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Thomas H. Kerr

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# Status of CR-Like Lower Bounds for Nonlinear Filtering

THOMAS H. KERR, Senior Member, IEEE
Lincoln Laboratory

**The motivation and mechanics of utilizing Cramer-Rao (CR)-type lower bounds are reviewed as used to gauge the performance of filters being evaluated in nonlinear estimation applications such as in sonar/sonobuoy/radar target tracking. The status of several similar alternative CR-type lower bounds that have been considered or used for this purpose are reviewed and certain limitations/caveats associated with their use are offered. These results should be of interest to sonar/sonobuoy/radar practitioners and Kalman filter or nonlinear filter theorists.**

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Author's address: M.I.T. Lincoln Laboratory, 244 Wood St., Lexington, MA 02173.

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## I. INTRODUCTION

The topic of Cramer-Rao (CR) lower bounds for nonlinear filtering applications is treated here as being timely since the subject was raised in [1, 2, 26–32, 42, 47] and continues to be utilized in applications (e.g., [28, 29, 32]). In [47, p. 98], the trajectory estimation problem is identified as being one of nonlinear estimation or nonlinear filtering as is fairly well known. It is acknowledged in [47, p. 98], that completely rigorous treatment of nonlinear filtering for completely general nonlinear state-variable representations of systems and measurement structures (actually only rigorously described by integral equations but symbolically denoted differentially by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\mathbf{w},\mathbf{t}) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x},\mathbf{v},t) \end{aligned} \tag{1}$$

(where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are zero-mean independent "derivatives" of Brownian noises, independent of the initial condition on $\mathbf{x}$, and $\mathbf{x}$ and $\mathbf{y}$ represent the system state and corresponding measurement sensor output, respectively, in a continuous-time representation where $\dot{\mathbf{x}}$ denotes differentiation of $\mathbf{x}$ with respect to time, $t$)) would involve manipulations of stochastic integrals (of the type known as either Ito, Stratonovich, or, as added here, of McShane [71] as a unifying simplification [70]). However, for the more restrictive but fairly prevalent special case of nonlinear systems of the form of

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},t) + B(t)\mathbf{w}(t) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x},t) + N(t)\mathbf{v}(t) \end{aligned} \tag{2}$$

where *only* additive noises are present and $\mathbf{B}(t)$ and $\mathbf{N}(t)$ are specified weighting matrices, it frequently suffices to merely engage in "formal" (i.e., nonrigorous) manipulations of white noise processes as a convenient mathematical fiction that avoids the unwieldy baggage of stochastic integrals as being "trees that might otherwise obscure the view of the forest". This simplifying philosophy is also subscribed to here and has been justified to an extent (with the notable exception of [91], which observes that the continuous-time formulation of an extended Kalman filter (EKF) involves a differential equation that should be interpreted as an "Ito integral" and as a consequence must include an additional term beyond what is usually prescribed for an EKF formulation) through use of mean-square calculus and the associated weaker convergence arguments ([39, 55]) than would be available if measure-theoretic and "martingale" arguments associated with rigorous handling of stochastic integrals were invoked. (Also see [63, 64, 66, 67, 69, and 86] for three other alternative approaches to handling stochastic integrals.)

The use of a CR lower bound methodology is considered here as a gauge of the quality of tracking filter performance. A high-level overview of the CR lower bound methodology and its benefits and limitations are as follows.

The statistically rigorous CR inequality [54, pp. 154–155] provides a lower bound on the mean-square estimation error achievable. No matter what estimator is ultimately selected, none can do better than what the rock-bottom nonnegative CR lower bound indicates (where such a bound exists, as is routinely available in many cases of practical interest [43]). Whether the bound is tight or not is another question that depends on problem structure for an answer [73]. Just as standard approaches to quantifying the performance of linear Kalman filters are susceptible to inadvertent or intentional modeling errors (e.g., use of over simplified "truth" models or the omitting of relevant states or instances of inadequate parameter selection), CR bounding techniques (as well as any other simulation approach) are susceptible to such oversights. In such a case, the lower bound could be either too high or too low.

In a particular application, the structural form of the CR lower bound is first obtained analytically, then explicitly evaluated using computational techniques that are usually implemented on a computer.

The simulation performance of any estimator selected for the particular application can be "gauged for goodness" as ascertained by its relative proximity to the CR lower bound. Alternate estimator designs can be traded off by proximity to the CR bound (as the best that can be done) versus computational burden associated with implementation. It is precisely this aspect of CR lower bounds that is of interest in most nonlinear filtering applications. Additionally, Monte-Carlo evaluation of estimator performance is required for the subsequent comparison with the CR lower bound for tightness of proximity as a "gauge of goodness."

The CR lower bound is of particular interest when the physics or geometric structure underlying a particular parameter identification problem prohibit estimation to the degree of accuracy sought or specified. That such an unfortunate circumstance is present in a particular application is reflected by the corresponding increase in size of the CR lower bound to reflect this fundamental decrease in the absolute accuracy achieveable. With such an obvious indication, the estimator is therefore not faulted for lack of estimation accuracy which is beyond its control. Otherwise, with no CR bound indication of a comprising situation being present, the estimator would probably be blamed. The utility of CR bounds as situation-dependent gauges for setting realistic estimation algorithm performance specifications or "fair" goals should be obvious.

This important topic of CR lower bounds is discussed here to offer critical comments and less well-known caveats on proper use/calculation of CR bounds in evaluating nonlinear filter performance that should be of interest to sonar/acoustic applications specialists and practitioners handling this evaluation aspect. This critique of CR bounding techniques is being undertaken here only because of the author's extensive long-term familiarity with this area of evolving technology (viz., [5], also see [2, 6, 7, Acknowledgments].

Unlike the relatively pleasant situation for both optimal Kalman filters and reduced-order Kalman filters where well-known rigorous techniques exist for evaluating the filter performance for applications involving only linear system and measurement models [8–10], there is currently no convenient single all-embracing method for evaluating the covariance of estimation error a priori for general nonlinear estimation applications. There are a few exceptions, a few tractable special case, degenerate, nonlinear filtering applications such as [11], with updated corrections as [12] and certain nonlinear systems of fairly restrictive special structure as identified in [13–17, 61, 62, and 68]. These have extensions and hopes for wider applicability via a contorted form of linearization in [18], as compared with how difficult the general nonlinear filtering problem is as explained in [25] and with a novel approximate finite-dimensional methodology offered in [53, 62, 78] with supporting niceties in [84], and a nice adaptive EKF in [36]). In the linear case, the associated Riccati equation can be solved explicitly off-line a priori for the covariance of mean-square estimation error. In the general nonlinear case, the differential or difference equations that describe the estimation error are inextricably mixed with higher order moments that in turn satisfy differential or difference equations involving even higher order moments [19]. Only fairly drastic truncation of higher moments or less severe truncation of cumulants (semiinvariants) [20, pp. 7–10] allows a simultaneous solution of the coupled set of matrix differential equations as an approximation to the actual error in estimation. However, the associated expense of the indicated large scale computations may preclude completion of such a horrendous evaluation task; thus, there is great motivation for a tractable alternative. One such prospective alternative is to employ the CR lower bound as an evaluation gauge of goodness of the results obtained from a likely suboptimal nonlinear filtering implementation as would be tested against through extensive realistic Monte-Carlo simulations.

The familiar fundamental scalar version of the CR lower bound inequality associated with the estimation or identification of an unknown parameter $x$ is [21, The use of a CR lower bound methodology is considered here as a gauge of the quality of tracking filter performance. A high-level overview of the CR lower bound methodology and its benefits and limitations are as follows.

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The familiar fundamental scalar version of the CR lower bound inequality associated with the estimation or identification of an unknown parameter $x$ is [21, p. 248]

$$E[(x-\hat{x})^2 \mid x]] \geq \frac{\left[\dfrac{\partial \Psi(x)}{\partial x}\right]^2}{E\left\{\left[\dfrac{\partial}{\partial x}\ln p(y \mid x)\right]^2\right\}} \tag{3a}$$

where the random variable $\hat{x}$ is an estimator of the parameter $x$ and the measurement $y(t)$ is a function of the unknown parameter $x$ as

$$y(t) = g(x,t,v(t)) \tag{4}$$

and

$$\Psi(x) \triangleq E[\hat{x} \mid x] = x + \phi(x) \tag{5}$$

where $\phi(x)$ is the bias in the particular estimator $\hat{x}$, $p(x \mid y)$ is the conditional probability density function of $x$ given $y$ and $v(t)$ is the measurement noise. The original CR formulation of (3a) is emphasized here as a reminder that many recent generalizations of the CR lower bound focus attention only on the denominator term of the right side of (3a) while totally ignoring the significant effect of the important numerator term (cf., [22, 42, 47]). The correct multidimensional generalization of (3a) (from M.I.T. lecture notes derived by Dr. Nils Sandell) should be

$$E[(x-\hat{x})(x-\hat{x})^T \mid x] \geq \left[I + \frac{\partial}{\partial x}\phi\right] \times \left[-E\left\{\left(\frac{\partial}{\partial x}\right)^T\left(\frac{\partial}{\partial x}\right)\ln p(y \mid x)\right\}\right]^{-1} \times \left(I + \frac{\partial}{\partial x}\phi\right)^T. \tag{3b}$$

On the question of tightness of CR-type bounds, few estimators can actually achieve the CR lower bound [73]. If a maximum likelihood estimator (MLE) exists for the particular problem, it is theoretically guaranteed to achieve its CR lower bound only asymptotically (as the amount of measured data collected and processed increases without limitation). Unfortunately, MLEs are usually only asymptotically unbiased estimators. However, for some particularly nice problem structures where the unknown parameter being sought enters the measurements linearly and the noise is additive and Gaussian (as identified in [46, 5, p. 99, 21, p. 252, 45, eq. (1)), the CR bound is achieved by MLE in processing each finite length segment of data, while the magnitude of the CR lower bound associated with these nice problems just decreases as the length of data record processed increases (cf., [33]). This well-known conclusion compares favorably with [33] which also claims achieving CR bounds for finite data length and has the requisite linear structure, as described above. Also see [92], [93] for follow-up.

Several different generalizations of the above classic CR bound exist for the multiparameter estimation/identification case (e.g., [5, 22, pp. 63–85, 33, 43]). Some early CR bound evaluation approaches utilized "extraneous" parameters, or "nuisance" parameters, or fiddle factors that were obviously perceived by many to be somewhat unsavory subjective "tuning factors" (or "fudge factors") and as such were controversial. Kullback's information-theoretic approach to statistical inference by utilizing the "discrimination information number" has also been interpreted (see [75]) to be a generalization of the CR inequality. Moreover, the interrelationship of CR bounds to other lower bounds such as those of Zacks and Barankin have in fact been unraveled [23] and a useful physical interpretation of the CR bound has been provided [24] as a structural sensitivity measure of an associated likelihood function to changes in the underlying parameters to be estimated. The relevance of such likelihood functions to the target tracking bearing estimation problem is exemplified in [28]. The constant value on the right side of (3a) is known explicitly as the CR lower bound and, as such, serves as a conservative bound on the mean-square estimation error that is incurred no matter what type of estimator $\hat{x}$ is used, as reflected on the left-hand side of (3a) and in a possible bias effect term in the numerator on the right-hand side.

The usual well-established statistical-based CR bound of (3a), as used for merely bounding the goodness of estimating or identifying a constant (but unknown) parameter or in estimating constant realizations of a random vector at a specified time as the parameters to be determined [22], has recently been extended by a significant leap to now bound the goodness of state estimates as time evolves. It is this fairly recent extension that is perceived here to be somewhat controversial. However, even with this tremendous jump in the asserted range of applicability, the same name as being a CR bound was ultimately retained (although a variation in its name termed a Bobrovsky-Zakai bound [1] was in vogue for awhile as a distinction). The ultimate retention of the name as a CR bound, although perhaps inadvertent, serves to bolster confidence (perhaps unjustified) in its use in this new area of applicability by serving to remind others of ties to the more familiar well-established completely rigorous CR-bound for the constant and random parameter cases. Since some type of performance evaluation methodology was sorely needed for nonlinear filters, the sonar/acoustic application area was quick to adopt use of the novel CR bounds for nonlinear estimation problems (e.g., [27–29] with clarifications on the tightness of these bounds being offered in [31, 42] as obtained by exploiting specific special case nonlinear filters having an exact finite-dimensional realization) and almost no investigation of recent vintage is without a CR evaluation of this new type (e.g., [32]). Less well-known quirks, softspots, and caveats in the use of the two new CR lower bounds [1, 2] (as acknowledged to be utilized in [28 and 29], respectively), are discussed in Section III. The indicated impact of this entire subject on sonar/sonobuoy applications is offered in Section IV. Evidently, radar practitioners who sometimes encounter similar nonlinear filter formulations have not yet succumbed to the lure of this type of CR evaluation (e.g., [34, 35]) to the same degree or along the same lines as pursued by the sonar/sonobuoy practitioners.

## II. REPRESENTATIVE APPLICATION EXAMPLES THAT UTILIZE THESE NEW CR BOUNDING TECHNIQUES

An approach is provided in [28] to the delay estimation of a noise-like random signal as observed at two or more spatially separated receivers that funnel their information to a centralized facility to process for subsequent target detection and target tracking using an EKF. The methodology employed in [28] is to extend the bound of [1] and provide estimation performance evaluations as a function of the signal-to-noise ratio (SNR) range anticipated for the particular application. The bound of [1] is extended in [28] in a fashion similar to how it was handled earlier in [30], (although unacknowledged in [28], where they start the derivation from scratch again). Somewhat startling was the fact that the now well-known close interrelationship between Barankin and CR bounds [23, 60, 77] was not invoked in [28], but plotted separately for the same span of magnitudes of SNR being considered.

A milestone discussion [29] of passive bearings-only target tracking applications (involving a ship-borne towed array) makes several significant contributions by providing an intuitive physical (and rigorous mathematical) understanding of the following.

1) The EKF (used as a practical approximation to a true nonlinear filter for this nonlinear application) is adversely affected unless an appropriate "modified polar" coordinate system (described below) is adopted as the convention to use; otherwise "filter divergence" (i.e., a disparity between the accuracy assessment of the estimate and its true accuracy) inevitably occurs.

2) Selection of the most appropriate states to be used in the EKF was eventually narrowed down to the preferred choice of:

$\beta$ = bearing of target
$\dot{\beta}$ = bearing-rate of target
$\frac{\dot{r}}{r}$ = range-rate of target divided by the range-to-target
$\frac{1}{r}$ = reciprocal of the range-to-target.

3) Explicit maneuvers of the friendly tow ship are required to enhance the "observability" of the otherwise obscured or inaccessible $1/r$ state to thereby facilitate more accurate target estimation. (For further insights into the observability availed in this type of application and how to handle it to an advantage see [3, 4, 34, 36, 37].)

As in most recent applications of Kalman filtering techniques (e.g., [27, 28]) to the fundamental nonlinear geometry of several different sonar applications, [29] proceeds to evaluate the realistic simulation performance of this approximate implementation of a nonlinear filter as gauged against the idealized optimal performance availed by the CR lower bound of [2]. It is claimed in [2] that this exact lower bound is provided from the EKF covariance propagation equations when linearized about the true trajectory (that is in fact unknown to the application filter, which must instead obtain an approximation to the requisite information by linearizing about its estimate of the true trajectory). However, no system process noise is allowable in this CR bound formulation of [2]. Practitioners rarely elaborate on how they were able to apply the bound of [2] to their applications involving process noise. More is said in Section III about limitations encountered in CR lower bound formulations of [1] and [2] as they are currently being used for nonlinear estimation performance evaluation in sonar/sonobuoy applications.

## III. QUESTIONABLE ASPECTS OF SOME CONVENTIONAL APPROACHES USED TO OBTAIN LOWER BOUNDS FOR NONLINEAR FILTERING PERFORMANCE

A promising approach is suggested in [1] for lower bounding the best possible estimation error to be incurred during nonlinear estimation by comparison in a CR lower bound inequality of the form

$$E[(x_i - \hat{x}_i(y))(x_i - \hat{x}_i(y))] \geq [J_T^{-1}]_{ii} \qquad (6)$$

where $p(x,y)$ is the joint probability density function (pdf) of vector random variables $x$ and $y$, and

$$[J_T]_{ij} = E\left\{\frac{\partial}{\partial_{x_i}} \ln p(x,y) \frac{\partial}{\partial_{x_j}} \ln p(x,y)\right\} \qquad (7a)$$

$$= -E\left\{\frac{\partial^2}{\partial_{x_i}\partial_{x_j}} \ln p(x,y)\right\}. \qquad (7b)$$

A CR lower bound of this form was already well known to be useful in parameter identification [21, 22, p. 72, 5] and depends only on system structure and on SNR present. Reference [1] considers both a discrete-time nonlinear system of the form of

$$x_{n+1} = \mu_n(x_n) + \beta\gamma_{n+1}, \qquad 0 \leq n \leq N-1 \qquad (8)$$

$$y_n = g_n(x_n) + \sqrt{N_0}\xi_n, \qquad 0 \leq n \leq N \qquad (9)$$

the two new CR lower bounds [1, 2] (as acknowledged to be utilized in [28 and 29], respectively), are discussed in Section III. The indicated impact of this entire subject on sonar/sonobuoy applications is offered in Section IV. Evidently, radar practitioners who sometimes encounter similar nonlinear filter formulations have not yet succumbed to the lure of this type of CR evaluation (e.g., [34, 35]) to the same degree or along the same lines as pursued by the sonar/sonobuoy practitioners.

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(a special case of (2) in Section I) and the following associated Gaussian linear system:

$$z_{n+1} = a_n z_n + \beta\gamma_{n+1}, \qquad 0 \leq n \leq N-1 \tag{10}$$

$$\tilde{y}_n = h_n z_n + \sqrt{N_0}\xi_n, \qquad 0 \leq n \leq N \tag{11}$$

where $\gamma_n \stackrel{\Delta}{=}$ zero-mean Gaussian white unit covariance process noise, but effective system process noise covariance in (10) is $\beta^2$, $\xi_n \stackrel{\Delta}{=}$ zero-mean Gaussian white unit covariance measurement noise, but effective measurement noise covariance in (11) is $N_0$, and

$$a_n \stackrel{\Delta}{=} \overline{\dot{\mu}_n(x_n)} \tag{12}$$

$$h_n^2 \stackrel{\Delta}{=} \frac{N_0}{\beta^2}\overline{[\dot{\mu}_n(x_n) - a_n]^2} + \overline{[\dot{g}_n(x_n)]^2} \tag{13}$$

where the vinculum denotes expectation and the following notation is used for convenience:

$$\dot{\mu}_n(x_n) \stackrel{\Delta}{=} \frac{\partial}{\partial u}\mu_n(u)\bigg|_{u=x_n}; \qquad \dot{g}_n(x_n) \stackrel{\Delta}{=} \frac{\partial}{\partial u}g_n(u)\bigg|_{u=x_n} \tag{14}$$

and similarly for higher derivatives.

Reference [1] proceeds to "prove" in its Theorem 1 that the linear Gaussian construction of (10)–(14) above provides a pdf, $p_g(x,y)$, that is "equivalent" to the pdf of the original system of (8), (9). This equivalence that is sought is denoted in [1] by

$$p_g(x,y) \sim p(x,y) \tag{15}$$

where the subscript $g$ signifies that the particular pdf is associated with the Gaussian linear system and where the $N+1$ term sequence $\{x_n\}_{n=0}^N$ is denoted by the augmented vector $x$ (and similarly for $y$ and $z$). Unfortunately, the concept of equivalence of pdfs that is invoked was not clarified in [1] (mutual absolute continuity of both probability measures being one possible but rather stringent interpretation that would allow a conclusion favoring the existence of pdfs as measure-theoretic Radon-Nikodym derivatives [38]). Unfortunately, there appear to be a few typos within the steps of the proof of [1] that if left unresolved or uncorrected appear to be logical contradictions or incompatibilities. These minor problems are now discussed. Within [1, eq. (8)] for $i = j = N$, the following should appear

$$\frac{\partial^2}{\partial x_N \partial x_N} \ln \frac{p(x,y)}{p_g(x,y)} = \frac{\sqrt{N_0}\xi_N \ddot{g}_N(x_N) - [\dot{g}_N(x_N)]^2 + h_N^2}{N_0}. \tag{16}$$

However, [17, eq. (8)] is missing the numerator factor $\sqrt{N_0}$ as a minor oversight.

According to the directions prescribed prior to [1, eq. (9)], taking expectations of (16) with respect to $p(x,y)$ provides the objective that the evaluation of $J_T$ as required to establish that $J$, as defined in the Lemma on [1, p. 786] is nonnegative definite. Moreover, as stated in [1], "since $x$ is independent of $\lambda_{i+1}$ or $\zeta_i$, the terms containing second derivatives vanish," thus performing the indicated expectation of (16) with respect to $p(x,y)$ yields [1, eq. (9c)]

$$[J]_{NN} = -E\left\{\frac{\partial}{\partial x_N}\frac{\partial}{\partial x_N} \ln \frac{p(x,y)}{p_g(x,y)}\right\} \tag{17a}$$

$$= \frac{0 - \overline{[\dot{g}_N(x_N)]^2} + h_N^2}{N_0}. \tag{17b}$$

Upon substituting the definition of the deterministic $h_N^2$ from (13) into (17b) yields

$$[J]_{NN} = \frac{-\overline{[\dot{g}_N]^2} + \frac{N_0}{\beta^2}\overline{[\dot{\mu}_N(x_N) - a_N]^2} + \overline{[\dot{g}_N]^2}}{N_0} \tag{18a}$$

$$= \frac{1}{\beta^2}\overline{[\dot{\mu}_N(x_N) - a_N]^2} \geq 0. \tag{18b}$$

Note that (18b), derived here by adhering exactly to the indicated steps of [1], differs from the final result of [1, unnumbered equation following eq. (9c)]

$$[J]_{NN} = \frac{N_0}{\beta^2}\overline{[\dot{\mu}_N(x_N) - a_N]^2} \geq 0 \tag{19}$$

by only the absence of the factor $N_0$ that appears in [1] as an apparent typo; however, the requisite nonnegative definiteness is established, nonetheless.

The seemingly slight incongruity of having only a missing factor between (18b) and (19) would ordinarily be considered of not much consequence except that it is precisely this presence (herein noted as unexplained) of the factor $N_0$ in the lower bound that is used in [1] to conclude general tightness of the lower bound. In [1, Section V], a scalar special case is investigated and the behavior as the controversial parameter $N_0 \to 0$ in the numerator is noted to be better (tighter) than an earlier bound for the same special case. This argument is apparently adversely affected by the correct result of (18b), which is devoid of any $N_0$ term.

On the other hand, if there were no term $N_0$ present in the denominator of (17b), then there would be no incongruity between the result of (18b) and the result of [1] (repeated here as (19)) because then the numerator $N_0$ could not divide out with a nonexistent term of $N_0$ in the denominator. However, the form of $p(y_n \mid x_n)$ as utilized in [1] and depicted following [1, eq. (7)] as

$$p(y_n \mid x_n) = \frac{1}{\sqrt{2\pi N_0}} \exp\left\{-\frac{[y_n - g_n(x_n)]^2}{2N_0}\right\} \tag{20}$$

does in fact clearly show the proper effect of the presence of the measurement noise as the term $N_0$, appearing in the denominator of the exponent. Once logarithms are taken, (20) is consistent with (16), clearly indicating that a term $N_0$ should rightly appear in the denominator of (17b) and that (18b) offered herein is the correct expression (and therefore (19) is incorrect). This same error of [1] persists unchanged in [85, paragraph prior to Appendix].

The symbol $J$ in (19) (and in [1]) is well known by somewhat universal convention to be the Fisher information matrix. However, just obtaining the correct form of the expression and establishing that it is nonnegative definite is not enough to guarantee that it can be inverted as needed in the CR lower bound comparisons of (6) (corresponding to [1, eq. (2)]. The utility of the nonnegative definiteness result of (18b) (as stated in [1, Theorem 1] and utilized within the Lemma on [1, p. 786]) is that attention can be focused on tightness comparisons on an equivalent inequality between mean-squre estimation errors $P_g$, for the linear Gaussian system of (10), (11) and the mean-square estimation errors $P$ in nonlinear estimation for the system of (8), (9), respectively, as

$$P_g \leq P$$

(as also quoted in [1, Theorem 1].

Apparently there is a lack of a precise definition of "equivalence of pdfs" in [1] and some contrary physically motivated intuition [25, 65, 72] that it is not very likely that a linear Gaussian dynamical system can provide a pdf that is equivalent (in any meaningful sense since multimodal pdfs can occur in arbitrary nonlinear systems and outputs can have nonzero means although inputs had zero means while the linear system will always yield a unimodal Gaussian of zero mean)[1] to that emanating from a fairly arbitrary general nonlinear system of the form of (8) and (9). A later attempt in [85] to be more precise in the definition of equivalence as being "mutually absolutely continuous" [85, following eq. (11)] still does not explicitly explain how it is established for the two measures under scrutiny but instead cites two 1960 papers by J. V. Girsanov and A. V. Skorokhod. Despite this unpleasant state of affairs, other researchers [27, 30, 42] have continued to work with CR lower bounds for nonlinear filtering applications to get tighter lower bounds than were offered in [1]. Apparently progress is being made in the discrete-time multidimensional formulation [30], where a linear system is specified such that its associated Fisher information matrix $J_L$ is greater (in the matrix positive definite sense) than $J_{NL}$ (the Fisher information matrix associated with the general nonlinear system). This relationship is represented by [30, eq. (6b)]

$$J_L \geq J_{NL}. \tag{21}$$

However, the target objective of a comparison that is being sought [30, eq. (3)] with the covariance of nonlinear filtering error $P_{NL}$ (in analogy to (6)) is

$$P_{NL} \geq [0 \mid I_n] J_{NL}^{-1} \left[\frac{0}{I_n}\right]. \tag{22}$$

Therefore, use of $J_L > 0$ in a comparison with $P_{NL}$ would have the following ordering:

$$P_{NL} \geq [0 \mid I_n] J_{NL}^{-1} \left[\frac{0}{I_n}\right] > [0 \mid I_n] J_L^{-1} \left[\frac{0}{I_n}\right]. \tag{23}$$

As noted [30, prior to eq. (5)], $J_{NL}$ may perhaps not be invertible in practice, giving rise to only the following comparison being tractably available

$$P_{NL} \geq [0 \mid I_n] J_L^{-1} \left[\frac{0}{I_n}\right] > 0. \tag{24}$$

It is speculated in [30] that nonlinear estimator design could eventually be performed by ignoring inversion entirely and just comparing the resulting $J$ for tightness to the easily accessible $J_L$ as

$$J \geq J_L. \tag{25}$$

Unfortunately, usually only the $P$s are readily available from Monte-Carlo simulations unless an information filter form is implemented.

On the other hand, [2] offers a completely different approach to the problem of specifying a CR lower bound for nonlinear estimation by considering a continuous-discrete system without process noise of the following form

$$\dot{x} = f(x,u,t) \tag{26}$$

$$z_k = h(x_{k,k}) + v_k \tag{27}$$

where $u(t)$ is the deterministic control, and $v_k$ is the zero-mean white Gaussian noise of covariance level $R_k$. However, the following two main results appeared previously as [39, eq. (7.43) and (7.49)] (combined with [39, eq. (7.50)]), respectively. These two results were derived as [2, eq. (13)] and [2, eq. (14)], respectively,

$$J_k = (\phi_{k-1}^{-1})^T J_{k-1} (\phi_{k-1}^{-1}) + H_k^T R_k^{-1} H_k \tag{28}$$

and

$$(P_k^*)^{-1} = (\phi_{k-1} P_{k-1}^* \phi_{k-1}^T)^{-1} + H_k^T R_k^{-1} H_k \tag{29}$$

where $\phi_k$ is the transition matrix associated with the linearization of $f(x,u,t)$ about the state $x$ and deterministic control $u$, as a corresponding Jacobian matrix evaluated along the true trajectory (known in

[1]In this same vein, it is well known (via R. S. Lipster and A. N. Shiryayev) that weak solutions of stochastic differential equations with a unit diffusion coefficient and a square integrable drift coefficient are the only processes the measure of which is absolutely continuous with respect to the Wiener measure [72].

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simulations). Furthermore, the device following [2, eq. (4)] of having an additional fictitious measurement[2]

$$z_0 = x_0 + v_0, \qquad v_0 \sim N(0, S_0) \tag{30}$$

is contrary to the formal definition of a dynamical system [40, Section 4–5] being required to have a measurement dimension that is time invariant. (At issue here is not a time-varying observation matrix but a time-varying dimension of the measurement noise and its associated covariance matrix, which of necessity is required to be nonsingular.) Use of this fictitious measurement has the adverse effect of possibly introducing a subjective or arbitrary bias in the calculation of $J_K$ using [2, eq. (9)] as

$$J_K = \left(\frac{\partial x_0}{\partial x_K}\right)^T S_0^{-1} \left(\frac{\partial x_0}{\partial x_K}\right) + \sum_{k=0}^{K} \left(\frac{\partial x_0}{\partial x_K}\right)^T H_k^T R_k^{-1} H_k \left(\frac{\partial x_0}{\partial x_K}\right). \tag{31}$$

(Equations (28) and (29) are just convenient recursive restatements of (31).) The first term in (31) contains the effect of this fictitious measurement and can be potentially finagled to be arbitrarily large. Thus, in situations corresponding to identification where the Fisher information matrix would ordinarily be singular (e.g., [42, Section VI, Part A, 43, 44]), the first term in (31) could be artificially induced to reflect a more benign structure than actually exists.

Snyder and Rhodes [58] offer a CR bound for the case of the system of (8) being completely linear in $x$ (i.e., $\mu_n(X_n) = F(n)X_n$) and for this special case, the results of [1] and [58] coincide [56, p. 453]. Moreover, the finite-dimensional nonlinear filter utilized in tightness arguments between alternative bounds in [42] corresponds to the following bilinear system with nilpotent Lie algebra of the following form as treated in [14].

System:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sqrt{q} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{32}$$

Measurements:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sqrt{r} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{33}$$

Additional nonlinearity:

$$\dot{y} = -\gamma y + x_1 \cdot x_2. \tag{34}$$

It is noted here that the above system of (32), (33) is linear and observable and the associated underlying pdfs are Gaussian and the conditional expectation as an optimal estimator of the state can be extracted from a standard linear Kalman filter. The effect of the additional output nonlinearity of (34) is essentially a nonlinear transformation after the fact, that is much easier to handle in forming an estimate of the noiseless $y(t)$ than could be done in a general nonlinear filtering problem [85]. It is claimed here that even this special case is especially nice and degenerate as being unrealistically easy to handle. Even when the measurements are nonlinear but the underlying system is linear, the situation is much more tractable (viz., observability can be successfully handled [81]) than in the general nonlinear case.

The Zakai-Ziv bounds of [57] are based on rate distortion theory and were applicable to autonomous nonlinear systems (i.e., nonlinearity cannot be a function of time) with additive white Gaussian system and measurement noise. The results of Galdos [56] also use rate distortion theory to extend the applicability of the results to a more general nonlinear filtering context. Comparisons are made in [56] of this bound to other "conventional" CR-based lower bounds for nonlinear filtering. However, it is the contention in [75], that Kullback's rate distortion theory itself and the discrimination information number in particular as a basis for statistical hypothesis testing harken back to yet another generalization of CR lower bounds.

The tightness of another CR-based lower bound of Galdos [30] is claimed in [30, paragraph prior to Section III] to be tighter than the CR-bound of [1], but this has been disputed in the evaluations of [31] by the thesis student of one of the authors of [1]. However, conclusions of relative tightness of alternative CR bounds based on evaluations of a particular tractable finite-dimensional nonlinear filter special case (as pursued in both [42] and [31] with different special cases and different rankings resulting) should probably be viewed as somewhat suspect just as was the case in early comparisons of alternative optimization approaches and alternative parameter identification approaches 20 years ago. Specific test problems or examples can be selected to "stack the deck" and make a particular approach "win". Unfortunately, the "winning" could be highly test-case dependent.

Another worry concerning the bound of [30] is the assumption that the dimension of the system and measurements are identical. While a contrived method is offered in [30] for creating this situation for any general problem, the resulting system could have a singular observation noise covariance matrix which reference [30] requires to be nonsingular and attempts to argue away any occurrence to the contrary.

Other alternative CR bounds of note are those of Rhodes and Gilman [59] that are applicable to nonlinear filtering for systems whose dynamics and measurement models have fairly mild nonlinearities

[2]Inclusion of an additional measurement is reminiscent of familiar procedures for converting Fisher models to Bayesian models as highlighted in [41, Section 5.3]. The distinction is that expectation for Fisher models should only be with respect to the noise $v_0$ (with $x_0$ assumed unknown), while expectation should be with respect to both $x$ and $v$ for the Bayesian approach.

that are close to linear with any discrepancies being confined within a cone or tight sector. The Naval Research Laboratory has recently further pursued such an approach.

Historical approaches to using CR lower bounds to conservatively gauge the uncertainty incurred in autoregressive (AR) parameter estimation previously encountered a barrier of having to perform multitudinous differentiations in order to obtain the intermediate result of computing the associated Fisher information matrix. However, the recent result of [90] circumvents this prior computational burden via a neat identity that allows mere shift-matrix operations to suffice (as long as the underlying process is stationary). Unfortunately, the form of the CR bound utilized in [90] assumes that the AR parameter estimator being utilized is unbiased, which is not the case for all applications.

Additional benefits of using the evaluation technique of [89] are that an explicit formula for the lower bound is available that does not involve numerical integration as most other evaluation approaches do. Indeed, in some applications one is interested not so much in the AR parameters themselves as in some useful function of these parameters such as in the center frequency, bandwidth, and power of narrowband spectral lines. Another beneficial aspect of [89] is in providing a simplified methodology for computing CR lower bounds on such general functions of the AR coefficients (and additive noise intensity). See follow-on in [92], [93].

The approaches of [79, 80, 89, and 90] are great from both the important aspect of providing significant rigorous results and from the aspect of being sufficiently tutorial that they are easy to read. However, the CR bounds utilized in [79, 80, 89, and 90] are apparently related to the fundamental geometry of the underlying time-invariant problem and do not include a consideration of the transient time-varying case, hence [79, 80, and 89] do not embrace techniques such as the CR bounds under scrutiny here that have been utilized in evaluating the performance of various approximate nonlinear filter formulations for the sonar/sonobuoy application.

## IV. CONCLUSION

In summary, the purpose of Section III is to caution the nonlinear filtering practitioners and applications engineers about less well-known caveats associated with two different CR bound extensions that have been developed relatively recently [1, 2] for the application of performance evaluation in nonlinear filtering scenarios. Such CR-like bounding techniques currently enjoying fairly widespread utilization appear to have been somewhat uncritically embraced for sonar-related applications as if these new bounds had automatically inherited the full rigor of their namesake, namely, the structurally simpler CR bounds arising in the identification of constant parameters, where similar comparisons and inequalities arise.

While [1 and 2] gave the impression of being iron clad, more recent investigations of CR-bounding technology by, say, Galdos [56] and Chang [42], also truthfully acknowledge the soft spots in the analysis such as in [56, paragraph following eq. (28)] where Galdos admits that "the ordering result of Appendix B is but a conjecture and that it is difficult if not impossible to ascertain whether it actually holds", while Chang admits in [42, Section C] that his two matrix bounds $B_1$ and $B_2$ are not as tight as a true CR bound and that any hypothesized ordering between $B_1$ and $B_2$ has eluded a rigorous proof. It is indeed unfortunate that sonobuoy practitioners apparently resort back to [1] or [2] for evaluating nonlinear filter performance instead of using the most up-to-date investigations of [56, 30, 42] for applying CR bounds that go further to say how the evaluation should be implemented and what to be wary of. Further rigorous investigations in this important area would indeed be useful.

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**Thomas Kerr** (S'67—M'74—SM'85) was born in Washington, D.C. on November 9, 1945. He received the B.S.E.E. (magna cum laude) from Howard University, Washington, D.C. in 1967 and the M.S. and Ph.D. degrees (via NSF) in the electrical engineering specialty of control/estimation from the University of Iowa, Iowa City, IA in 1969 and 1971, respectively.

A staff member at MIT Lincoln Laboratory since October 1986 in the area of spectral estimation (which uses the same estimation/detection/identification/pattern recognition tools) for radar tracking applications, he had been a senior analyst/systems engineer at Intermetrics from 1979 with interests in navigation system and sonobuoy system tracking/implementation/Kalman filtering/failure detection. His previous experience includes university teaching/research (1967–1971), analysis, computer simulation, and real-time implementation at General Electriic's Corporate R&D Center in Schenectady, NY (1971–1973), and navigation/filtering/failure detection applications at TASC (1973–1979).

His DOD application involvements include: investigations and computer implementations of parameter identification, mathematical and algorithmic aspects of estimation and control, decentralized and nonlinear filtering, and quantifying associated computer burdens, hypothesis testing/failure detection, effects of failure detection on the reliability/availability of reconfigurable modular composite systems, bicriteria (Pareto-) optimization algorithms, and procedures for Kalman filter sensor selection and scheduling, tractable applications of level-crossing theory, point-process detection in antisubmarine warfare, optimal search and screening techniques, multi-target tracking, and pattern recognition. These above investigations were peformed for various applications including Poseidon and Trident SINS/ESGN navigation systems, JTIDS RelNav, ICNIA and other GPS-aided multi-sensor avionics navigation applications, minesweeper PINS, PTA sonobuoy tracking, Post-Coherence Function sonobuoy target tracking, ASW search, submarine antenna detectability to radar surveillance, and helicopter-based Missile Warning System development and refinement.

Dr. Kerr is also a member of Tau Beta Pi, Eta Kappa Nu, Sigma Pi Sigma, Pi Mu Epsilon, AIAA, ION, AAAS, the Naval Institute, ADPA, NSIA, and IEEE AC, IT, AES, and ASSP groups.