00
 Y        4.0000E 00

 CASE   1   CØNVERGENCE ØF TWØ LØØP S.S. ITERATIØNS FØR I.C.

   TIME         U            Y            Z            E

 0.          4.0000E 00   4.0000E 00   1.0000E 00   1.0000E 00
 0.          4.0000E 00   4.0000E 00   1.0000E 00   1.0000E 00
E1.0000E 00  4.0000E 00   4.0000E 00   1.0000E 00   1.0000E 00
```

$g\ (1,\ 1) = -\ 2\ -\ 2 + 4 = 0$

$g\ (2,\ 2) = -\ 2\ -\ 2\ (2) + 4\ (2 \times 2) = -\ 6 + 10 = 10 > 0$

$p\ (1,\ 1) = 1 + 1 + 2 + 4 = 8$

$q\ (3,\ 3) = -\ 9\ -\ 2\ (3) + 4\ (9) = 36 - 9 - 6 > 0$

The above indicates that choosing convergence factors anywhere within the shaded region should cause the iterations to converge.

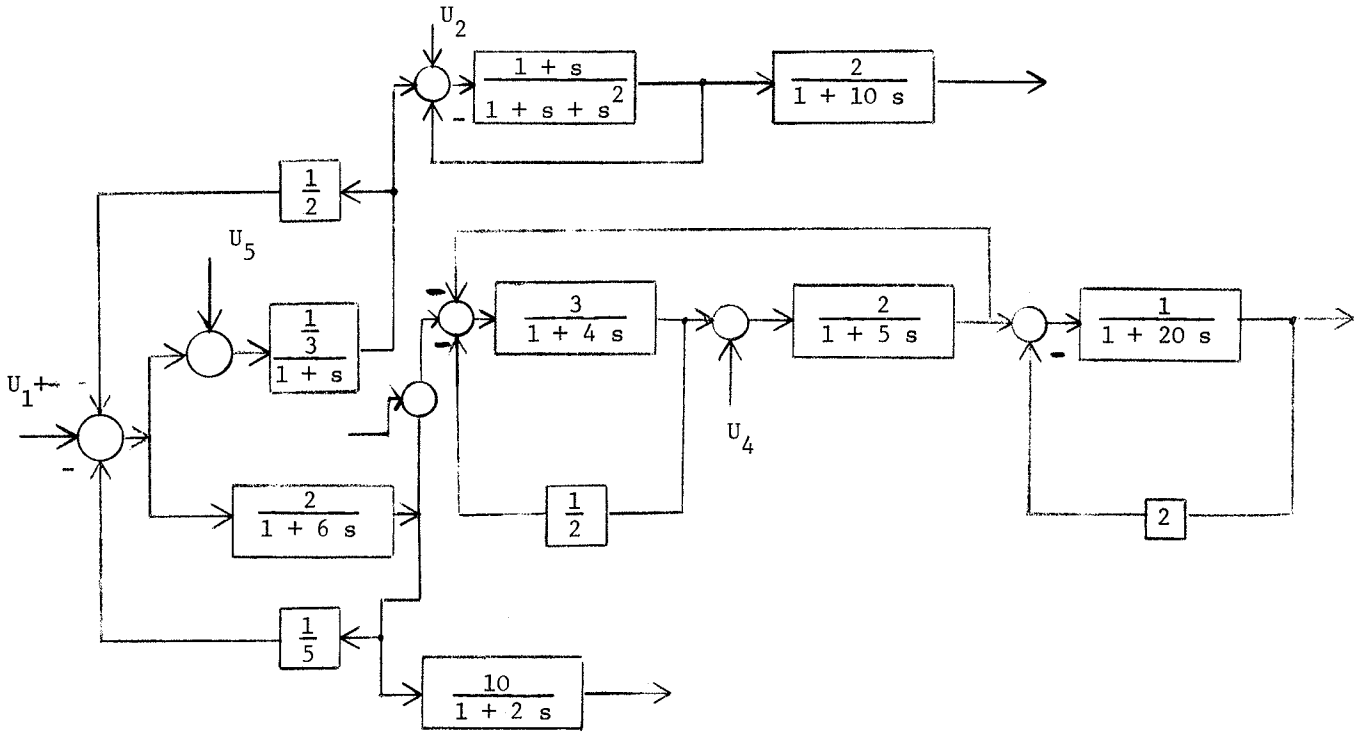
D. General Approach for Linear Systems

Thus far this report has given explicit results for all possible configurations involving one and two D.E.'s. What does one do if one wishes to specify the convergence acceleration factors for a more complicated system?

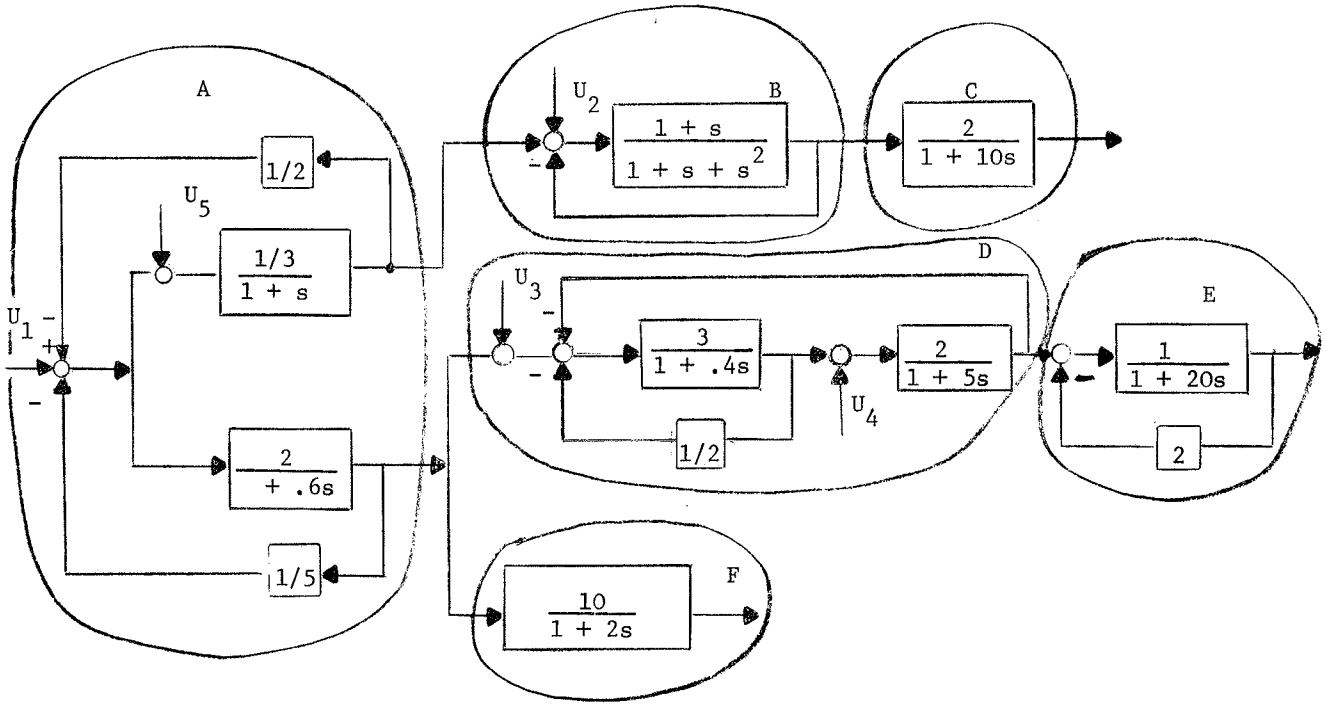
When one encounters a more complicated system one partitions it up into smaller component subsystems (please see Example 6) which have been treated in this report. If the component subsystems look exactly like the one and two D.E. configurations explicitly treated in this report except for the presence of additional outside additive inputs, the results of the analysis of the configuration without the additional inputs still applies. The theory still applies because convergence is assured on the basis of the characteristic equation of the difference equations which describe the iterations; additional additive inputs will not change the characteristic equations. Example 6 illustrates this approach.

Example 6: Partitioning a Complicated Linear System

An example will now be presented of a linear system containing more than two differential equations for which the graphical method for the selection of iteration convergence acceleration factors is applicable. Consider the following system simulation:



Convergence acceleration factors will be chosen by partitioning the general simulation into several smaller more fundamental block diagram configurations which are special cases of the configuration treated in the report. The partitioning is as follows:

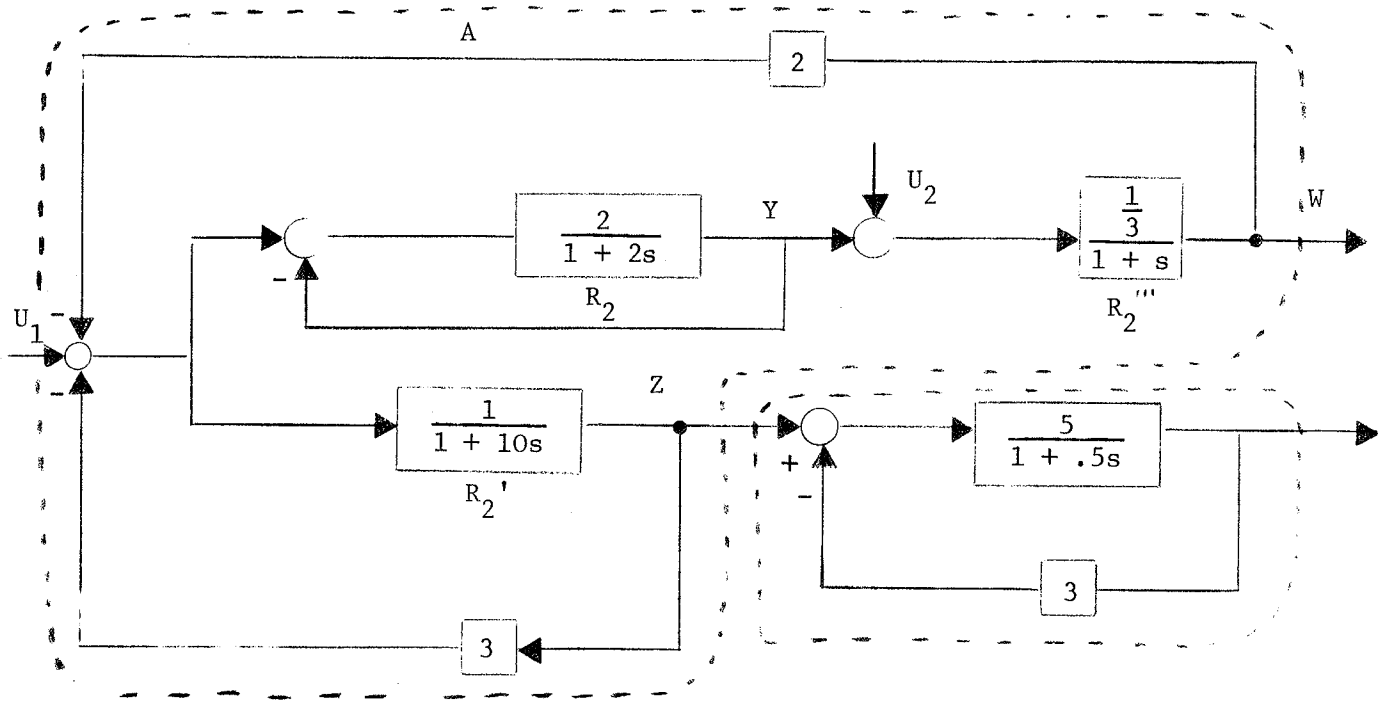


Section A can be considered to be a special case of the general 2 D.E. parallel arrangement. The input U_5 doesn't cause any added difficulties because if one thinks of the underlying difference equations that describe the steady-state iterations, the presence of U_5 just means that there is another forcing function present; the characteristic equation is still the same, and convergence acceleration factors are chosen on the basis of considering where the roots of the characteristic equations lie. Section B and section E are special cases of the single loop D.E. Section C and section F are special cases of the no loop case. Section D is a special case of the general 2 D.E. series case. By finding the convergence acceleration factors for each of these component configurations by the methods presented in the report, we have the convergence acceleration factors for the whole system.

Suppose that after a complicated system is partitioned into smaller component subsystems, some of the subsystems are not special cases of the general cases treated in this report. How does one proceed to obtain the convergence acceleration factors? First write the difference equations that describe the iterations for the individual subsystem and obtain the characteristic equation. Now, using properties of the roots of polynomial equations or constructions, graphic or analytical, based on the Jury stability criterion, find the convergence acceleration factors that will yield roots of the characteristic equation that are only within the unit circle. This approach is illustrated in Example 7 and Example 7b.

Example 7

An example will now be presented of what approach to take when one encounters a system that has component systems which are not all special cases of the general cases treated earlier in this report. Consider the following system simulation and its partitions:



Now Section B is a special case of the general one loop D.E. that has been treated in the report. Section A, however, has not been treated in the report. To determine the proper convergence acceleration factor, first write the difference equations which describe the iterations (neglecting constant inputs which don't affect the characteristic equations of linear systems).

$$w\ (n+1) = \frac{1}{3}\ Y\ (n) + R_2''\ [\frac{1}{3}\ Y\ (n) - w\ (n)]$$

$$Y\ (n+1) = 2\ [-\ 2\ w\ (n) - 3\ Z\ (n) - Y\ (n)] +$$

$$R_2\ \{2\ [-\ 2\ w\ (n) - 3\ Z\ (n) - Y\ (n)] - Y\ (n)\}$$

$$Z\ (n+1) = [-\ 2\ w\ (n) - 3\ Z\ (n)] + R_2'\ \{[-\ 2\ w\ (n) - 3\ Z\ (n)] - Z\ (n)\}$$

or

$$(E + R_2'')\ w_n - \frac{1}{3}\ (1 + R_2'')\ Y_n + 0 = 0\ ,$$

$$4\ (1 + R_2)\ w_n + (E + [2\ (1 + R_2) + R_2])\ Y_n + 6\ (1 + R_2)\ Z_n = 0\ ,$$

$$2\ (1 + R_2')\ w_n + 0 + (E + [3\ (1 + R_2') + R_2']) = 0,$$

which has the characteristic equation

$$0 = \Delta = \begin{vmatrix} (E + R_2'') & -\frac{1}{3}\ (1 + R_2'') & 0 \\ 4\ (1 + R_2) & (E + [2\ (1 + R_2) + R_2]) & 6\ (1 + R_2) \\ 2\ (1 + R_2') & 0 & E + [3\ (1 + R_2') + R_2'] \end{vmatrix}$$

$$= (E + R_2'') \; (E + [2 \; (1 + R_2) + R_2] \; (E + [3 \; (1 + R_2') + R_2'])$$

$$- \frac{1}{3} \; (1 + R_2'') \; 6 \; (1 + R_2) \; 2 \; (1 + R_2')$$

$$+ \frac{1}{3} \; (1 + R_2'') \; 4 \; (1 + R_2) \; (E + [3 \; (1 + R_2') + R_2'])$$

Considerable simplification occurs if we take $R_2 = R_2' = R_2''$ (the assumption that all acceleration factors can be the same is only reasonable when there is no positive feedback), the characteristic equation is now

$$0 = (E + R_2)(E + [2 \; (1 + R_2) + R_2])^2 - 4 \; (1 + R_2)^3 +$$

$$\frac{4}{3} \; (1 + R_2)^2 \; (E + [3 \; (1 + R_2) + R_2])$$

Let $1 + R_2 = H$, the characteristic equation becomes

$$0 = (E + [H - 1])(E + [3 \; H - 1)^2 - 4 \; H^3 + \frac{4}{3} \; H^2 \; (E + [4 \; H - 1])$$

from which a root may be factored as

$$(E + [H - 1]) \; \{(E + [3 \; H - 1])^2 + \frac{4}{3} \; H^2\} = 0$$

One root of the characteristic equation is $E_1 = - H + 1$, and the other two roots, obtained from the formula for the roots of a quadratic equation, are

$$E_{2,3} = \frac{2 - 6 \; H \pm j \; 4 \; H}{2} \; .$$

In order that all the roots lie within the unit circle, it must be that

$$1 > \left| E_{2,3} \right|^2 = 1 - 6\,H + 13\,H_2 \text{ and } 1 > \left| E_1 \right| = \left| 1 - H \right|$$

or, equivalently, we must find H so that the following inequalities are satisfied:

$$(-6 + 13\,H)\,H < 0,$$

$$0 < H < 2\ .$$

Now since $0 < H < 2$, the only way the first inequality may be satisfied is for $-6 + 13\,H < 0$ or $H < \frac{6}{13}$. Just pick H in $(0, \frac{6}{13})$ and all the inequalities required for convergence of the iteration scheme are satisfied; this corresponds to a convergence acceleration of $-1 < R_2 < -\frac{7}{13}$. For instance, take $R_2 = -\frac{10}{13}$, $(R_2 = R_2' = R_2'' = -\frac{10}{13})$, and R_2'' is obtained from the general one loop D.E. considered in the report (an acceptable value is $R_2'' = -\frac{15}{16}$).

Example 7b: Another Analytic Technique for Solving Example 7

Solving for convergence acceleration factors of Section A in Example 7 by the method used in the proof of Section IV:

Case 2:b.

From Example 7, the characteristic equation is

$$0 = (E + H_3 - 1)\,(E + [3\,H_1 - 1])\,(E + [4\,H_2 - 1]) - \frac{1}{3}\,H_3\,6\,H_1\,2\,H_2$$

$$+ \frac{1}{3}\,H_3\,4\,H_1\,(E + [4\,H_2 - 1])\ .$$

Let $H_1 = X$, $H_2 = \alpha X$, $H_3 = \beta X$, then the characteristic equation becomes

$$0 = [E + (\beta X - 1)]\,[E + (3X - 1)]\,[E + (4\,\alpha X - 1)] - 4\,\alpha\beta\,X^3 + \frac{4}{3}\,\beta X^2\,[E + (4\,\alpha X - 1]$$

Now $\frac{4}{3}\,\beta X^2\,[E + (4\,\alpha X - 1)] - 4\,\alpha\beta\,X^3 = \frac{4}{3}\,\beta X^2\,E + \frac{4}{3}\,\beta[\alpha\,X^3 - X^2]$,

$$E + (3\,X - 1)\overline{)\frac{4}{3}\,\beta X^2\,E + \frac{4}{3}\,\beta\alpha X^3 - \frac{4}{3}\,\beta X^2}\;\frac{\frac{4}{3}\,\beta X^2}{}$$

$$\frac{\frac{4}{3}\,\beta X^2\,E + 4\,\beta X^3 - \frac{4}{3}\,\beta X^2}{(\frac{4}{3}\,\beta\alpha - 4\,\beta)\,X^3}$$

If $(\frac{4}{3}\,\beta\alpha - 4\,\beta)\,X^3 = 0$ (we have zero remainder in the above equation), then

$[E + (3\,X - 1)]\,\{[E + (\beta X - 1)]\,[E + (4\,\alpha X - 1)] + \frac{4}{3}\,\beta X^2\} = 0$ and for stability $-1 < 1 - 3\,X < 1$ or, equivalently, $0 < X < \frac{2}{3}$.

Now $X \neq 0$, so the condition that $\beta\,(\frac{4}{3}\,\alpha - 1) = 0$ requires that either $\beta = 0$ or that $\frac{4}{3}\,\alpha - 1 = 0$. Look at what occurs if $\beta = 0$; the second order characteristic equation after one root has been extracted is

$$[E - 1]\,[E + (4\,\alpha X - 1)] = 0\ ;$$

a root of this equation is 1, hence this is unstable; hence we desire that $\beta \neq 0$. Since we cannot have $\beta = 0$, we must take $\frac{4}{3}\,\alpha = 1$ or $\alpha = \frac{3}{4}$. The second order characteristic equation becomes

$$[E + (\beta X - 1)]\,[E + (3\,X - 1)] + \frac{4}{3}\,\beta X^2 = 0$$

or

$$E^2 + [(\beta + 3)\, X - 2]\, E + [\frac{13}{3}\, \beta X^2 - (3 + \beta)\, X + 1] = 0.$$

By Jury's stability test we must have that

$$h\ (X, \beta) = \frac{13}{3}\, \beta X^2 > 0$$

$$g\ (X, \beta) = 4 - 2\ (3 + \beta)\ X + \frac{13}{3}\, \beta X^2 > 0$$

$$p\ (X, \beta) = 2 - (3 + \beta)\ X + \frac{13}{3}\, \beta X^2 > 0$$

$$0 > -\ (3 + \beta\ )\ X + \frac{13}{3}\, \beta X^2 = q\ (X, \beta)$$

Recall that already X is such that $0 < X < \frac{2}{3}$; let $X = k\ (\frac{2}{3}) + (1 - k)\ (0)$, $0 < k < 1$. Now Jury's stability condition becomes

$$h\ (k, \beta) = \frac{40}{3}\, (\frac{2}{3})^2\, \beta\, k^2 > 0\ ,$$

$$g\ (k, \beta) = 4 - \frac{4}{3}\, (12 + \beta)\, k + \frac{40}{3}\, \beta\, k^2 > 0\ ,$$

$$p\ (k, \beta) = 2 - \frac{2}{3}\, (12 + \beta)\, k + \frac{40}{3}\, \beta\, k^2 > 0\ ,$$

$$q\ (k, \beta) = -\frac{2}{3}\, (12 + \beta)\, k + \frac{40}{3}\, \beta\, k^2 < 0\ .$$

Adjust β until $(12 + \beta) = W$, a constant. The constant will be determined later. Jury's conditions become

$$h\ (k, W) = \frac{40}{3}\, (\frac{2}{3})^2\, [W - 12]\, k^2 > 0\ ,$$

$$g\ (k, W) = 4 - \frac{4}{3}\, W\, k + \frac{40}{3}\, [W - 12]\, k^2 > 0\ ,$$

$$p\ (k, W) = 2 - \frac{2}{3} W k + \frac{40}{3} [W - 12] k^2 > 0 ,$$

$$q\ (k, W) = - \frac{2}{3} W k + \frac{40}{3} [W - 12] k^2 < 0 .$$

Note that $q\ (k, W) < 0$ for $0 < k < \frac{1}{20} \frac{W}{W - 12}$. Since k must be such that $0 < k < 1$, we have that $(\frac{20}{19})\ 12 < W$.

Let $k = \frac{2}{3} \frac{1}{W}$, then Jury's stability test becomes

$$q = - 1 + \frac{40}{3} (\frac{3}{2})^2 \frac{[W - 12]}{W^2} < 0 ,$$

$$p = 1 + \frac{40}{3} (\frac{3}{2})^2 \frac{[W - 12]}{W^2} > 0 ,$$

$$g = 2 + \frac{40}{3} (\frac{3}{2})^2 \frac{[W - 12]}{W^2} > 0 ,$$

$$h = \frac{40}{3} (\frac{3}{2})^2 \frac{[W - 12]}{W^2} > 0$$

Since $0 < k < 1$, it must be that $\frac{3}{2} < W$; combining this with the last condition we obtain

$$W > \max [\frac{3}{2}, (\frac{20}{19})\ 12] = (\frac{20}{19})\ 12$$

For any $W > (\frac{20}{19})\ 12$, all the conditions of Jury's test are satisfied. Now $H_1 = \frac{1}{W}$, $H_2 = \frac{3}{W}$, $H_3 = \frac{W - 12}{W}$, for all $W > (\frac{20}{19})\ 12$ yield convergent iterations.

Example 7
Example 7b

```
$LIST IT18
01/03/72   09:55

00010 STRP,/,1, ,8,16,;,7,13,19,25,31,37,43,:,14,26,38,50,62,',0
00020 /$ SNUMB THKAA
00030 /$ IDENT T,KERR,KERR  37  578D  5-4171
00040 /$ PRØGRAM RLHS
00050 /$ LIMITS 1,16000,,1000
00060 /$ PRMFL H*,Z,R,WATSØN/ADA70A
00070 /$ DISC Q2,X2R,20L
00080 /$ PRMFL P*,R/A/S,L,*/P
00090 /$ NØTIFY */S
00100 0;40;2;1
00110 'T,KERR 9/21/71
00120 'R&DC 37 578D 8*235-4171
00130 'GENERAL APPRØACH TØ FINDING CØNV. ACCEL FACTØRS FØR ARB. SYSTEM
00140 1;1;500;1;5;1
00150 2
00160 1=5:1=5:1=6:1-2:10
00170 0:1:1:0:0
00180 ;2;Y;Z;W;X;U1;U2
00190 U1;6
00200 1
00210 U2;6
00220 2
00230 E1;11;3;U1;Z;W
00240 0:1:-3:-2
00250 E2;11;2;E1;Y
00260 0:1:-1
00270 Y;1;E2;1;1
00280 1:-.959:2:1:2
00290 E3;11;2;U2;Y
00300 0:1:1
00310 W;1;E3;1;1
00320 2:-.5:.3333:1:1
00330 Z;1;E1;1;1
00340 3:-.875:1:1:10
00350 E4;11;2;Z;X
00360 0:1:-3
00370 X;1;E4;1;1
00380 .1:-.937:5:1:.5
00390 0;40;1
00400 /$ ENDJØB

READY
```

01/03/72 10:07

$R_2 = -.769$

$R_2' = -.769$

$R_2'' = -.769$

ADA70 01/03/72 10.067

CASE 1 STEADY-STATE INITIAL VALUES ITERATION 57

ALGEBRAIC FUNCTIONS

NAME	VALUE	NAME	VALUE	NAME	VALUE
E1	-7.4972E-02	E2	-2.4992E-02	E3	1.9500E 00
E4	-4.6862E-03				

CONSTANT, STEP, AND TABULAR FUNCTIONS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U1	1.0000E 00	U2	2.0000E 00		

DIFFERENTIAL EQUATIONS

NAME	Y	DY1 ...
X	-2.3431E-02	
Z	-7.4972E-02	
W	6.4994E-01	
Y	-4.9984E-02	

CASE 1 FINAL VALUES AT TIME 1.0000E 00

ALGEBRAIC FUNCTIONS

NAME	VALUE	NAME	VALUE	NAME	VALUE
E1	-7.4969E-02	E2	-2.4989E-02	E3	1.9500E 00
E4	-4.7130E-03				

CONSTANT, STEP, AND TABULAR FUNCTIONS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U1	1.0000E 00	U2	2.0000E 00		

DIFFERENTIAL EQUATIONS

NAME	Y	DY1 ...
X	-2.3419E-02	
Z	-7.4971E-02	
W	6.4994E-01	
Y	-4.9980E-02	

CASE 1 GENERAL APPROACH TO FINDING CONV. ACCEL FACTORS FOR ARB. SY
**********TEM

TIME Y
STOP

READY

01/03/72 09:39

ADA70 01/03/72 9.412

W = 24
R_2 = - .959
R_2' = - .875
R_2'' = - .5

CASE 1 STEADY-STATE INITIAL VALUES ITERATION 99

ALGEBRAIC FUNCTIONS

NAME	VALUE	NAME	VALUE	NAME	VALUE
E1	-7.4972E-02	E2	-2.4990E-02	E3	1.9500E 00
E4	-4.6863E-03				

CONSTANT, STEP, AND TABULAR FUNCTIONS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U1	1.0000E 00	U2	2.0000E 00		

DIFFERENTIAL EQUATIONS

NAME	DY1 ...
X	-2.3431E-02
Z	-7.4972E-02
W	6.4994E-01
Y	-4.9980E-02

CASE 1 FINAL VALUES AT TIME 1.0000E 00

ALGEBRAIC FUNCTIONS

NAME	VALUE	NAME	VALUE	NAME	VALUE
E1	-7.4968E-02	E2	-2.4990E-02	E3	1.9500E 00
E4	-4.7621E-03				

CONSTANT, STEP, AND TABULAR FUNCTIONS

NAME	VALUE	NAME	VALUE	NAME	VALUE
U1	1.0000E 00	U2	2.0000E 00		

DIFFERENTIAL EQUATIONS

NAME	Y	DY1 ...
X	-2.3403E-02	
Z	-7.4972E-02	
W	6.4994E-01	
Y	-4.9978E-02	

CASE 1 GENERAL APPROACH TO FINDING CONV. ACCEL FACTORS FOR ARB. SY
**********TEM

TIME	Y	Z	W

STOP

READY

Even for nonlinear system simulations, the methods of this Section may often be successfully used. A nonlinear system simulation may be partitioned into component subsystems. Some of these subsystems may be linear and the techniques of this section should be applied to them first. Convergence factors for the strictly nonlinear component subsystems may sometimes be determined by, first, obtaining a linear model to represent the nonlinearity by a linearization technique and, second, by trial and error perturbations within and near the allowable region for convergence factors that are established for the linearized model of the nonlinearity.

Section III: Computing Initial Values Directly
(linear systems and some nonlinear systems)

As will be discussed in more detail in Section IV, at the initial time, a linear system can be represented in state vector notation (DeRusso, Roy, Close, 1965, p. 329) as

(eq. II) $$\dot{X} = A X + B u_o,$$

$$Y = H X + D u_o, \text{ where } u_o \text{ is a constant.}$$

The problem of solving for the steady-state initial values can be posed algebraically as trying to find the X_o and Y_o such that

(eq. III) $$0 = A X_o + B u_o,$$

$$Y_o = H X_o + D u_o.$$

Under the assumption that A is non-singular, the solution of the above problem is

(eq. IV) $$X_o = - A^{-1} B u_o \ ,$$

$$Y_o = [- H A^{-1} B + D] u_o \ ;$$

if A is singular, the solution of the above problem is

(eq. VII) $$X_o = - A^{\#} B u_o \ ,$$

$$Y_o = [- H A^{\#} B + D] u_o \ ,$$

where $A^{\#}$ is the pseudo inverse.

At the initial time, a nonlinear (i.e., not necessarily linear) system can be represented in state variable notation as

(eq. VIII) $$\dot{X} = f\ (X,\ t_o,\ u_o)\ ,$$

$$Y = h\ (X,\ Y,\ t_o,\ u_o)\ .$$

The problem of solving for the steady-state initial values can be posed algebraically as trying to find the X_o and Y_o such that

(eq. IX) $$o = f\ (X_o,\ t_o,\ u_o)\ ,$$

$$Y_o = h\ (X_o,\ Y_o,\ t_o,\ u_o)\ .$$

It is impossible to write out an analytic solution for the general nonlinear case.

When confronted with the problem of calculating the steady-state initial values of a linear system, the simulation engineer may choose between several alternatives in calculating the initial values directly:

(1) He may perform the calculations directly by hand since (eq. III) is just a set of simultaneous linear algebraic equations and there is ample theory available on the existence and uniqueness of solutions of simultaneous linear algebraic systems (Kreyszig, 1962, p. 423). The problem is tractable by hand if the dimensionality of the problem is not too great.

(2) He may write a program in some other computer language such as in, say, FØRTRAN to solve the simultaneous linear algebraic equations that arise.

(3) He may use a packaged program such as LINEQ/CLINEQ to solve the simultaneous linear systems of algebraic equations.

Given enough time and patience, each of the above choices is guaranteed to yield correct results (ignoring ill-conditioned matrices).

When confronted with the problem of calculating the steady-state initial values of a nonlinear system, the simulation engineer may choose between several alternatives in calculating the initial values directly:

(1) He may perform the calculations directly by hand; (eq. IX) is a set of simultaneous nonlinear algebraic equations which sometimes succumb to hand calculation if the dimensionality of the problem is not too great.

(2) He may write a program in some other computer language such as in, say, FØRTRAN to solve the simultaneous nonlinear algebraic equations that arise by, say, a generalized Newton-Ralphson or steepest descent technique.

(3) He may use a packaged program such as SECANT to solve the simultaneous nonlinear system of algebraic equations.

(4) If he has table data or even if he doesn't, he may resort to a graphical technique of some type.

Unfortunately, with nonlinear systems, there does not exist the pleasant state of affairs that is associated with systems of linear equations. None of the

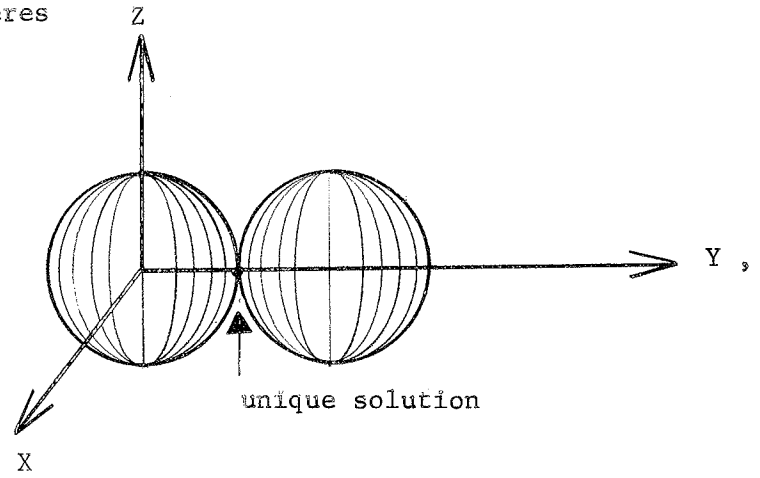
above four approaches can be guaranteed to work for nonlinear systems just because nonlinear theory is not cleaned up enough to allow anything to be guaranteed. (It is probable that a general nonlinear theory will never emerge, but for a variety of special cases, some of the above approaches to handling the nonlinear problem can be guaranteed).

As an aside, consider the impossibility of predicting uniqueness or the multiplicitly of solutions based on the number of nonlinear equations and the number of unknowns (an area that has been explored completely for linear systems with many fruitful results). Consider the two nonlinear equations in three unknowns

$$X^2 + (Y - a)^2 + Z^2 = b^2 ,$$

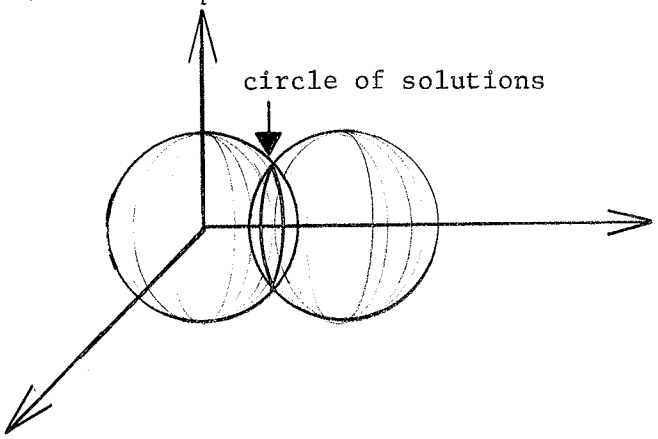
$$X^2 + Y^2 + Z^2 = b^2 ,$$

where a, b are elements of the real number field. For $a = 2\,b > 0$, the above system can be represented in Euclidean three space as the intersection of two spheres



which has a unique solution. For $a = \frac{3}{2}\,b$, the above system can be represented

as the intersection of two spheres



which results in a circle of solutions, an infinite number of solutions. For the single equation $X^2 + Y^2 + Z^2 = b^2$, a nonlinear equation in three unknowns, if $b = 0$, the equation in three unknowns has the unique solution $(0, 0, 0)$. The nonlinear equation $X^2 = -1$ has no solutions over the real field. The above four examples have been mentioned to point up the unsatisfactory state of affairs that must be lived with when working with nonlinear systems and how nothing may be said about existence and uniqueness from the number of nonlinear equations and the number of unknowns. The four options available to the simulation engineer for handling nonlinear systems will never work if no solutions exist to the nonlinear algebraic system. Existence <u>cannot</u> be argued from the point of view that the physical system has a steady-state equilibrium point; therefore, the nonlinear equations which represent the physical system must have an equilibrium point solution because the simulation is a mathematical idealization of the physical system and this particular simulation mathematical model may fail to capture the essence of the physical problem which ensures existence of an equilibrium. An example of demonstrating the nonexistence of a solution for a simultaneous

nonlinear algebraic system is given in the Appendix in the hope that the general technique employed will be useful to others.

Example 8: Nonlinear System by Hand Calculations

An example of a nonlinear system will now be presented which has steady-state initial conditions which may be found by direct hand calculation. Consider the system of Example 10,

$$\ddot{Y} + 2\dot{Y} - Y + 400\, y^3 = 0$$

Let $X_1 = Y$, $X_2 = \dot{Y}$, the state variable representation for the system is

$$\begin{bmatrix} \dot{X}_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_2 \\ -400\, X_1^3 + X_1 - 2\, X_2 \end{bmatrix} .$$

Steady state initial conditions occur where the derivatives are zero,

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \text{ hence}$$

$$0 = X_2$$

$$0 = -400\, X_1^3 + X_1 - 2\, X_2$$

can be solved simultaneously to yield the following three singular points (solutions)

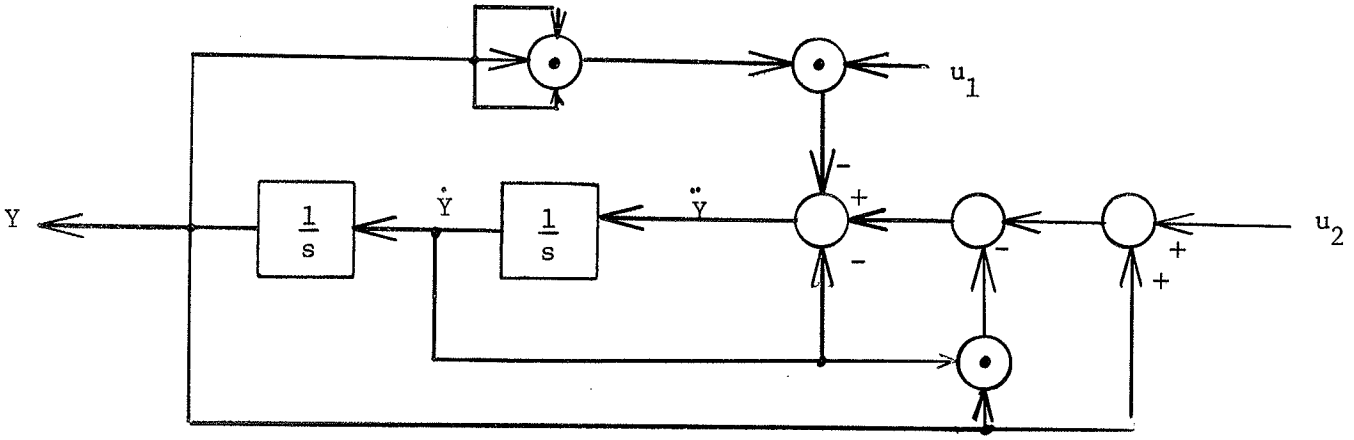
$$(-0.05, 0) , (0, 0) , (+0.05, 0) .$$

Upon further investigation (DeRusso, Roy, and Close, 1965, p. 485) it is found that (0, 0) cannot be the steady-state equilibrium point for a dynamic physical system; therefore, either (- 0.05, 0) or (+ 0.05, 0) is the

equilibrium point (i.e., the steady-state initial condition).

Example 9: Nonlinear System by a Graphical Analysis

An example of a nonlinear system will now be presented which has steady-state initial conditions which may be found directly by graphical analysis. Consider the following system:



$$\ddot{Y} + \dot{Y} - (1 - Y)\, Y = u_2 - u_1\, Y^3 \text{ , where } u_1\,(0) < 0 \text{ ,}$$

$$u_2\,(0) < 0 \text{ .}$$

Let $X_1 = Y$; $X_2 = Y$; hence

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} X_2 \\ - X_2 + (1 - X_2)\, X_1 + u_2 - u_1\, X_1^{\,3} \end{bmatrix} .$$

Steady state initial conditions occur where the derivatives are zero, $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; hence

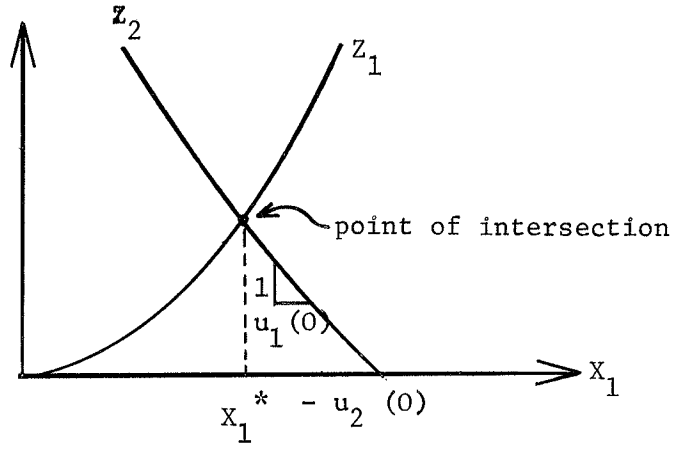
$$0 = X_2$$

$$0 = - X_2 + (1 - X_2)\, X_1 + u_2\,(0) - u_1\,(0)\, X_1^{\,3}$$

or $X_2 = 0$ and $0 = X_1 + u_2(0) - u_1(0) X_1^3$

or $X_2 = 0$ and $X_1^3 = \frac{1}{u_1(0)} (X_1 + u_2(0))$.

The above equation can be solved for the required X_1 by a graphical analysis. Let $Z_1 = X_1^3$ and $Z_2 = \frac{1}{u_1(0)} (X_1 + u_2(0))$, which may be plotted as



The point of intersection of the two graphs when projected to the X_1 - axis is X_1^*, the solution of both equations and so $(X_1^*)^3 = \frac{1}{u_1(0)} (X_1^* + u_2(0))$.

Section IV: Computing Initial Values by Integration (stable linear systems and asymptotically stable nonlinear systems)

If the system to be simulated is a linear system, there is yet another method that may be used to obtain the steady-state initial conditions. One may just run the system simulation while keeping the inputs constant at their initial values until there ceases to be any change in the variables other than what could properly be attributed to roundoff and truncation effects. This method may be preferable when trying to obtain the steady-state initial conditions for a linear system of high dimensionality. The justification for this method will now be discussed.

Any linear system may be represented in state variable notation (DeRusso, Roy, Close, 1965, p. 329) as

(eq. I) $$\dot{X}(t) = A(t)\,X(t) + B(t)\,u(t)$$

$$Y(t) = H(t)\,X(t) + D(t)\,u(t)\ .$$

At $t = 0$, the system becomes

(eq. II) $$\dot{X}(0) = A(0)\,X(0) + B(0)\,u(0)$$

$$Y(0) = H(0)\,X(0) + D(0)\,u(0)\ .$$

If the system is initially in the steady-state, $\dot{X}(0) \equiv 0$.

Let

$$A \triangleq A(0)\ ;\ B \triangleq B(0)\ ;\ H \triangleq H(0)\ ;\ D \triangleq D(0)\ .$$

The algebraic problem is the following: Solve for X (0) and Y (0) given that

(eq. III)
$$0 = A\,X\,(0) + B\,u\,(0)$$
$$Y\,(0) = H\,X\,(0) + D\,u\,(0)\ .$$

Under the assumption that A is non-singular, the solution of the above problem is

(eq. IV)
$$X\,(0) = -\,A^{-1}\,B\,u\,(0)$$
$$Y\,(0) = [-\,H\,A^{-1}\,B + D]\,u\,(0)\ ,$$

which are the steady-state initial values. This is the most direct way of obtaining the steady-state initial values.

The simulation language ADA is designed to efficiently solve dynamic problems, problems involving integration, rather than algebraic problems; however, the dynamic approach may be used to advantage in solving the algebraic problem of the preceding paragraph. The solution of equation II is

(eq. V)
$$X\,(t) = e^{At}\,X\,(0) + e^{At}\int_0^t e^{-A\tau}\,B\,u\,(0)\,d\tau\ ,$$
$$Y\,(t) = H\,e^{At}\,X\,(0) + H\,e^{At}\int_0^t e^{-A\tau}\,B\,u\,(0)\,d\tau + D\,u\,(0)\ .$$

As in the above algebraic solution, if we assume that A is non-singular, we have

$$X\,(t) = e^{At}\,X\,(0) + e^{At}\,[I - e^{-At}]\,A^{-1}\,B\,u\,(0)\ ,$$
$$Y\,(t) = H\,e^{At}\,X\,(0) + H\,e^{At}\,[I - e^{-At}]\,A^{-1}\,B\,u\,(0) + D\,u\,(0)\ .$$

If A has all eigenvalues with negative real parts, then $\lim_{t\to\infty} e^{At} = \underline{0}$. There is no fear that a stable time varying system will have an A (0) which is unstable since the transfer function of a stable variable system is analytic and bounded in the right half plane and on the imaginary axis for all t. (Gibson, 1963, p. 193). This result yields that

(eq. VI)
$$\lim_{t\to\infty} X(t) = 0 + 0 - A^{-1} B u(0)$$

$$\lim_{t\to\infty} Y(t) = 0 + 0 - H A^{-1} B u(0) + D u(0) = [- H A^{-1} B + D] u(0)$$

In performing integrations on the computer using ADA, it is ridiculous to think of integrating until $t \to \infty$, but it is only necessary to integrate for $t \geq \frac{10}{re\ (\lambda_{smallest})}$ to reach steady-state, where $\lambda_{smallest}$ is the eigenvalue of A having the smallest negative real part. Notice that eq. IV and eq. VI yield the same results, so that "integrating out" is an effective method of obtaining the steady-state initial conditions for a stable linear system. In the solution of eq. IV, if A was singular, the solution would be

(eq. VII)
$$X(0) = - A^{\#} B u(0)$$

$$Y(0) = [- H A^{\#} B + D] u(0),$$

where $A^{\#}$ is the pseudo inverse.

The decision as to what method should be used in calculating the steady-state initial condition should be made after considering the cost of man-hours required for fairly accurate initial guesses as to what the initial conditions should be and the proper selection of the convergence acceleration factors against the cost of running the computer simulation until all the values reach steady-state.

The steady-state initial values may always be found for stable linear systems by "integrating out" until balance is achieved; however, the same may not be claimed in general for nonlinear systems. Both linear and nonlinear systems may be simulated on ADA, so the results of a further investigation into when the method of "integrating out" to attain the steady-state values for nonlinear systems will now be discussed.

For nonlinear systems there are three possible, mutually exclusive, responses to an infinitessimal displacement from an equilibrium state (DeRusso, Roy, Close, 1965, p. 501):

(1) the state may ultimately return to the neighborhood of the equilibrium point,

(2) it may happen that the state does not return to the neighborhood of the equilibrium point, but its distance from the equilibrium point remains finite,

(3) the state vector may grow without bound. For the second and third possibility, the method of integrating out to achieve balance is not feasible.

The method of "integrating out" to achieve balance is feasible for the first possibility only if $\lim_{t\to\infty} X(t) = X_{equilibrium}$, the condition of asymptotic stability. Whereas for linear systems, stability was independent of the initial displacement, for nonlinear systems the type of stability may depend upon the displacement from the equilibrium. Systems which exhibit asymptotic stability for small displacements from the equilibrium position are said to be asymptotically stable in the small. Systems which exhibit asymptotic

stability for large displacements from the equilibrium position are said to be asymptotically stable in the large.

If the nonlinear system being simulated using ADA is asymptotically stable in the large (synonym: absolutely stable), then the method of "integrating out" to obtain balance will yield the steady-state initial values independent of the initial guesses. If the nonlinear system being simulated using ADA is only asymptotically stable in the small, then the method of "integrating out" to obtain balance will yield the correct steady-state initial values only if the initial guesses are fairly accurate.

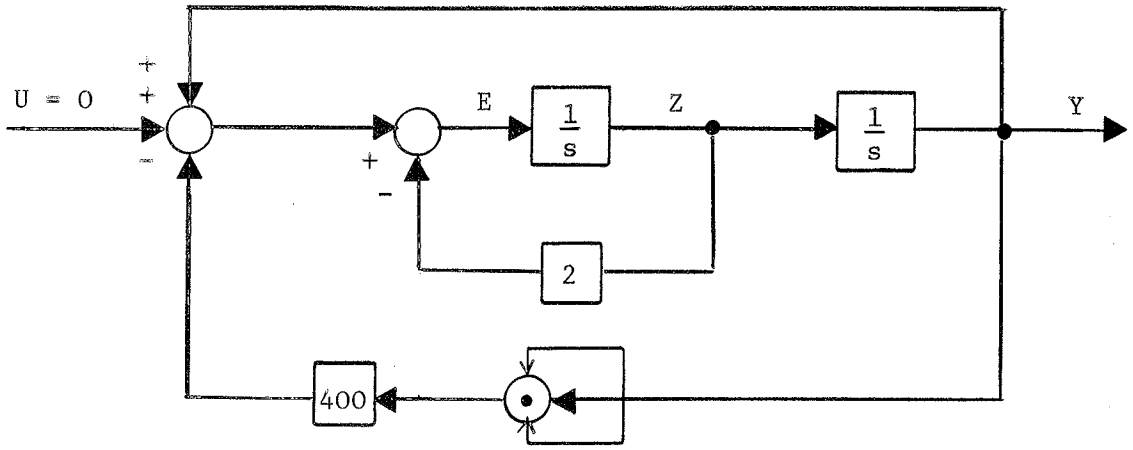
As a result of investigations in the area of nonlinear system stability theory, there are a variety of tests which may be applied to special configurations to determine if they are asymptotically stable. One such test, which is sufficient but not necessary for asymptotic stability in the large, is the first Canonic form of Lur'e (J.P. LaSalle, 1962, pp. 600-603). Unfortunately, since the test is only a sufficient condition, the system under consideration may actually be asymptotically stable in the large while the test does not confirm this. The second method of Lyapunov (LaSalle and Lefschetz, 1961) is both necessary and sufficient for absolute stability, but here one has the difficulty of finding the correct Lyapunov function to demonstrate stability.

In conclusion, the method of "integrating out" to find the steady-state initial values is not universally applicable to nonlinear systems. Further investigation on the part of the simulation engineer is required to assure that the method will yield correct results.

Example 10: A Nonlinear Example

To emphasize the conclusion that the method of "integrating out" can not be universally applied to nonlinear systems to find the steady-state initial values, an example of a system for which this method fails will now be given.

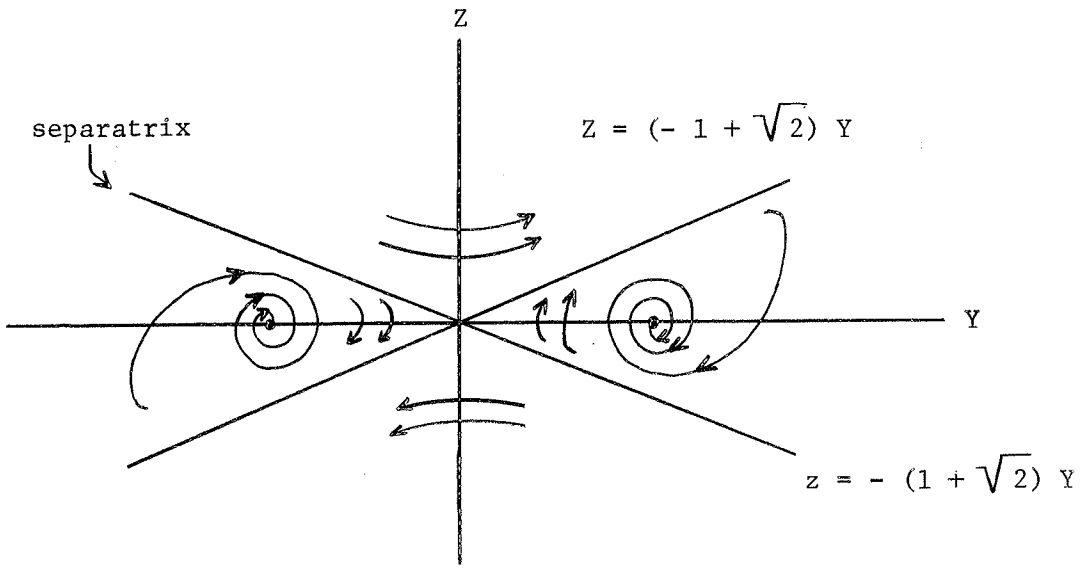
Consider the system below.



It is a special case of the following nonlinear differential equation

$$\ddot{Y} + 2a\dot{Y} - \frac{b^2}{2}Y + C^2 Y^3 = 0 \quad ,$$

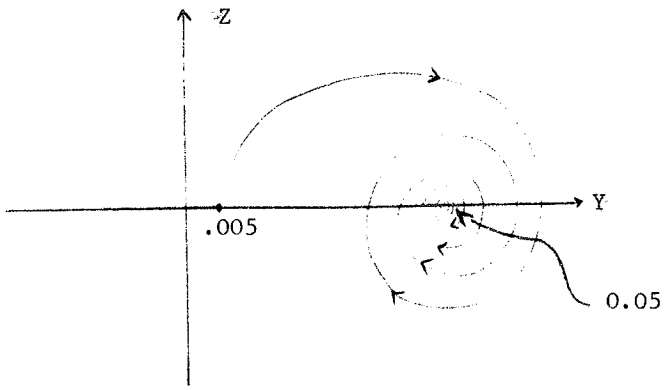
where $E \triangleq \ddot{Y}$; $Z \triangleq \dot{Y}$, $Y \triangleq Y$, $a = 1$, $b = \sqrt{2}$, and $C = 20$. This nonlinear system has singular points at (0, 0), (- 0.05, 0), and (+ 0.05, 0) in the Z - Y - plane, (see Example 8). Locally, (0, 0) is a saddle point while both (- 0.05, 0) and (+ 0.05, 0) are stable focii. In the Z - Y phase plane, the situation can be portrayed as follows (DeRusso, Roy, Close, 1965, p. 485):



As a physical system, the system can only assume one stable equilibrium point or the other; it cannot be both at the same time. For convenience, assume that the actual equilibrium that the physical system attains is at (- 0.05,0). Now suppose one were interested in obtaining the steady-state initial values by integration. If one begins the integration procedure with an initial guess that is above the separatrix the integration will converge to (+ 0.05, 0) rather than to (- 0.05, 0). Hence, this example is not asymptotically stable. However, this system is asymptotically stable in the small and if the initial guess were close enough to be on the correct side of the separatrix, then the method of integrating out will yield the correct initial values.

```
$LIST ITER7
01/03/72   10:34                    -83-

00010 STRP,/,1, ,8,16,:,7,13,19,25,31,37,43,:,14,26,38,50,62,',0
00020 /$ SNUMB THKAA
00030 /$ IDENT T.KERR,KERR  37  578D  5-4171
00040 /$ PRØGRAM RLHS
00050 /$ LIMITS 2,16000,,1000
00060 /$ PRMFL H*,Z,R,WATSØN/ADA70A
00070 /$ DISC 02,X2R,20L
00080 /$ PRMFL P*,R/A/S,L,*/P
00090 /$ NØTIFY */S
00100 0;40;2;1
00110 'T.KERR 9/21/71
00120 'R&DC 37 578D 8*235-4171
00130 'NØ S.S. I.C. BY INTEGATIØN FØR NØNLINEAR SYSTEM
00140 2;1;50;1;5;2
00150 2
00160 1-4:1-4:1-6:1-2:20
00170 0:30:1
00180 ;2;Y;Z;E;W
00190 U;6
00200 0
00210 E;11;4;U;Y;W;Z
00220 0:1:1:-1:-2
00230 Z;1;E;1;1
00240 0:0:1:0:1
00250 Y;1;Z;1;1
00260 -0.005:0:1:0:1
00270 W;13;3;0;Y;Y;Y
00271 400
00280 0;40;1
00290 /$ ENDJØB

READY
```

```
    ENTIAL EQUATIØNS

NAME        Y              DY1 ...
Y         4.9897E-02
Z        -2.9135E-04


CASE   1   NØ S.S. I.C. BY INTEGATIØN FØR NØNLINEAR SYSTEM

  TIME          Y             Z             E             W

0.          5.0000E-03  0.           4.9500E-03  5.0000E-05
1.0000E 00  6.4929E-03  2.5060E-03   1.3713E-03  1.0949E-04
2.0000E 00  9.6480E-03  3.9667E-03   1.3554E-03  3.5923E-04
3.0000E 00  1.4406E-02  5.6387E-03   1.9324E-03  1.1958E-03
4.0000E 00  2.0969E-02  7.8412E-03   1.5987E-03  3.6881E-03
5.0000E 00  2.9439E-02  9.4609E-03   3.1173E-04  1.0206E-02
6.0000E 00  3.8839E-02  8.6130E-03  -1.8220E-03  2.3435E-02
7.0000E 00  4.5990E-02  6.1084E-03  -5.1356E-03  3.8909E-02
8.0000E 00  4.9808E-02  1.8088E-03  -3.2356E-03  4.9426E-02
9.0000E 00  5.0459E-02 -1.1302E-04  -7.0489E-04  5.1390E-02
1.0000E 01  5.0187E-02 -2.5649E-04   1.3627E-04  5.0564E-02
1.1000E 01  5.0018E-02 -8.4201E-05   1.3230E-04  5.0054E-02
1.2000E 01  4.9985E-02 -3.9510E-06   3.7715E-05  4.9955E-02
1.3000E 01  5.0013E-02 -1.5557E-05   5.5797E-06  5.0038E-02
1.4000E 01  4.9988E-02  2.1825E-05  -2.0017E-05  4.9965E-02
1.5000E 01  5.0006E-02 -5.2258E-05   9.1745E-05  5.0019E-02
1.6000E 01  5.0020E-02  2.8016E-05  -9.5273E-05  5.0059E-02
1.7000E 01  4.9911E-02  4.1859E-05   9.3666E-05  4.9734E-02
1.8000E 01  5.0153E-02 -1.6658E-04   2.5592E-05  5.0461E-02
1.9000E 01  4.9813E-02  3.5368E-04  -3.3481E-04  4.9440E-02
2.0000E 01  5.0093E-02 -5.1573E-04   8.4457E-04  5.0280E-02
2.1000E 01  4.9889E-02 -2.0476E-05   2.6207E-04  4.9668E-02
2.2000E 01  4.9936E-02  7.1099E-05  -1.5073E-05  4.9809E-02
2.3000E 01  4.9988E-02  3.0931E-05  -3.8159E-05  4.9964E-02
2.4000E 01  5.0002E-02  5.3547E-06  -1.5221E-05  5.0007E-02
2.5000E 01  4.9990E-02 -2.8351E-05   7.7455E-05  4.9969E-02
2.6000E 01  5.0039E-02 -7.1075E-06  -6.4315E-05  5.0118E-02
2.7000E 01  4.9906E-02  9.8287E-05  -1.0057E-05  4.9720E-02
2.8000E 01  5.0118E-02 -1.9765E-04   1.5862E-04  5.0355E-02
2.9000E 01  4.9926E-02  2.8869E-04  -4.2970E-04  4.9778E-02
E3.0000E 01 4.9897E-02 -2.9135E-04   7.8894E-04  4.9690E-02


CPU TIME - TØTAL =    1.196 SECØNDS.

READY
```



IUATIØNS

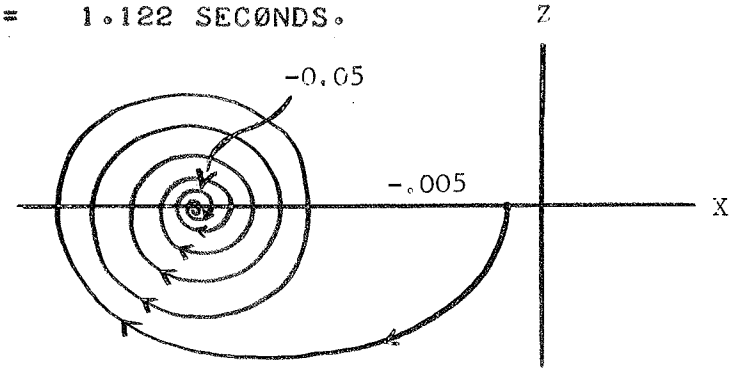
NAME	Y	DY1 ...
Y	-4.9897E-02	
Z	2.9140E-04	

CASE 1 NØ S.S. I.C. BY INTEGATIØN FØR NØNLINEAR SYSTEM

TIME	Y	Z	E	W
0.	-5.0000E-03	0.	-4.9500E-03	-5.0000E-05
1.0000E 00	-6.4929E-03	-2.5060E-03	-1.3713E-03	-1.0949E-04
2.0000E 00	-9.6480E-03	-3.9667E-03	-1.3554E-03	-3.5923E-04
3.0000E 00	-1.4406E-02	-5.6387E-03	-1.9324E-03	-1.1958E-03
4.0000E 00	-2.0969E-02	-7.8412E-03	-1.5987E-03	-3.6881E-03
5.0000E 00	-2.9439E-02	-9.4609E-03	-3.1173E-04	-1.0206E-02
6.0000E 00	-3.8839E-02	-8.6130E-03	1.8220E-03	-2.3435E-02
7.0000E 00	-4.5990E-02	-6.1084E-03	5.1356E-03	-3.8909E-02
8.0000E 00	-4.9808E-02	-1.8088E-03	3.2356E-03	-4.9426E-02
9.0000E 00	-5.0459E-02	1.1302E-04	7.0489E-04	-5.1390E-02
1.0000E 01	-5.0187E-02	2.5649E-04	-1.3627E-04	-5.0564E-02
1.1000E 01	-5.0018E-02	8.4201E-05	-1.3230E-04	-5.0054E-02
1.2000E 01	-4.9985E-02	3.9510E-06	-3.7715E-05	-4.9955E-02
1.3000E 01	-5.0013E-02	1.5556E-05	-5.5774E-06	-5.0038E-02
1.4000E 01	-4.9988E-02	-2.1825E-05	2.0014E-05	-4.9965E-02
1.5000E 01	-5.0006E-02	5.2261E-05	-9.1743E-05	-5.0019E-02
1.6000E 01	-5.0020E-02	-2.8022E-05	9.5276E-05	-5.0059E-02
1.7000E 01	-4.9911E-02	-4.1848E-05	-9.3680E-05	-4.9734E-02
1.8000E 01	-5.0153E-02	1.6657E-04	-2.5566E-05	-5.0461E-02
1.9000E 01	-4.9813E-02	-3.5367E-04	3.3478E-04	-4.9440E-02
2.0000E 01	-5.0093E-02	5.1575E-04	-8.4454E-04	-5.0280E-02
2.1000E 01	-4.9889E-02	2.0494E-05	-2.6208E-04	-4.9668E-02
2.2000E 01	-4.9936E-02	-7.1093E-05	1.5062E-05	-4.9809E-02
2.3000E 01	-4.9988E-02	-3.0931E-05	3.8157E-05	-4.9964E-02
2.4000E 01	-5.0002E-02	-5.3555E-06	1.5220E-05	-5.0007E-02
2.5000E 01	-4.9990E-02	2.8356E-05	-7.7454E-05	-4.9969E-02
2.6000E 01	-5.0039E-02	7.0984E-06	6.4323E-05	-5.0118E-02
2.7000E 01	-4.9906E-02	-9.8275E-05	1.0035E-05	-4.9720E-02
2.8000E 01	-5.0118E-02	1.9764E-04	-1.5859E-04	-5.0355E-02
2.9000E 01	-4.9926E-02	-2.8870E-04	4.2968E-04	-4.9778E-02
E3.0000E 01	-4.9897E-02	2.9140E-04	-7.8895E-04	-4.9690E-02

CPU TIME - TØTAL = 1.122 SECØNDS.

READY

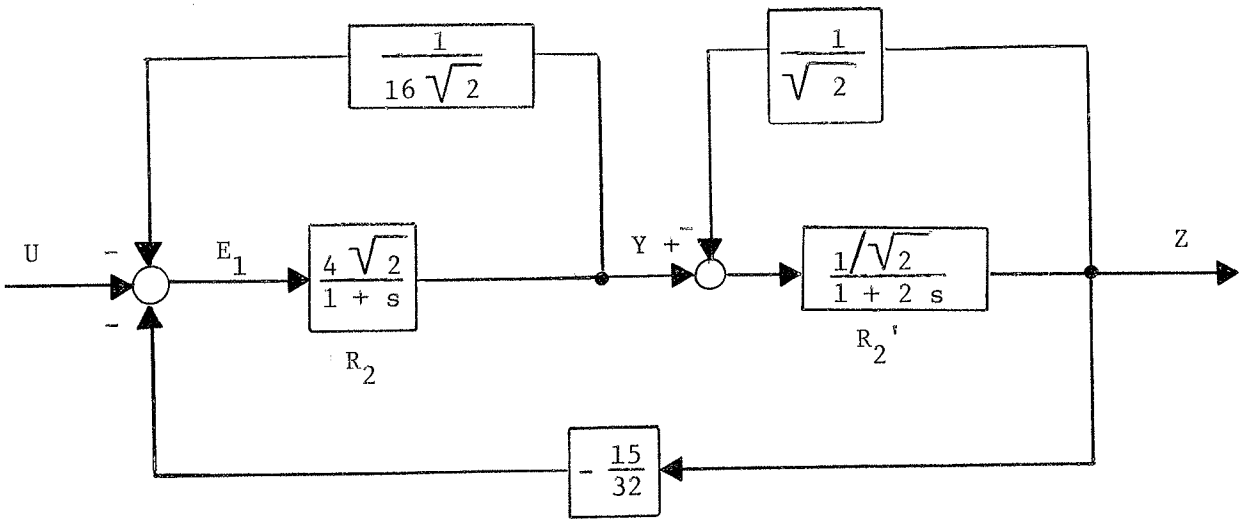


Section V: Linear System Configuration for which both Integration and Iteration Fail to Yield the Steady-State Initial Values

A. The Problem

There do exist linear systems for which both the method of iteration and the method of integration fail to yield the steady-state initial conditions.

Example 11: Initial Value Calculations Fail for Iteration and Integration



Now applying the notation of Section II: C: Case 1,

$$m_3 k_5 = \frac{1}{2} ; \; k_3 k_6 = \frac{1}{4} ; \; m_3 k_3 k_4 = -\frac{15}{8} ;$$

$B = \frac{3}{2}$; $C = \frac{5}{4}$; $B C = \frac{15}{8}$; $A = 0$. By Jury's stability test, the iterations fail to converge for any H_1, H_2 since $h(H_1, H_2) = A H_1 H_2 > 0$ is not satisfied for any H_1, H_2 when $A \equiv 0$.

As shown in Theorem 2, Case 1, of this section,

$$E_1(s) = \frac{[B + s D_2][1 + s D_1]}{A + s[B D_1 + C D_2 + s D_1 D_2]} U \quad .$$

Now with A = 0, and the input kept constant, as is done when obtaining initial conditions by integration, the output is

$$E_1(s) = \frac{[B + s D_2][1 + s D_1]}{s[B D_1 + C D_2 + s D_1 D_2]} \cdot \frac{U_o}{s} \quad .$$

Applying the Laplace final value theorem, we have that $\lim_{t\to\infty} e_1(t) = \lim_{s\to 0} s E_1(s) = \infty$, which indicates that this variable does not settle out as the integration proceeds; therefore, obtaining steady-state initial values by integration fails.

The salient features of a simulation that cause both of these methods of determining the steady-state initial conditions to fail will now be investigated.

For one D.E., it is easily established that the iterations do not converge for any choice of convergence factor if and only if the loop gain $k_3 k_4 = -1$. Sufficiency is established by observing that when $k_3 k_4 = -1$, the root of the characteristic equation is not within the unit circle, as shown in Section II: B:, Case 5. Necessity is established by observing that if $k_3 k_4 \neq -1$, Cases 1-4 of Section 1:B yield regions in which convergence of the iterations is guaranteed. That the method of integration fails to yield the initial conditions can be seen by observing that the transfer function for the closed loop system is $\frac{Y}{U} = \frac{k_3}{s(T + \ldots)}$, as pointed out in Section II: B:, Case 5, and which is a type one system. Applying the

Laplace final value theorem when the input is constant, we obtain

$$\lim_{t\to\infty} Y(t) = \lim_{s\to 0} s \frac{k_3}{s\,(T + \ldots)} \cdot \frac{U_o}{s} = \infty \text{ , which indicates that the}$$

method of integration will never settle out.

We now want to establish that, for two D.E.'s, we can have the steady-state initial condition iterations not converging only for either

(1) $A = 0$

or (2) $B = 0$, $C = 0$, $A \neq 0$.

We now present the proof of necessity for the iterations not to converge for two D.E.'s. The approach taken will be to prove the contrapositive, that is, it will be shown that if $A \neq 0$ and not both $A = 0$, $B = 0$, then convergence acceleration factors may be produced which ensure that the iteration will converge (or, equivalently, that the roots of the characteristic equation will all lie within the unit circle).

Theorem 1: Necessary conditions for Iteration Failure of 2 D.E. Configurations

Proof: Assume that $A \neq 0$; to establish necessity it is enough to produce an R_2, R_2' or a non-null region for R_2, R_2' which will cause the characteristic equation associated with the 2 D.E. configuration of Section I: C: Case 1 to have all roots within the unit circle.

The characteristic equation associated with the iteration is

$$0 = E^2 + [B H_2 + C H_1 - 2] E + A H_1 H_2 - C H_1 - B H_2 + 1 ,$$

where $B = (m_3 k_5 + 1)$, $C = (k_3 k_6 + 1)$, $A = B C + m_3 k_3 k_4$. The inequalities that must be satisfied for stability by the Jury test are

$$A H_1 H_2 > 0 ,$$

$$4 - 2 B H_2 - 2 C H_1 + A H_1 H_2 > 0 ,$$

$$2 + A H_1 H_2 - C H_1 - B H_2 > 0 ,$$

$$A H_1 H_2 - C H_1 - B H_2 < 0 .$$

Since Jury's stability test is satisfied if and only if the roots of the characteristic equation are within the unit circle, we will concentrate all our attention on the inequalities of the Jury test. As done in Section I: C: Case 1,

$$h (H_1, H_2) \triangleq A H_1 H_2 ,$$

$$g (H_1, H_2) \triangleq 4 - 2 B H_2 - 2 C H_1 + A H_1 H_2 ,$$

$$p (H_1, H_2) \triangleq 2 - B H_2 - C H_1 + A H_1 H_2 ,$$

$$q (H_1, H_2) \triangleq - B H_2 - C H_1 + A H_1 H_2 .$$

Now let $H_1 = X$, $H_2 = \alpha X$, where X and α are real numbers to be determined later.

For Jury's stability test to be satisfied it must be that

$$h (X, \alpha X) = \alpha A X^2 > 0 ,$$

$$g (X, \alpha X) = 4 - 2 [B \alpha + C] X + A \alpha X^2 > 0 ,$$

$$p\ (X, \alpha X) = 2 - [B\ \alpha + C]\ X + A\ \alpha\ X^2 > 0\ ,$$

$$q\ (X, \alpha X) = -\ [B\ \alpha + C]\ X + A\ \alpha\ X^2 < 0\ .$$

All of the above functions are parabolas.

From the condition that $h\ (X, \alpha X) = \alpha\ A\ X^2 > 0$, it is seen that for this inequality to hold for any X, it must be that $\alpha\ A > 0$. The inequality $q\ (X, \alpha X) < 0$ holds only for X such that

$$0 < X < \frac{B\ \alpha + C}{A\ \alpha}\ .$$

Represent X as $X = k\ (\frac{B\ \alpha + C}{A\ \alpha}) + (1 - K)\ 0\ , \quad 0 < k < 1\ .$

Jury's conditions now become

$$h\ (\alpha, k) = \frac{[B\ \alpha + C]^2}{A\ \alpha}\ k^2 > 0\ ,$$

$$g\ (\alpha, k) = 4 - \frac{2\ [B\ \alpha + C]^2}{A\ \alpha}\ k + \frac{[B\ \alpha + C]^2}{A\ \alpha}\ k^2 > 0\ ,$$

$$p\ (\alpha, k) = 2 - \frac{[B\ \alpha + C]^2}{A\ \alpha}\ k + \frac{[B\ \alpha + C]^2}{A\ \alpha}\ k^2 > 0\ ,$$

$$q\ (\alpha, k) = -\ \frac{[B\ \alpha + C]^2}{A\ \alpha}\ k + \frac{[B\ \alpha + C]^2}{A\ \alpha}\ k^2 < 0\ .$$

Case 1: $A \neq 0\ ,\ B = 0,\ C \neq 0$

Proof:

Pick $\alpha = \overline{\alpha} = \frac{C^2}{2\ A}$. Jury's conditions become

$$h\ (\overline{\alpha}, k) = 2\ k^2 > 0\ ,$$

$$g\ (\bar{\alpha},\ k) = 4 + \frac{C^2}{A\ \alpha}\ (k^2 - 2\ k) > 0\ ,$$

$$p\ (\bar{\alpha},\ k) = 2 + \frac{C^2}{A\ \alpha}\ (k^2 - k) > 0\ ,$$

$$q\ (\bar{\alpha},\ k) = \frac{C^2}{A\ \alpha}\ (k^2 - k) < 0\ .$$

Let $k = \frac{1}{2}$; the above conditions become

$$h\ (\frac{C^2}{2\ A}\ ,\ \frac{1}{2}) = \frac{1}{2} > 0\ ,$$

$$g\ (\frac{C^2}{2\ A}\ ,\ \frac{1}{2}) = 3\ \frac{1}{2} > 0\ ,$$

$$p\ (\frac{C^2}{2\ A}\ ,\ \frac{1}{2}) = 1\ \frac{1}{2} > 0\ ,$$

$$q\ (\frac{C^2}{2\ A}\ ,\ \frac{1}{2}) = -\ \frac{1}{2} < 0\ .$$

Since all the inequalities are satisfied for this choice of α and k, the iterations converge for $H_1 = \frac{B\ C + 2\ A}{2\ A\ C}$, $H_2 = \frac{B\ C^2 + 2\ A\ C}{4\ A^2}$ or, equivalently,

$$R_2 = \frac{B\ C + 2\ A - 2\ A\ C}{2\ A\ C}\ ,\ R_2' = \frac{B\ C^2 + 2\ A\ C - 4\ A^2}{4\ A^2}\ .$$

Case 2: $A \neq 0$, $B \neq 0$, $C = 0$

Proof: By symmetry, if we had, instead, let $H_2 = X$, $H_1 = \alpha\ X$, we would have obtained an expression identical to the one obtained for $B = 0$, $C \neq 0$. This guarantees convergence just as it did in Case 1.

Case 3: $A \neq 0$, $B \neq 0$, $C \neq 0$.

Proof:

$$\text{Let } U \triangleq \frac{[B\,\alpha + C]^2}{A\,\alpha} \text{ ; now } U > 0 \text{ since } A\,\alpha > 0 \text{ .}$$

The Jury stability conditions now become

$$h\,(\alpha, k) = U\,k^2 > 0 \text{ ,}$$

$$g\,(\alpha, k) = 4 + U\,(k^2 - 2\,k) > 0 \text{ ,}$$

$$p\,(\alpha, k) = 2 + U\,(k^2 - k) > 0 \text{ ,}$$

$$q\,(\alpha, k) = U\,(k^2 - k) < 0 \text{ .}$$

We want to manipulate α and k so that the above four inequalities are satisfied and so that $0 < k < 1$ and α is a real number. This last condition that α be real is very important since complex convergence factors are not acceptable to ADA. Considerable manipulation will be done to assure that all of the above conditions are satisfied.

Adjust α so that $U = \frac{[B\,\alpha + C]^2}{A} = W > 0$, where W will be taken large enough to ensure that α is real. If $U = W$, then

$$\frac{[B\,\alpha + C]^2}{A} = W > 0$$

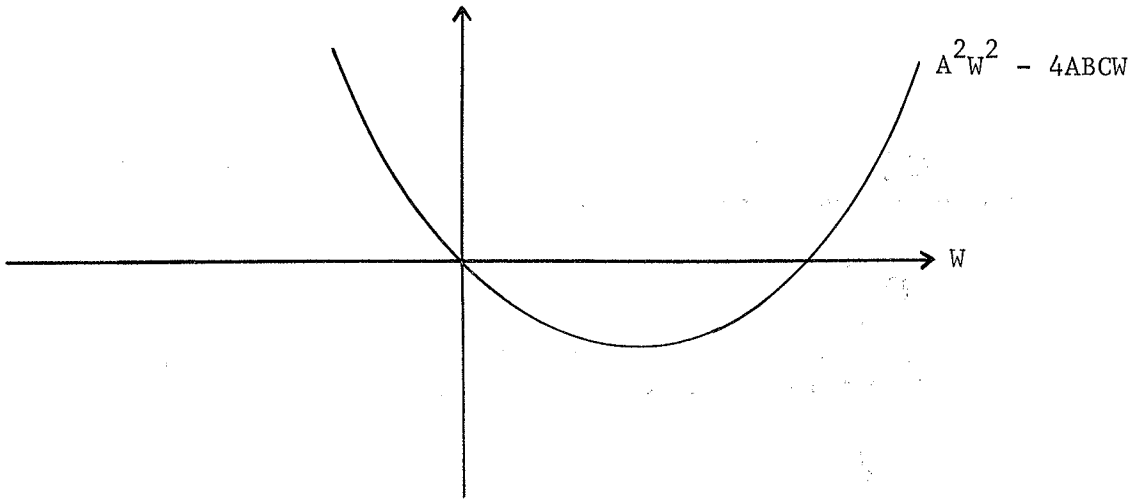
or

$$B^2\,\alpha^2 + 2\,B\,C\,\alpha + C^2 = W\,A\,\alpha \text{ ,}$$

which has the solution

$$\alpha = \frac{W\,A - 2\,B\,C \pm \sqrt{W^2\,A^2 - 4\,W\,A\,B\,C}}{2\,B^2}$$

Now α is real only if the discriminant $W^2 A^2 - 4 W A B C > 0$. Notice that as a function of W, the function $W^2 A^2 - 4 W A B C$ will be positive for a large enough W since the coefficient, A^2, of the squared power term is positive. The situation may be graphically portrayed as



Notice that it is of no concern that $A B C > 0$ or $A B C < 0$; for large enough W, the function $A^2 W^2 - 4 A B C W > 0$; hence, α is real. Summarizing, we pick W large enough so that the following inequality is satisfied

$$A^2 W^2 - 4 A B C W > 0 \text{ , let } W^* \text{ be such a W,}$$

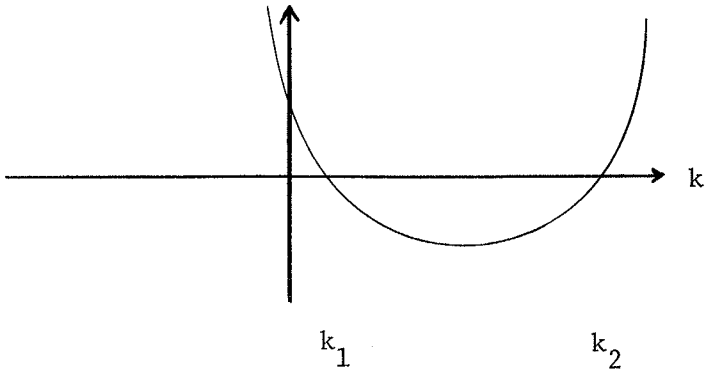
then $\alpha^* = \dfrac{W^* A - 2 B C - \sqrt{W^{*2} A^2 - 4 W^* A B C}}{2 B^2}$, a real number, we will later put the two more conditions on W^*.

Now we want to satisfy the four inequalities of the Jury stability test. Notice that $h(\alpha^*, k) = W^* k^2 > 0$ is satisfied for all k, $0 < k < 1$; similarly $q(\alpha^*, k) = W^*(k^2 - k) < 0$ for all k, $0 < k < 1$. We must now pick k^*, $0 < k^* < 1$ so that $p(\alpha^*, k^*) > 0$ and $g(\alpha^*, k^*) > 0$.

In other words, we want to pick k^*, so that $g(\alpha^*, k^*) = 4 + W^*(k^{*2} - 2k^*) > 0$ and $p(\alpha^*, k^*) = 2 + W^*(k^{*2} - k^*) > 0$. It is reemphasized that k^* must be within the prescribed region, $0 < k^* < 1$. The above two inequalities yield that we need

$$k^2 - 2k > -\frac{4}{W^*} \text{ and } k^2 - k > -\frac{2}{W^*} \quad .$$

Each of these inequalities can be represented as a parabola of the following form



To satisfy the inequalities, it is only necessary to choose $0 < k < k_1$ (it will be shown that $0 < k_1 < 1$). For $k^2 - 2k > -\frac{4}{W^*}$, $k_1 = 1 - \sqrt{1 - \frac{4}{W^*}}$

$k^2 - k > -\frac{2}{W^*}$, $k_1' = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{8}{W^*}}$. Here is where we obtain our other conditions on W^*. The expression for k_1 and k_1' must be real numbers, which is assured if

$1 - \frac{4}{W^*} > 0$ and $1 - \frac{8}{W^*} > 0$, this dictates that $W^* > 8$, $W^* > 4$, and, from before, we still require that $W^* > \frac{4BC}{A}$. All these conditions together require that $W^* > \max(8, \frac{4BC}{A})$.

Now we will establish the required properties of k_1 and k_1'.

Since $1 = 1$, $W^* > 0$, we have that

$1 - \frac{4}{W^*} < 1$; now since $W^* > \max\left(8, \frac{4\,B\,C}{A}\right)$,

$0 < 1 - \frac{4}{W^*}$; the above two conditions imply that $0 < 1 - \frac{4}{W^*} < 1$ or $0 < \sqrt{1 - \frac{4}{W^*}} < 1$. We can now say that $0 > -\sqrt{1 - \frac{4}{W^*}} > -1$ and $1 > 1 - \sqrt{1 - \frac{4}{W^*}} > 0$, which is what we set out to prove, Similarly, for k_1',

$$0 < \sqrt{1 - \frac{8}{W^*}} < 1$$

$$0 > -\frac{1}{2}\sqrt{1 - \frac{8}{W^*}} > -\frac{1}{2}$$

$$1 > \frac{1}{2} > \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{8}{W^*}} > 0 .$$

Now how do we know whether to pick $0 < k < k_1$ or $0 < k < k_1'$? We pick k such that $0 < k < k_1'' = \min(k_1, k_1')$. (Actually, for $W^* > \frac{16}{3}$, $k_1'' = k_1'$; since we pick W^* such that $W^* > \max\left(8, \frac{4\,B\,C}{A}\right) > \frac{16}{3}$, it will always happen that $k_1'' = k_1'$).

In summary, for $A \neq 0$, $B \neq 0$, $C \neq 0$, take $W^* > \max\left(8, \frac{4\,B\,C}{A}\right)$,

$\alpha^* = \frac{W^* A - 2\,B\,C - \sqrt{W^{*2} A^2 - 4\,W^*\,A\,B\,C}}{2\,B^2}$, a real number, and any k within the region $0 < k < \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{8}{W^*}}$; or, equivalently

$$R_2 = H_1 - 1 = k\left(\frac{B\alpha^* + C}{A\alpha^*}\right) - 1 ,$$

$$R_2' = H_2 - 1 = \alpha^* k\left(\frac{B\alpha^* + C}{A\alpha^*}\right) - 1 ,$$

for any k such that $0 < k < \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{8}{W^*}}$, then the four inequalities of Jury's stability test, namely, $h > 0$, $g > 0$, $p > 0$, $q < 0$, are satisfied and the roots of the characteristic equation are within the unit circle so the iterations converge. ■

It is relatively easy to show that, for two D.E.'s, having

(1) $A = 0$

(2) $A \neq 0$, $B = 0 = C$

are sufficient conditions to cause the iteration not to converge.

Theorem 2: Sufficient Condition for Iteration Failure of 2 D.E. Configuration

Proof: As before, since Jury's stability test is satisfied if and only if the roots of the characteristic equation are within the unit circle, we will concentrate all our attention on the inequalities of the Jury test.

The inequalities are:

$$h(H_1, H_2) \triangleq A H_1 H_2 > 0 ,$$

$$g(H_1, H_2) \triangleq 4 - 2 B H_2 - 2 C H_1 + A H_1 H_2 > 0 ,$$

$$p\ (H_1, H_2) \triangleq 2 - B\ H_2 - C\ H_1 + A\ H_1\ H_2 > 0\ ,$$

$$q\ (H_1, H_2) \triangleq - B\ H_2 - C\ H_1 + A\ H_1\ H_2 < 0.$$

Case 1: For $A = 0$, $h\ (H_1, H_2) = 0$ for all H_1, H_2 and hence can never satisfy the strict inequality $h\ (H_1, H_2) > 0$.

Case 2: $A \neq 0$, $B = 0$, $C = 0$.

Now $h\ (H_1, H_2) = A\ H_1\ H_2$ and $q\ (H_1, H_2) = A\ H_1\ H_2$. Since now $h\ (H_1, H_2) = A\ H_1\ H_2 = q\ (H_1, H_2)$, it can never happen that, simultaneously, $h\ (H_1, H_2) > 0$ and $q\ (H_1, H_2) < 0$ be satisfied, since this would require that a number be strictly greater than and less than zero simultaneously. Hence, Jury's stability test can never be satisfied. ■

Theorem 1 and 2 together show that, for a 2 D.E., configuration, to have the steady-state initial value calculations fail to converge for any choice of R_2, R_2' it is necessary and sufficient that $A = 0$, or $A \neq 0$, $B = 0$, $C = 0$. (The condition $A = 0$, $B = 0$, or $C = 0$ is equivalent to having a loop gain of -1 [i.e., unity positive feedback yielding an effective type 1 or greater system somewhere within the whole system].)

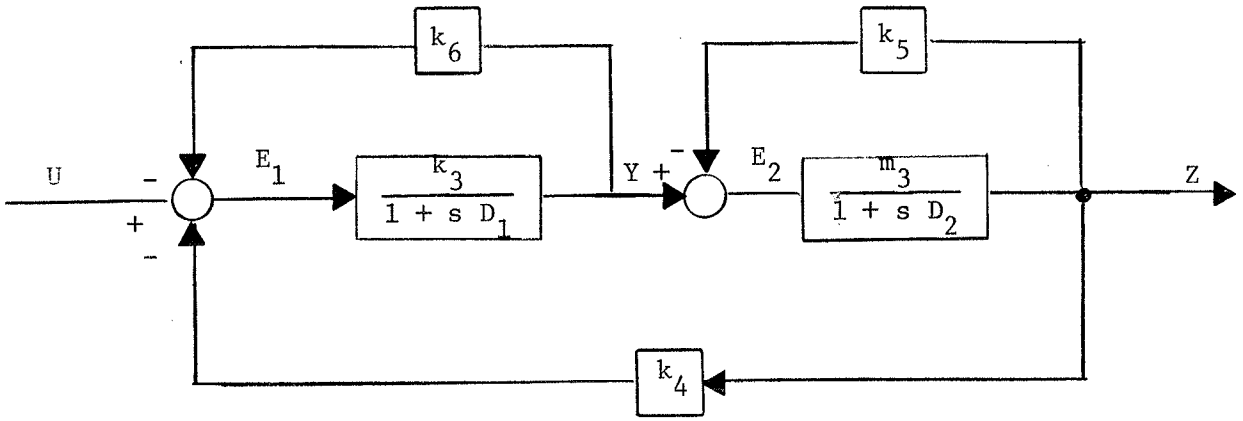
We now want to establish that, for two D.E.'s, we can have the method of obtaining the steady-state initial values by integration fail if

(1) $A = 0$

(2) $B = 0 = C$

Theorem 3: Sufficient Conditions for Integration Failure of 2 D.E. Configuration

Proof: Consider the configuration



$$E_1 = U - k_6 Y - k_4 Z$$

$$E_2 = Y - k_5 Z$$

$$Y = \frac{k_3}{1 + s D_1} E_1 \; ; \; Z = \frac{m_3}{1 + s D_2} E_2 \; ,$$

where $D_1 = d_{10} + d_{11}s + d_{12}s^2 + \ldots$

$D_2 = d_{20} + d_{21}s + d_{22}s^2 + \ldots$

and the d_{ij}'s are arbitrary real numbers.

By calculation,

$$E_1 = \frac{[1 + m_3 k_5 + s D_2] [1 + s D_1]}{A + s [B D_1 + C D_2 + s D_1 D_2]} U$$

The denominator is the characteristic equation of the entire system. If A = 0, this system is at least a type one system and could possibly be of higher type depending on D_1 and D_2. If A = 0, then in applying the method of obtaining the steady-state initial values by integration, the inputs are kept constant $U(s) = \frac{U_o}{s}$.

Therefore, $E_1(s) = \frac{[1 + m_3 k_5 + s D_2] [1 + s D_1]}{s^2 [B D_1 + C D_2 + s D_1 D_2]}$

and applying the Laplace transform final value theorem, $\lim_{t\to\infty} e_1(t) = \lim_{s\to 0} s E_1(s) = \infty$. This indicates that no matter how long the integration is carried out the variable $e_1(t)$ refuses to converge to a constant value.

Now $E_1(s) = \frac{[B + s D_2] [1 + s D_1]}{A + s [B D_1 + C D_2 + s D_1 D_2]} U$

Case 2: If B = 0 = C, then

$$E_1(s) = \frac{s D_2 [1 + s D_1]}{A + s^2 D_1 D_2} .$$

The denominator has a missing power of s, and is therefore unstable. As shown in Section IV, the method of determining initial conditions by integration fails for unstable systems. ■

Based on the proofs that both the method of integration and the method of iteration fail to converge if and only if we have systems that are inherently of type n ($n > 0$), which were done for all one D.E. and 2 D.E. linear systems, we now use a plausibility argument and conclude that the same can be said for linear systems containing any number of D.E.'s.

When methods of obtaining steady-state initial conditions fail, it should be looked upon as a blessing because, by our previous proofs, it indicates that we have an inherent type n ($n > 0$) system. For a type n ($n > 0$) system, the initial conditions on the integrators should all be specified (an integrator never has a steady-state unless the input is zero).

B. Detection of the Problem: An Application of Signal Flow-Graph Techniques

Can we determine whether a linear system simulation is inherently a type n ($n > 0$) system without having to first see that the method of determining steady-state initial conditions fail? Yes, by appealing to the techniques of signal flow-graph analysis (Kuo, 1967, p. 193) one can quickly determine if the system is of type n ($n > 0$) merely by considering the loop gains.

From Mason's general gain formula (Kuo, 1967, p. 193) one can obtain the input-output relationship for a signal flow graph as simply

$$G = \sum_n \frac{G_n \Delta_n}{\Delta} = \frac{\text{output-node variable}}{\text{input-node variable}} \quad ,$$

where G_n = gain of n^{th} forward path,

Δ = 1 - (sum of all individual loop gains)

+ (sum of gain products of all possible combinations of two nontouching loops)

- (sum of gain products of all possible combinations of three nontouching loops) + ...,

Δ_n = value of Δ for that part of the signal flow graph not touching the n^{th} forward path. The Δ is called the signal flow graph determinant.

Without calculating G, the system under consideration is an inherent type n ($n > 0$) system if $\Delta = 0$.

Section VI: Conclusion

Three distinct approaches have been presented for obtaining the ADA steady-state initial values. The choice as to which of these three alternatives to use should be based on economics. For a particular simulation situation, one method may be much less expensive than the others, hence, one should choose the method that is the least costly for the situation.

The general purpose simulation language ADA is designed to most efficiently solve dynamic problems, problems involving integration. The steady-state initial value problem is an algebraic problem and the steady-state initial value calculation feature of ADA solves this algebraic problem by iteration. Two new methods, based on the characteristic equation of the system of iteration equations and a graphical interpretation of Jury's stability test, were presented in this report for determining the user specified convergence factors. Prior to this report, the approach that had been taken was to try to force convergence by a motley assortment of techniques based on experience and tinkering with what were believed to be the "important" integrators in a simulation. By understanding the underlying mathematics of the problem, specification of the convergence factors, which was once done by art, can now be done by science. The extension of the technique to more complicated systems both linear and nonlinear was given in Section II:D.

In Section III, the state variable formulation of the steady-state initial value problem was shown to be an algebraic problem. In Section IV,

it was shown that this algebraic problem for linear systems and asymptotically stable nonlinear systems may be solved by integration.

Another technique for solving systems of nonlinear equations and an example of its application to a high dimensional problem were presented in the Appendix.

Finally, three new theorems were presented to characterize a simulation situation where both the methods of iteration and integration fail to converge to steady-state initial values.

Concrete examples of how the various techniques are applied to actual computer simulations are interspersed throughout this report in the belief that they will facilitate the reading.

Although the three approaches to obtaining the steady-state initial values, especially the method of choosing convergence factors, are slanted toward the ADA user, the results of the analysis should be applicable to any general purpose continuous simulation program since calculating the steady-state initial values is a universal problem. The predecessor of ADA, Dynasar, also incorporated a steady-state initial value iteration routine, but the iteration equations were of a form different from ADA's. However, the idea still applies; pick convergence acceleration factors to force all the roots of the characteristic equation associated with the system of difference equations to lie within the unit circle.

Appendix: A Technique for Solving Nonlinear Algebraic Systems or Demonstrating that no Solution Exists

The technique to be described is not very sophisticated and may best be shown by demonstrating its use.

Consider the following nonlinear algebraic system of equations over the complex field:

$$X\,Y - X^2 = 3 ,$$

$$X^2 - Y^2 = 2 .$$

Solutions may be found easily by making the substitution $Y = \alpha X$, where α is some complex scalar. Substituting, we now have

$$\alpha X^2 - X^2 = 3$$

$$X^2 - \alpha^2 X^2 = 2$$

or, equivalently, $X^2 = \dfrac{3}{(\alpha - 1)}$

$$X^2 = \frac{2}{(1 - \alpha^2)} .$$

This leads to the equation $\dfrac{3}{(\alpha - 1)} = \dfrac{2}{(1 - \alpha^2)}$, which has the solution $\alpha = -\dfrac{5}{3}$. Now $X^2 = -\dfrac{9}{8}$, or $X = \pm j \dfrac{3}{2\sqrt{2}}$. The two solutions for the above system of nonlinear algebraic equations over the complex field are

$$(X, Y) = \left(+ j \frac{3}{2\sqrt{2}} , - j \frac{5}{2\sqrt{2}}\right) \text{ and } (X, Y) = \left(- j \frac{3}{2\sqrt{2}} , + j \frac{5}{2\sqrt{2}}\right) .$$

If we were only interested in real solutions (i.e., considering the system of nonlinear algebraic equations over the real field), no solutions exist for this system.

Example 12: Application of the Technique to Demonstrate the Inconsistency of a Particular System of 10 Nonlinear Algebraic Equations in 12 Unknowns (High Dimensionality)

Consider the following nonlinear system of algebraic equations:

(1) $3 = f_{11} + f_{22}$,

(2) $2 = f_{11}f_{22} - f_{12}f_{21}$,

(3) $-\sqrt{7}/2 = - g_{11}h_{11}f_{22} + g_{11}h_{12}f_{21} - g_{21}h_{12}f_{11} + g_{21}h_{11}f_{12}$,

(4) $- 1 = g_{11}h_{11} + g_{21}h_{12}$,

(5) $- 1/2 = - g_{12}h_{11}f_{22} + g_{12}h_{12}f_{21} - g_{22}h_{12}f_{11} + g_{22}h_{11}f_{12}$,

(6) $0 = g_{12}h_{11} + g_{22}h_{12}$,

(7) $-\sqrt{7}/2 = - g_{11}h_{21}f_{22} + g_{11}h_{22}f_{21} - g_{21}h_{22}f_{11} + g_{21}h_{21}f_{12}$,

(8) $- 1 = g_{11}h_{21} + g_{21}h_{22}$,

(9) $3/2 = - g_{12}h_{21}f_{22} + g_{12}h_{22}f_{21} + g_{22}h_{21}f_{12} - g_{22}h_{22}f_{11}$,

(10) $0 = g_{12}h_{21} + g_{22}h_{22}$. It is assumed that this is a consistent system.

Additional equations can be derived from these 10 equations which perhaps give more insight into the interrelationships that exist.

Combining (4) and (8) yields $g_{11}h_{11} + g_{21}h_{12} = - 1 = g_{11}h_{21} + g_{21}h_{22}$ which reduces to

(11) $g_{11}(h_{11} - h_{21}) = g_{21}(h_{22} - h_{12})$.

Combining (6) and (10) yields $g_{12}h_{11} + g_{22}h_{12} = 0 = g_{12}h_{21} + g_{22}h_{22}$ which reduces to

(12) $g_{12}(h_{11} - h_{21}) = g_{22}(h_{22} - h_{12})$.

Now if $g_{12} \neq 0$, $(h_{22} - h_{12}) \neq 0$, and $g_{11} \neq 0$, (11) and (12) can be combined to yield

$$\frac{g_{21}}{g_{11}} = \frac{(h_{11} - h_{21})}{(h_{22} - h_{12})} = \frac{g_{22}}{g_{12}}$$

or

(13) $g_{11}g_{22} = g_{12}g_{21}$. Notice that the equation (13) is only valid if $g_{11} \neq 0$, $(h_{22} - h_{11}) \neq 0$, and $g_{12} \neq 0$. The conditions for which this hypothesis holds true will now be investigated.

If, contrary to the hypothesis of (13), $h_{22} - h_{12} = 0$, then

$$(4') \quad 1 = g_{11}h_{11} + g_{21}h_{22}$$

$$\Leftarrow (8') \quad 1 = g_{11}h_{21} + g_{21}h_{22}$$

$$0 = g_{11}(h_{11} - h_{21})$$

From the above, either $g_{11} = 0$ or $(h_{11} - h_{21}) = 0$.

CaseI: $h_{22} - h_{12} = 0$ and $h_{11} - h_{12} = 0$.

Substituting into (5) and (9), we have that

(5') $-\frac{1}{2} = -g_{12}h_{11}f_{22} + g_{12}h_{22}f_{21} - g_{22}h_{22}f_{11} + g_{22}h_{11}f_{12}$

and

(9') $\frac{3}{2} = -g_{12}h_{11}f_{22} + g_{12}h_{22}f_{21} + g_{22}h_{11}f_{12} - g_{22}h_{22}f_{11}$,

which serve as a contradiction. Therefore, this case cannot happen.

Case II: $h_{22} - h_{12} = 0$ and $g_{11} = 0$.

From (11), we have that $0 = g_{21}(h_{22} - h_{12})$. It is already assumed that $(h_{22} - h_{12}) = 0$, but a question that may now be asked is does $g_{21} = 0$? If $g_{21} = 0$, then (4') yields the desired contradiction since $-1 = g_{11}h_{11} + g_{21}h_{12} = 0 + 0 = 0$. Therefore, $g_{21} \neq 0$. By substituting into (12), we have that $g_{12}(h_{11} - h_{21}) = 0$; so either $g_{12} = 0$ or $h_{11} - h_{21} = 0$.

Subcase IIa: $h_{22} - h_{12} = 0$, $g_{11} = 0$, and $h_{11} - h_{21} = 0$.

This contradicts (5') and (9') as in case I; so this cannot happen.

Subcase IIb: $h_{22} - h_{12} = 0$, $g_{11} = 0$, and $g_{12} = 0$.

From (6), we have that $0 = g_{12}h_{11} + g_{22}h_{12} = 0 + g_{22}h_{12}$; so either $g_{22} = 0$ or $h_{12} = 0$.

Subcase IIb-(i): $h_{22} - h_{12} = 0$, $g_{11} = 0$, $g_{12} = 0$, and $g_{22} = 0$.

Substituting into (5), we have that

$-1/2 = -g_{12}h_{11}f_{22} + g_{12}h_{12}f_{21} - g_{22}h_{12}f_{11} + g_{22}h_{11}f_{12} = 0 + 0 + 0 + 0$ or $-1/2 = 0$; this is a contradiction so that this case cannot happen.

Subcase IIb-(ii): $h_{22} - h_{12} = 0$, $g_{11} = 0$, $g_{12} = 0$, and $h_{12} = 0$.

Since $h_{22} = h_{12}$, we have that $h_{22} = 0$.

From (8), $-1 = g_{11}h_{21} + g_{21}h_{22} = 0 + 0 = 0$; this is a contradiction so this case cannot happen. Therefore, subcase IIb cannot happen so it is impossible that $h_{22} - h_{12} = 0$.

Now if $g_{11} = 0$, then by substituting in (11) we have that $g_{21}(h_{22} - h_{12}) = 0$; so that either $g_{21} = 0$ or $h_{22} - h_{12} = 0$.

Case I: $g_{11} = 0$ and $g_{21} = 0$.

From (3), we have the necessary contradiction, $-\sqrt{7}/2 = -0 + 0 - 0 + 0 = 0$; therefore, this case cannot happen.

Case II: $g_{11} = 0$ and $(h_{22} - h_{12}) = 0$.

This is the same as case II in the above discussion and so yields a contradiction. Therefore, it is impossible for $g_{11} = 0$.

Now if $g_{12} = 0$, from (12), we have that $0 = g_{22}(h_{22} - h_{12})$.

Case I: $g_{12} = 0$ and $g_{22} = 0$.

From (5), we have that $-1/2 = 0 + 0 + 0 + 0 = 0$, a contradiction. Therefore, this case cannot happen.

Case II: $g_{12} = 0$ and $(h_{22} - h_{12}) = 0$.

From (6), we have that $0 = g_{12}h_{11} + g_{21}h_{12} = 0 + g_{21}h_{12}$; so either $g_{21} = 0$ or $h_{12} = 0$.

Subcase IIa: $g_{12} = 0$, $h_{22} - h_{12} = 0$, $g_{21} = 0$.

From (6), we have that $0 = g_{22}h_{12}$, so either $g_{22} = 0$ or $h_{12} = 0$.

Subcase IIa(i): $g_{12} = 0$, $h_{22} - h_{12} = 0$, $g_{21} = 0$, $g_{22} = 0$.

Substituting into (9), we have that $3/2 = 0 + 0 + 0 + 0 = 0$, a contradiction; so subcase IIa(i) cannot happen.

Subcase IIa(ii): $g_{12} = 0$, $h_{22} - h_{12} = 0$, $g_{21} = 0$, $h_{12} = 0$. It is enough to contradict subcase IIb.

Subcase IIb: $g_{12} = 0$, $h_{22} - h_{12} = 0$, $h_{12} = 0$.

From (4) and (8), we have that $-1 = g_{11}h_{11}$ and $-1 = g_{11}h_{21}$. Since $h_{11} \neq 0$, $g_{11}h_{11} = -1 = g_{11}h_{21}$ or $h_{11} = h_{21}$. Now the problem has $h_{22} - h_{12} = 0$ and $h_{11} - h_{21} = 0$; this is the same as case I of the first argument and hence yields a contradiction.

By the above three arguments, it is always true that $g_{11} \neq 0$, $h_{22} - h_{12} \neq 0$, and $g_{12} \neq 0$ and that (13) is valid.

Since (13) could have been derived from the ratio $\frac{g_{11}}{g_{21}} = \frac{(h_{22} - h_{12})}{(h_{11} - h_{21})} = \frac{g_{12}}{g_{22}}$, it can be argued from symmetry that $g_{21} \neq 0$, $h_{11} - h_{21} \neq 0$, and $g_{22} \neq 0$.

It is restated for emphasis that all the arguments presented in the above were made using the assumption that the system of equations is consistent. It was then proved that $g_{11} \neq 0$, $g_{21} \neq 0$, $g_{12} \neq 0$, $g_{22} \neq 0$

along with the fact that $h_{11} \neq h_{21}$ and $h_{22} \neq h_{12}$.

Since any complex number can be expressed as the product of two complex numbers, since $g_{11} \neq 0$, every other variable in the above system can be represented as some complex constant multiplied by g_{11}.

Let $g_{12} = a\, g_{11}$,

$g_{21} = b\, g_{11}$,

$g_{22} = c\, g_{11}$,

$h_{11} = d\, g_{11}$,

$h_{12} = e\, g_{11}$,

$h_{21} = j\, g_{11}$,

$h_{22} = k\, g_{11}$,

$f_{11} = m\, g_{11}$,

$f_{12} = n\, g_{11}$,

$f_{21} = p\, g_{11}$,

$f_{22} = q\, g_{11}$.

Then substituting this into the system of equations (1) through (13) yields:

(1) $3 = (m + q)g_{11}$

(2) $2 = (mq - np)g_{11}^2$

(3) $-\sqrt{7}/2 = [-dq + en - bem + bdn]g_{11}^3$

(4) $-1 = [d + be]g_{11}^2$

(5) $-1/2 = [-adq + aep - cem + cdn]g_{11}^3$

(6) $0 = [ad + ke]g_{11}^2 \Longrightarrow 0 = ad + ke$

(7) $-\sqrt{7}/2 = [-jq + kp - bkm + bjn]g_{11}^3$

(8) $-1 = [j + bk]g_{11}^2$

(9) $3/2 = [-abq + akp + cjn - ckm]g_{11}^3$

(10) $0 = [aj + ck]g_{11}^2 \Longrightarrow 0 = aj + ck$

(11) $g_{11}^2 (d - j) = g_{11}^2 b (k - e) \Longrightarrow (d - j) = b (k - e)$

(12) $g_{11}^2 e (d - j) = g_{11}^2 c (k - e) \Longrightarrow e(d - J) = b (k - e)$

(13) $g_{11}^2 c = g_{11}^2 ab \Longrightarrow c = ab$

and $a \neq 0$, $b \neq 0$, $c \neq 0$, $d - j \neq 0$, $k - e \neq 0$.

Now (13) implies $c = ab$; and (10) implies that $j = \frac{-ck}{a}$. Substituting for $c = ab$ in $j = \frac{-ck}{a}$ yields $j = \frac{-ck}{a} = \frac{-abk}{a} = -bk$. When this expression for j is substituted in (8) we have $-1 = [j + bk]g_{11}^2 = [-bk + bk]g_{11}^2 = 0$, or $-1 = 0$; this is a contradiction so the system of nonlinear equations must be inconsistent.

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