where μ is some scalar constant (positive or negative), and w and b are independent standard Wiener processes. Also of interest for comparison is the corresponding quadratic cost functional, meaning the expectation of the quantity in square brackets in the exponent of J. This gives the limiting form of the problem (and solution) as $\mu \to 0$ because it is approached in this limit by $2(J - \mu)/\mu^2$, the minimization of which is equivalent to that of J for any $\mu \neq 0$.

The notation here is what is standard for such problems in the usual finite-dimensional real vector-matrix context, and the matrices (always denoted by capital letters) may be time varying when that is meaningful. Also M, $Q(t)$, P_0, $N(t)$, and $R(t)$ are symmetric and positive semidefinite, with the last three positive definite. Complete notational details and some further restrictions are specified in [1], and further explanation is not repeated here.

III. Optimal Control Law Implementations

Bensoussan and van Schuppen [1] have shown under these conditions that if P and S are symmetric solutions to

$$\dot{P} = FP + PF^* + GG^* - P(H^*R^{-1}H - \mu Q)P; \qquad P(0) = P_0 \qquad (1)$$

and

$$\dot{S} = -S(F + \mu PQ) - (F^* + \mu QP)S + S(BN^{-1}B^* - \mu PH^*R^{-1}HP)S - Q;$$

$$S(t_1) = \frac{1}{2}\{[I - \mu MP(t_1)]^{-1}M + M[I - \mu P(t_1)M]^{-1}\} \qquad (2)$$

such that $P^{-1}(t)$ exists for all $t \in T$, then the control law defined by

$$u(t) = -N^{-1}(t)B^*(t)S(t)r(t) \qquad (3)$$

$$dr = [(F + \mu PQ)r + Bu]dt + PH^*R^{-1}(dy - Hrdt); \qquad r(0) = \mu_0 \qquad (4)$$

is optimal (and also conditionally optimal in the sense of Striebel [6]). This control law has a finite-dimensional form, but differs markedly from solutions that had been obtained for $Q = 0$ in that r and P are not, in general, the conditional mean and covariance matrix of the state x.

The purpose of this note is to point out that an extension of Speyer's control-law construction [5] nevertheless has similar properties for nonzero Q under the additional condition that $[I + \mu S(t)P(t)]^{-1}$ exists for all $t \in T$. In this case, one can define

$$W(t) = [I + \mu S(t)P(t)]^{-1}S(t), \qquad (5)$$

for which it follows from the matrix inversion lemma applied to $(I - \mu WP)$ that

$$S(t) = [I - \mu W(t)P(t)]^{-1}W(t). \qquad (6)$$

With the matrix inversion lemma applied to $(I + \mu SP)$, it follows from (5) that

$$W = S - \mu S(P^{-1} + \mu S)^{-1}S$$

and that W is therefore symmetric, and fairly straightforward substitution and matrix manipulation show that (1), (2), and (5) imply

$$\dot{W} = -WF - F^*W - Q + W(BN^{-1}B^* - \mu GG^*)W; \qquad W(t_1) = M. \quad (7)$$

Furthermore, it follows similarly that if W is any symmetric solution to (7) such that $[I - \mu W(t)P(t)]^{-1}$ exists for all $t \in T$, then the function S constructed from it by (6) is a symmetric solution to (2) such that $[I + \mu S(t)P(t)]^{-1}$ exists for all $t \in T$.

IV. Discussion

The significance of these results is that the control $u(t)$ can be constructed from (1), (3), (4), (6), and (7) without using $H(\tau)$ or $R(\tau)$ for any $\tau \in (t, t_1]$, which are needed if $u(t)$ is obtained from (1)–(4). This also means that the optimality of this control is unaffected by any changes in these future values of H or R for which (with the same $P(\tau)$ for $\tau \leq t$) the resulting future values of P are still such that $[I - \mu W(\tau)P(\tau)]^{-1}$ and $P^{-1}(\tau)$ exist for all $\tau \in (t, t_1]$ and the restrictions of [1] still hold. These properties can be important for application because H and R, which are parameters of the measurement process, may in effect be part of the measurement data in the sense that the controller only knows their values with meaningful precision for past times [5].

In addition, the form of this other control law implementation is appealing because the Riccati equations for $\dot{P}$ and $\dot{W}$ are mutually dual, in the sense of Kalman [7], with respect to the problem parameters in the same way that the equations for $\dot{P}$ and $\dot{S}$ are for the corresponding quadratic cost (i.e., in the limiting case of $\mu \to 0$). Unlike this quadratic-cost case, however, the $\dot{P}$ equation here contains the cost parameter Q and the $\dot{W}$ equation contains the process-noise parameter G. Therefore, although the optimal control here is similar to the familiar optimal control for the quadratic cost in being independent of future values of H and R, it is different in depending on future values of G and past values of Q. A further difference is the restriction on the range of future H and R values for which this independence has been established, namely, to those values for which P^{-1} and the inverses in (5) and (6) exist and the other restrictions imposed by Bensoussan and van Schuppen [1] still hold.

If these latter restrictions and the existence of P^{-1} throughout T implied that $(I + \mu SP)^{-1}$ exists throughout T for any symmetric solution S of (2), then any of the optimal control laws established by Bensoussan and van Schuppen [1] could be implemented with (6) and (7) in place of (2). However, it is an open question at this point whether that is true or even whether $(I - \mu WP)^{-1}$ exists throughout T for every symmetric solution W to (7) under these conditions.

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Instability of a Decoupled Kalman Tracking Filter

STEVEN R. ROGERS

Abstract—**The performance of a decoupled Kalman tracking filter is studied by means of a suboptimal covariance analysis. Analytical and numerical results are presented that demonstrate filter instability over a range of target-sensor line-of-sight (LOS) rates. The instability is shown to occur only when the ratio of the decoupled filter gains exceeds a specific threshold value.**

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I. Introduction

In real-time tracking applications, decoupling of the Kalman filter is frequently employed to reduce computational requirements [1]–[3]. For example, a filter for tracking a constant velocity target in three dimensions would normally require six state variables. If the target LOS rate is small with respect to the sensor, it is advantageous to compute the Kalman filter gains in the rotating reference frame attached to the sensor. In this frame, the six-state filter can be approximated by three decoupled alpha–beta filters, one for each of the sensor axes.

Thus far, the performance of decoupled tracking filters has been studied in the literature, primarily by Monte Carlo simulation. The only analytical result obtained [4] has been for a sensor rotating about a fixed target, without process noise. In this case, the rms position error of the decoupled filter approaches zero as the tracking time increases, regardless of the LOS rate.

This note analyzes decoupled tracking of a constant-velocity target with white noise acceleration. For simplicity, a continuous-time formulation of the filter has been assumed.

II. Decoupled Filter Formulation

Consider a constant-velocity target being tracked by a sensor that continuously measures target position along three orthogonal axes. In order to maintain the target within its field of view, the sensor rotates with angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. We shall assume that $\boldsymbol{\omega}$ is roughly constant during the time required by the filter to reach steady state.

The target state in a coordinate frame attached to the sensor will be denoted by $s = (x_1, x_2, x_3, v_1, v_2, v_3)'$ and the measurement vector will be denoted by $y = (y_1, y_2, y_3)'$. The dynamics and measurement equations are given by

$$ds/dt = Fs + Gu \tag{1}$$

$$y = Hs + n$$

$$H = [I \; 0], \; G = [0 \; I]'$$

$$F = \begin{bmatrix} \Omega & I \\ 0 & \Omega \end{bmatrix}, \; \Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$E[u(t_1)u'(t_2)] = Q\delta(t_1 - t_2)$$

$$E[n(t_1)n'(t_2)] = R\delta(t_1 - t_2)$$

$$Q = \text{diag}\,[q_1, q_2, q_3]$$

$$R = \text{diag}\,[r_1, r_2, r_3] \tag{2}$$

where 0 and I are the 3×3 zero and identity matrices, respectively. When $\Omega = 0$, (1) reduces to the usual dynamic equation for straight-line motion in a nonrotating frame of reference.

The tracking filter produces a smoothed estimate $\hat{s}$ of the target state vector as follows:

$$d\hat{s}/dt = [F - KH]\hat{s} + Ky \tag{3}$$

where K is a 6×3 gain matrix. For arbitrary gain K, the covariance matrix P of the filter estimate propagates according to the Joseph equation [5]:

$$dP/dt = [F - KH]P + P[F - KH]' + GQG' + KRK'. \tag{4}$$

The time-varying gains of the decoupled filter are computed by integrating (4) with $K = PH'R^{-1}$ and $\Omega = 0$ in the state transition matrix F. Note that sensor rotation is only neglected in the gain computation—*not* in the state update equation (3). If the initial covariance matrix $P(0)$ is decoupled along the three sensor axes (as it normally is), the time-varying gain $K(t)$ will have the diagonal form

$$K = \begin{bmatrix} A \\ B \end{bmatrix}, \quad \begin{matrix} A = \text{diag}\,[a_1, a_2, a_3] \\ B = \text{diag}\,[b_1, b_2, b_3] \end{matrix}. \tag{5}$$

The six-state filter is thus reduced to three independent alpha–beta filters. The steady-state gains are given by [6]

$$a_i = (4q_i/r_i)^{1/4}$$

$$b_i = a_i^2/2 \qquad (i = 1, 2, 3). \tag{6}$$

III. Suboptimal Covariance Analysis of the Decoupled Filter

Having obtained the suboptimal gain matrix K of the decoupled filter, the performance of the filter is determined by integrating (4) with nonzero Ω. The steady-state covariance is found by setting $dP/dt = 0$ in (4), and solving the linear Lyapunov matrix equation

$$[F - KH]P + P[F - KH]' + GQG' + KRK' = 0 \tag{7}$$

for the symmetric matrix P. The solution will be positive definite if and only if the matrix $[F - KH]$ is stable, e.g., all its eigenvalues have negative real parts. Using the expressions for R, K, and H given above, one finds the characteristic equation

$$\begin{aligned} d(\lambda) &= \det\,[F - KH - \lambda I] \\ &= \det\,[(\lambda I - \Omega)^2 + (\lambda I - \Omega)A + B] = 0. \end{aligned} \tag{8}$$

To simplify further analysis, consider the special case $\boldsymbol{\omega} = (0, 0, \omega)$, which corresponds to rotation in the $x - y$ plane. For this case,

$$d(\lambda) = d_3(\lambda)[(d_1(\lambda) + \omega^2)(d_2(\lambda) + \omega^2) + \omega^2(a_1a_2 - 2(b_1 + b_2))]$$

$$d_i(\lambda) = \lambda^2 + a_i\lambda + b_i \qquad (i = 1, 2, 3). \tag{9}$$

The Routh–Hurwitz criterion (7) applied to $d(\lambda)$ indicates that there is an unstable eigenvalue whenever $d(0) < 0$. Recalling that $b_i = a_i^2/2$, the condition for instability is

$$[4\omega^2 - (a_1 - a_2)^2]^2 < (a_1^2 + a_2^2)(a_1^2 - 4a_1a_2 + a_2^2)$$

which can be simplified to

$$\omega_- < \omega < \omega_+$$

$$2\sqrt{2}\omega_\pm = \sqrt{a_1^2 + a_2^2} \pm \sqrt{a_1^2 - 4a_1a_2 + a_2^2}. \tag{10}$$

Note that there is no region of instability for real values of ω unless the second square root is real. This occurs only when

$$a_{\max}/a_{\min} > 2 + \sqrt{3} \tag{11}$$

where $a_{\max} = \max\,(a_1, a_2)$ and $a_{\min} = \min\,(a_1, a_2)$.

Thus, when the ratio of the decoupled filter gains exceeds about 3.73, there is a range of angular rates (ω) for which the decoupled filter is unstable as shown in Fig. 1. The instability region can be understood physically by examining (10) in the limit $a_{\max} \gg a_{\min}$. In this case, the range of unstable angular rates is approximately

$$a_{\min}/\sqrt{2} < \omega < a_{\max}/\sqrt{2}. \tag{12}$$

That is, instability occurs when the LOS rate "resonates" between the minimum and maximum filter gains (divided by $\sqrt{2}$).

Numerical solutions of the Lyapunov equation are shown in Fig. 2 for the case $q_1 = q_2 = 1$, $r_1 = 0.05$, and three values of r_2: 8, 10, and 20. The numerical solution was obtained by directly solving (7) as a system of 21 linear algebraic equations for the elements of P [8]. Fig. 2 shows the steady-state rms position error in the horizontal plane $\sqrt{P_{11} + P_{22}}$. For the case $r_2 = 8$, the steady-state gain ratio is $a_1/a_2 = 3.56$, and the decoupled filter is stable for all (ω), as predicted by (11). For the cases $r_2 = 10$ and $r_2 = 20$, the corresponding gain ratios are $a_1/a_2 = 3.76$ and $a_1/a_2 = 4.47$, both of which exceed the threshold for instability. The unstable regions occur at precisely those values of (ω) given by (10). The rms velocity error exhibits the same qualitative behavior as that shown in Fig. 2.

One approach to removing instability of the decoupled filter at nonzero LOS rates is simply to modify the filter gains in accordance with (11). For example, one might choose to leave the minimum gains unchanged and to

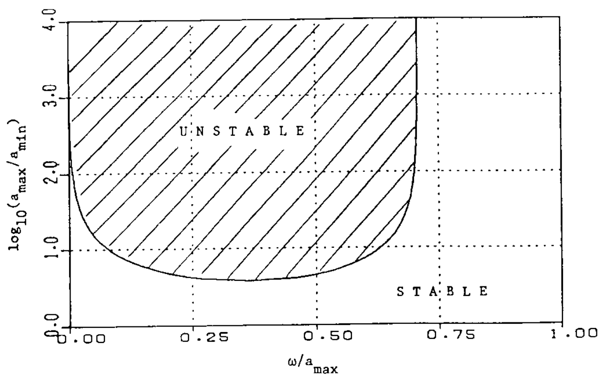

Fig. 1. Region of instability.

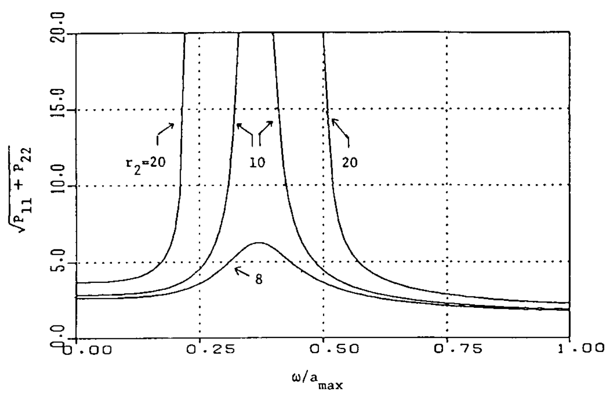

Fig. 2. rms position error in horizontal plane for three values of $r_2(q_1 = q_2 = 1, r_1 = 0.05)$.

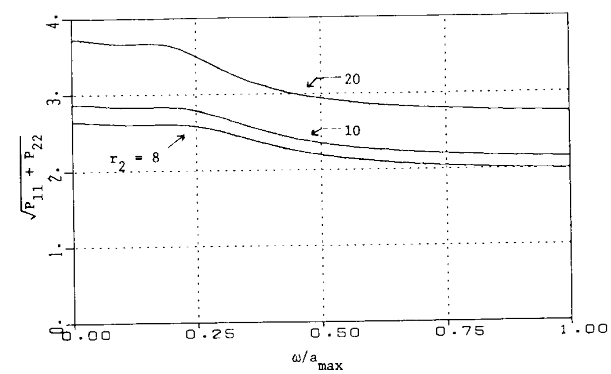

Fig. 3. rms position error in horizontal plane for three values of $r_2(q_1 = q_2 = 1, r_1 = 0.05)$ and *modified* filter gains.

replace the maximum gains by

$$a''_{max} = \min [a_{max}, \gamma a_{min}]$$

$$b''_{max} = a''^2_{max}/2 \tag{13}$$

where, for stability, the constant γ must be in the range $1 \le \gamma < 2 + \sqrt{3}$. Within this interval, smaller values of γ provide a greater "margin of stability." Fig. 3 shows the performance of the decoupled filter with the modified gains of (13) for the case $\gamma = 2$. The values of the process and measurement noise variances are the same as in Fig. 2. To be sure, there is a slight loss in position accuracy (about 1 percent) at $\omega = 0$ where the original (unmodified) filter gains are optimal. However, this slight loss in accuracy is more than offset by the improved performance for nonzero ω.

IV. Conclusion

In designing a tracking filter, it is important that regions of instability be avoided. The above analysis for a continuous, constant-velocity tracker indicates that decoupling leads to instability whenever 1) the ratio of decoupled filter gains exceeds a certain threshold, and 2) the angular rate of the sensor falls in a specific range of values. Similar results have been obtained for a discrete decoupled filter [9].

The analytical derivation presumes that ω is constant in time. In a realistic situation, the LOS rate of a crossing target reaches its maximum value at the point of closest approach to the sensor, and then decreases to zero as the target recedes. In such a case, filter instability will appear as a large, but not unbounded, increase in the rms tracking error at close range.

The analysis also suggests methods to avoid instability in a given application. By modifying the filter gains, in accordance with (11), one can force the decoupled filter to be stable for *all* (ω). The penalty for modifying the gains is usually a slight degradation in tracking accuracy when $\omega = 0$. In most practical cases, however, the loss in accuracy is more than offset by the guarantee of filter stability at nonzero LOS rates.

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A Tracking Filter for Maneuvering Sources

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***Abstract*—An adaptive filter, that is based on the Pontryagin minimum principle and the method of invariant imbedding, is introduced and applied to the problem of tracking maneuvering sources. The major advantages of this adaptive filter are: 1) no model for the target dynamics**

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