non-anticipatory functions by defining the Ito integral for arbitrary non-anticipatory functions to be

$$\int_0^b \zeta(t,\omega)d\beta(\tau,\omega) \triangleq \text{l.i.m.} \int_0^b \zeta_n(t,\omega)d\beta(\tau,\omega).$$

It is shown in Jazwinski (1970, pp. 99-100) and in Varadhan (1968, pp. 129-131) that the Ito integral of a simple random process has the following properties:

(i) $\int_a^b [f(t,\omega) + g(t,\omega)]d\beta(t,\omega) = \int_a^b f(t,\omega)d\beta(t,\omega) + \int_a^b g(t,\omega)d\beta(t,\omega)$,

(ii) $\{\int_a^t f(\tau,\omega)d\beta(\tau,\omega)\}_{t\varepsilon T}$ is a martingale process,

(iii) $\{\int_a^t f(\tau,\omega)d\beta(\tau,\omega)\}_{t\varepsilon T}$ has sample functions that are continuous with probability one,

(iv) $E\{\int_a^b f(t,\omega)d\beta(t,\omega)\} = 0$,

(v) $E[\{(I) \int_a^t f(\tau,\omega)d\beta(\tau,\omega)\} \cdot \{(I) \int_a^s g(u,\omega)d\beta(u,\omega)\}^T] = \int_a^{\min(t,s)} f(u,\omega)Q\, g^T(u,\omega)du$, where Q is the covariance matrix of $\{\beta_t\}_{t\varepsilon T}$ and Q is positive definite.

The above properties are preserved when the domain of the integral is extended to the wider class of arbitrary non-anticipatory functions.

Stochastic Integral Equations

Now that the Ito stochastic integral has been defined for simple random processes and has been extended to general random processes, random integral equations may now be considered. In real variable theory, to every differential equation $\frac{dx}{dt} = f(x,t)$, with $f(x,t)$ continuous in x and t and Lipschitz in x over the appropriate region and having an initial condition $x(t_0) = x_0$, there corresponds an integral equation $x(t) = x_0 + \int_{t_0}^{t} f(x(t),t)dt$, where the solution of the differential equation is the solution of the integral equation. In working with systems driven by a Wiener process, $\beta(t,\omega)$, only the stochastic integral equation,

0

$$x(t,\omega) = x(t_0,\omega) + \int_{t_0}^{t} f(x(s,\omega),s)ds + (I) \int_{t_0}^{t} \sigma(x(s,\omega),s)d\beta(s,\omega),$$

with $f(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ continuous in both arguments over the appropriate region and Lipschitz in x, and $\sigma(\cdot,\cdot)$, $f(\cdot,\cdot)$ measurable in x and t, has meaning. An often encountered expression is $dx(t,\omega) = f(s(t),t)dt + \sigma(x(t),t)d\beta(t,\omega)$ which is just notation to represent the above defined integral equation and nothing more. In eq. 0 , the first integral is just the ordinary Lebesgue integral, but the second integral is the Ito integral. It is seen that it would be impossible to define the above stochastic integral equation without first having the Ito integral defined. In order to establish results concerning stochastic integral equations it is necessary to use the properties of the Ito integral elaborated upon earlier.

In analogy to the method of using Picard iteration to establish the existence and uniqueness of a solution to the deterministic differential equation or integral equation (Goldberg, 1963, p. 266), Picard iteration will again be used to establish the existence and uniqueness of a solution to the stochastic integral equation, but the arguments are complicated by the fact that the solution will be a random process and convergence of the solution must be argued from the standpoint of convergence with probability 1 for the most powerful results.

The proof of the following theorem will be elaborated on in detail since it involves many of the concepts that have been defined so far and will serve to illustrate the probabilistic techniques needed in studying random processes and stochastic integrals from a rigorous point of view.

<u>Theorem 3.4:</u> If $f(x)$ and $\sigma(x)$ are Lipschitz in x then the integral equation $x(t,\omega) - x(s,\omega) = \int_s^t f(x(u,\omega))du + \int_s^t \sigma(x(u,\omega))d\beta(u,\omega)$ has a unique global solution $x(t,\omega)$ which is a Markov process having almost all sample functions continuous and passing through any random initial point α, where α is a random variable, $\alpha \in L^2(P)$, with distribution independent of $\beta(t,\omega)$ for all t, and this unique global solution $x(t,\omega)$ is an element of $L^2(\Omega \times [\tau_1,\tau_2])$ where $0 < \tau_1 \leq \tau_2 < \infty$.

<u>Elaborated Proof:</u> (Modeled after sketch in Bucy, 1969, p. 19.) Consider the time interval $[0,T]$ where T is an arbitrary fixed finite value. Consider the iteration equation, which gives the result of the n-th iteration to be

1
$$x_n(t,\omega) = \alpha(\omega) + \int_0^t f(x_{n-1}(s,\omega))ds + (I) \int_0^t \sigma(x_{n-1}(s,\omega)d\beta(s,\omega)$$

for each fixed $t \in [0,T]$ and where $s = 0$ for convenience.

Similarly, the result of the n-1[st] iteration is

2
$$x_{n-1}(t,\omega) = \alpha(\omega) + \int_0^t f(x_{n-2}(s,\omega))ds + (I) \int_0^t \sigma(x_{n-2}(s,\omega))d\beta(s,\omega).$$

Subtracting eq. 2 from eq. 1 yields

$$[x_n(t,\omega) - x_{n-1}(t,\omega)] = \int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds + (I) \int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega)]d\beta(s,\omega)$$

which when squared on both sides yields

$$(x_n(t,\omega) - x_{n-1}(t,\omega))^2 = \{\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds + (I)\int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)\ ^2.$$

Now from the fact that $(a + b)^2 = |a + b|^2$, it follows that

3 $$|x_n(t,\omega) - x_{n-1}(t,\omega)|^2 = \{\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds + (I)\int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)\}^2$$

(1)
$$\leqq 2|\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))ds|^2 + 2\ (I)\int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)|^2$$

(2)
$$\leqq 2\int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2 ds \cdot \int_0^t 1^2 ds + 2|(I)\int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)|^2.$$

---

(1) The above is obtained by an application of the inequality $|a + b|^2 \leqq 2|a|^2 + 2|b|^2$ which is a consequence of the triangle inequality as is shown below. The fact that $|a + b| \leqq |a| + |b| \Rightarrow (|a + b|)^2 \leqq (|a| + |b|)^2 = |a|^2 + 2|a| \cdot |b| + |b|^2$ and the fact that $0 \leqq (|a| - |b|)^2 = |a|^2 - 2|a| \cdot |b| + |b|^2 \Rightarrow 2|a||b| \leqq |a|^2 + |b|^2$ taken together imply that $|a + b|^2 \leqq |a|^2 + 2|a| \cdot |b| + |b|^2 \leqq |a|^2 + |a|^2 + |b|^2 + |b|^2$ or $|a + b|^2 \leqq 2|a|^2 + 2|b|^2$.

(2) The above is obtained by an application of the Cauchy-Schwarz inequality $|\int_0^t hg\, ds|^2 \leqq \int_0^t |h|^2 ds \cdot \int_0^t |g|^2 ds$ with $g = 1$ and $h = f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))$.

Since eq. 3 holds and taking expected values on both sides preserves the inequality, it follows that

4

$$E|x_n(s,\omega) - x_{n-1}(s,\omega)|^2 \leqq E\, 2t \int_0^t |f(x_{n-1}(s,\omega) - f(x_{n-2}(s,\omega))|^2\, ds + E\, (I) \int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)|^2$$

③

$$= 2t\, E \int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2\, ds + \int_0^t E|\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))|^2\, ds$$

④

$$= 2t \int_0^t E|f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2\, ds + \int_0^t E|\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))|^2\, ds.$$

---

③ The above is a result of using the Ito integral in that $E|(I) \int_0^t h(s,\omega)d\beta(s,\omega)|^2 = \int_0^t E|h(s,\omega)|^2\, ds$ is a property of the Ito integral as demonstrated in a previous section.

④ The above is obtained by applying Fubini's theorem (Rudin, 1966, p. 140) and recognizing the fact that E[·] is just an integral,

$$E \int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2\, ds$$

$$= \int_\Omega dP \int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2\, ds.$$

The assumption that the first integral exists resulting in the function being absolutely integrable in one of the three possible forms justifies the interchange of the order of

Both $f(\cdot)$ and $\sigma(\cdot)$ satisfy a Lipschitz condition, so $\exists\, M_1 > 0 \ni |f(x) - f(y)| \leqq M_1 |x - y|$ and $\exists\, M_2 > 0 \ni |\sigma(x) - \sigma(y)| \leqq M_2 |x - y|$. Now letting $K = \max(M_1, M_2)$, it is true that $|f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))| \leqq K|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|$ and $|\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))| \leqq K|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|$ and that squaring both sides preserves the direction of the inequalities to yield

$$|f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2 \leqq K^2|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2$$

and

$$|\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))|^2 \leqq K^2|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2.$$

Since $0 \leqq a \leqq b$ and P is a positive measure $\Rightarrow 0 \leqq \int_\Omega a\,dP \leqq \int_\Omega b\,dP$, taking expectations of both sides of the above preserves the inequality to yield

$$0 \leqq E|f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2 \leqq K^2\, E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2$$

and

$$0 \leqq E|f(x_{n-1}(s,\omega) - f(x_{n-2}(s,\omega))|^2 \leqq K^2\, E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2$$

---

integration.

$$\int_\Omega dP \int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2\, ds = \int_0^t ds \int_\Omega |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2\, dP = \int_0^t E|f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2\, ds.$$

for each s ε [0,T]. Similarly, since $0 \leqq c \leqq D \Longrightarrow$ $\int_0^t c\, ds \leqq \int_0^t D\, ds$, integrating both sides of the above with respect to s yields

$$\int_0^t E|f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|^2\, ds \leqq K^2 \int_0^t E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2\, ds$$

and

$$\int_0^t E|\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))|^2\, ds \leqq K^2 \int_0^t E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2\, ds$$

for each fixed t ε [0,T].

The above results from using the Lipschitz hypothesis and yields a bound for $E|x_n(t,\omega) - x_{n-1}(t,\omega)|^2$. The expression for this bound is

5

$$E|x_n(t,\omega) - x_{n-1}(t,\omega)|^2 \leqq 2(t+1)K^2 \int_0^t E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2\, ds.$$

Using eq. 5 to obtain a bound on $E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2$ yields

$$E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2 \leqq 2(t+1)K^2 \int_0^s E|x_{n-2}(u,\omega) - x_{n-3}(u,\omega)|^2\, du.$$

So now the bound in eq. 5 can be rewritten as

6

$$E|x_n(t,\omega) - x_{n-1}(t,\omega)|^2$$
$$\leqq 2(t+1)K^2 \int_0^t ds\; 2(t+1)K^2 \int_0^s E|x_{n-2}(u,\omega) - x_{n-3}(u,\omega)|^2\, du.$$
$$= [2(t+1)K^2]^2 \int_0^t ds \int_0^s E|x_{n-2}(u,\omega) - x_{n-3}(u,\omega)|^2\, du.$$

Proceeding to iterate the above expression n-1 times yields

$$E|x_n(t,\omega) - x_{n-1}(t,\omega)|^2 \leqq [2(t+1)K^2]^{n-1} \cdot \int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-2}} E|x_1(s_{n-1},\omega) - x_0(s_{n-1},\omega)|^2 \, ds_{n-1}$$

for each fixed $t \in [0,T]$ where $x_0(t,\omega) = \alpha(\omega)$, $\forall\, t \in [0,T]$.

Now from the property of supremums,

$$E|x_1(s_{n-1},\omega) - x_0(s_{n-1},\omega)|^2 \leqq \sup_{0 \leqq u \leqq s_{n-1}} E|x_1(u,\omega) - x_0(u,\omega)|^2$$

for all $s_{n-1} \in [0,s_{n-2}]$. By another property of supremums, the supremum of a quantity over a larger set is greater than or equal to the supremum over the smaller set and since $[0,s_{n-1}] \subseteq [0,T]$ it is true that

$$\sup_{0 \leqq u \leqq s_{n-1}} E|x_1(u,\omega) - x_0(u,\omega)|^2 \leqq \sup_{0 \leqq u \leqq T} E|x_1(u,\omega) - x_0(u,\omega)|^2.$$

The above two facts together imply that $\forall\, s_{n-1} \in [0,s_{n-2}]$

$$E|x_1(s_{n-1},\omega) - x_0(s_{n-1},\omega)|^2 \leqq \sup_{0 \leqq u \leqq T} E|x_1(u,\omega) - x_0(u,\omega)|^2$$

(a finite constant). It therefore follows that

$$\int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-2}} E|x_1(s_{n-1},\omega) - x_0(s_{n-1},\omega)|^2 \, ds_{n-1} \leqq \int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-2}} \sup_{0 \leqq u \leqq T} E|x_1(u,\omega) - x_0(u,\omega)|^2 \, ds_{n-1} =$$

$$\sup_{0 \leqq u \leqq T} E|x_1(u,\omega) - x_0(u,\omega)|^2 \int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-2}} ds_{n-1}.$$

A result from advanced calculus is that an expression for the (n-1)-fold iterated integral is

$$\int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-2}} ds_{n-1} = \int_0^t \frac{(t-u)^{n-2}}{(n-2)!} du.$$

The expression on the right hand side is easily integrated to yield

$$\int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-2}} ds_{n-1} = \int_0^t \frac{(t-u)}{(n-2)!} du = \frac{t^{n-1}}{(n-1)!}.$$

The bound in eq. 7 can now be extended to

8
$$E|x_n(t,\omega) - x_{n-1}(t,\omega)|^2 \leqq \frac{[2(t+1)tK^2]^{n-1}}{(n-1)!} \cdot \sup_{0 \leqq u \leqq T} E|x_1(u,\omega) - x_0(u,\omega)|^2$$

for each fixed $t \varepsilon [0,T]$. It should be noted that for fixed $t \varepsilon [0,T]$, $|x_n(t,\omega) - x_{n-1}(t,\omega)|$ is a random variable and so the bound that has been obtained is simply a bound on the second moment of the difference of the random variables $x_n(t,\omega)$ and $x_{n-1}(t,\omega)$.

The bound in eq. 8 and the fact that $t \varepsilon [0,T]$ so $t \leq T$ imply that, all the more,

9
$$E|x(t,\omega) - x(t,\omega)|^2 \leqq \frac{[2(T+1)TK^2]^{n-1}}{(n-1)!} \cdot \sup_{0 \leqq u \leqq T} E|x_1(u,\omega) - x_0(u,\omega)|^2$$

for each fixed $t \varepsilon [0,T]$. Rewriting the above for $x_{n-1}(t,\omega) - x_{n-2}(t,\omega)$, we have

$$E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2 \leqq \frac{[2(T+1)TK^2]^{n-2}}{(n-2)!} \cdot \sup_{0 \leqq u \leqq T} E|x_1(u,\omega) - x_0(u,\omega)|^2$$

for each fixed $t \varepsilon [0,T]$. Now taking the supremum of the left hand side of the above over t, $0 \leqq t \leqq T$, where the right hand side is independent of t yields

10

$$\sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2 \leq \frac{[2(T+1)TK^2]^{n-2}}{(n-2)!} \cdot \sup_{0 \leq u \leq T} E|x_1(u,\omega) - x_0(u,\omega)|^2.$$

This bound will be used in the next part of the proof.

In order to establish almost sure convergence of the iteration scheme the following device can be used. Define

$$R_n(\omega) = \sup_{0 \leq t \leq T} \frac{|x_n(t,\omega) - x_{n-1}(t,\omega)|^2}{2} \geq 0$$

for each $\omega \in \Omega$. For each $\omega \in \Omega$, $R_n(\omega)$ will be an indication of the non-closeness of $x_n(t,\omega)$ and $x_{n-1}(t,\omega)$ over the whole interval $[0,T]$. If sequences $\{\varepsilon_n\}_{n=1}^{\infty}$ and $\{N_n\}_{n=1}^{\infty}$, each consisting of nonnegative terms exist $\cdot\ni\cdot$ $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, $\sum_{n=1}^{\infty} N_n < \infty$, and $P\{\omega \mid |R_n(\omega)| > \varepsilon_n\} < N_n \; \forall \; n \in I^+$, then $\exists$ $x(t,\omega)$ $\cdot\ni\cdot$ $x_n(t,\omega)$ converges to $x(t,\omega)$ with probability 1 (Cramer and Ledbetter, 1966, p. 42, expression 3.5.5).

It will now be established that $P\{\omega \mid |R_n(\omega)| > \varepsilon_n\} < N_n$ $\forall n \in I^+$. From eq.3 it is true that for each fixed $\omega \in \Omega$,

$$\frac{|x_n(t,\omega) - x_{n-1}(t,\omega)|^2}{2} \leq \left|\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds\right|^2 + \left|(I)\int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)\right|^2,$$

and by taking the supremum on both sides

$$R_n(\omega) = \sup_{0 \leq t \leq T} \frac{|x_n(t,\omega) - x_{n-1}(t,\omega)|^2}{2} \leq$$

$$\sup_{0 \leq t \leq T} \left|\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds\right|^2 +$$
$$\sup_{0 \leq t \leq T} \left|(I) \int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)\right|^2$$

for each $\omega \in \Omega$. For any $\varepsilon > 0$, since $R_n(\omega) \geq 0$ it follows that

$$P\{\omega \,|\, |R_n(\omega)| > \varepsilon\} = P\{\omega \,|\, R_n(\omega) > \varepsilon\}$$

(5)
$$\leq P\{\omega \,|\, \sup_{0 \leq t \leq T} \left|\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds\right|^2 + \sup_{0 \leq t \leq T} \left|(I) \int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)\right|^2 > \varepsilon\}$$

(6)
$$\leq P\{\omega \,|\, \sup_{0 \leq t \leq T} \left|\int [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds\right|^2 > \varepsilon\}$$
$$+ P\{\omega \,|\, \sup_{0 \leq t \leq T} \left|(I) \int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)\right|^2 > \varepsilon/2\}.$$

---

(5) The above is obtained from the fact that given $A(\omega) \leq B(\omega) + C(\omega)$, for each $\omega' \in \{\omega|A(\omega) > \varepsilon\}$ since $\varepsilon < A(\omega') \leq B(\omega') + C(\omega')$ then $\omega' \in \{\omega|B(\omega) + C(\omega) > \varepsilon\}$; so $\{\omega|A(\omega) > \varepsilon\} \subset \{\omega|B(\omega) + C(\omega) > \varepsilon\}$. Now by the monotone property of positive measures, $P\{\omega|A(\omega) > \varepsilon\} \leq P\{\omega|B(\omega) + C(\omega) > \varepsilon\}$.

(6) The above is obtained from the fact that $\{\omega|C(\omega) \leq \varepsilon/2\} \subset \{\omega|B(\omega)\varepsilon/2\} \cap \{\omega|B(\omega) + C(\omega)\varepsilon\}$, which is seen by considering an arbitrary $\omega' \in \{\omega|C(\omega) \leq \varepsilon/2\}$

It has now been established that

11

$$P\{\omega \,|\, |R_n(\omega)| > \varepsilon\} \leq P\{\omega \,|\, \sup_{0 \leq t \leq T} |\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds| > \varepsilon/2\} + P\{\omega \,|\, \sup_{0 \leq t \leq T} |(I) \int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)| > \varepsilon/2\}.$$

Now bounds will be established on both of the two distinct terms on the right hand side of eq. 11 .

For the first term it is true that

$$P\{\omega \,|\, \sup_{0 \leq t \leq T} |\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds|^2 > \varepsilon/2\}$$

$\{\omega | B(\omega) \leq \varepsilon/2\}$. Now $B(\omega') \leq \varepsilon/2$ and $C(\omega') \leq \varepsilon/2 \Rightarrow B(\omega') + C(\omega') \leq \varepsilon$ so $\omega' \in \{\omega | B(\omega) + C(\omega) \leq \varepsilon\}$. Since this was for arbitrary ω', containment is proved. Now observe that $\{\omega | B(\omega) > \varepsilon/2\}^c \cap \{\omega | C(\omega) > \varepsilon/2\}^c = \{\omega | B(\omega) \leq \varepsilon/2\} \cap \{\omega | C(\omega) \leq \varepsilon/2\} \subset \{\omega | B(\omega) + C(\omega) \leq \varepsilon\} = \{\omega | B(\omega) + C(\omega) > \varepsilon\}^c$; and by taking complements on both sides $\{\omega | B(\omega) + C(\omega) > \varepsilon\} \subset \{\omega | B(\omega) > \varepsilon/2\} \cup \{\omega | C(\omega) > \varepsilon/2\}$. Now since $\{\omega | B(\omega) > \varepsilon/2\}$ and $\{\omega | C(\omega) > \varepsilon/2\}$ are not known to be disjoint, it follows from the finite subadditivity of positive measures that $P\{\omega | B(\omega) + C(\omega) > \varepsilon\} \leq P\{\omega | B(\omega) > \varepsilon/2\} + P\{\omega | C(\omega) > \varepsilon/2\}$.

⑦

$$= P\{\omega | \sup_{0 \leq t \leq T} |\int_0^t f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))ds| > \sqrt{\varepsilon}/2\}$$

⑧

$$\leq P\{\omega | \sup_{0 \leq t \leq T} \int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|ds > \sqrt{\varepsilon}/2\}$$

⑨

$$\leq P\{\omega | \int_0^T |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|ds > \sqrt{\varepsilon}/2\}$$

---

⑦ The above is obtained since for any ω for which $A^2(\omega) > b > 0$ it is also true that $+A(\omega) > \sqrt{b} > 0$ and vice-versa so $\{\omega | |A(\omega)|^2 > b\} = \{\omega | |A(\omega)| > \sqrt{b}\}$ where $b > 0$.

⑧ The above is obtained since $|\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds| \leq \int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|ds$ and consequently $\sup_{0 \leq t \leq T} |\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega)]ds| \leq \sup_{0 \leq t \leq T} \int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|ds$ so for each $\omega' \varepsilon \{\omega | \sup_{0 \leq t \leq T} |\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds| > \sqrt{\varepsilon}/2\}$ means that $\sqrt{\varepsilon}/2 < \sup_{0 \leq t \leq T} |\int_0^t [f(x_{n-1}(s,\omega') - f(x_{n-2}(s,\omega'))]ds| \leq \sup_{0 \leq t \leq T} \int_0^t |f(x_{n-1}(s,\omega')) - f(x_{n-2}(s,\omega'))$ ds so that $\omega' \varepsilon \{\omega | \sup_{0 \leq t \leq T} \int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|ds > \varepsilon\}$ and $\{\omega | \sup_{0 \leq t \leq T} |\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds| > \varepsilon\} \subset \{\omega | \sup_{0 \leq t \leq T} \int_0^t |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|ds > \varepsilon\}$. Using $A \subset B \Longrightarrow P(A) \leq P(B)$, the result follows.

⑨ The above is obtained since for $|g| \geq 0$ and $[0,t]$ $[0,T]$ for all $t \varepsilon [0,T] \Longrightarrow \int_{[0,t]} |g|ds \leq \int_{[0,T]} |g|ds$ and

(10) $P\{\omega | K \int_0^T |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)| ds > \sqrt{\varepsilon}/2\}$

(11) $= P\{\omega | |K \int_0^T |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)| ds| > \sqrt{\varepsilon}/2\}$

(12) $\leq \dfrac{E|\int_0^T K\, x_{n-1}(s,\omega) - x_{n-2}(s,\omega)| ds|^2}{(\sqrt{\varepsilon}/2)^2}$

(13) $\leq \dfrac{E[T \int_0^T K^2 |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2\, ds]}{\varepsilon/2}$

$= 2/\varepsilon\ TK^2\ E \int_0^T |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2\, ds$

---

consequently $\sup_{0 \leq t \leq T} \int_{[0,t]} |g| ds \leq \int_{[0,T]} |g| ds.$

(10) For arbitrary $\omega' \varepsilon\{\omega | \int_0^T |f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))| ds > \sqrt{\varepsilon}/2\}$ and since $\sqrt{\varepsilon}/2 < \int_0^T |f(x_{n-1}(s,\omega')) - f(x_{n-2}(s,\omega'))| ds \leq K \int_0^T |x_{n-1}(s,\omega') - x_{n-2}(s,\omega')| ds$, so $\omega' \varepsilon\{\omega | K \int_0^T |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)| ds > \sqrt{\varepsilon}/2\}$. Since $A' \subset B' \Longrightarrow P(A') \leq P(B')$, the result follows.

(11) The above is obtained since $0 \leq K \int_0^T |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)| ds = |K \int_0^T |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)| ds|$.

(12) The above is obtained by the use of the Chebyshev inequality, $P[|y| > \lambda] \leq \dfrac{E|y|^2}{\lambda^2}$.

(13) The above is obtained by use of the Cauchy-Schwarz inequality $|\int_0^T hg\, ds|^2 \leq \int_0^T h^2\, ds \cdot \int_0^T g^2\, ds$ where $h = K|f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))|$ and $g = 1$.

(14)
$$= 2/\varepsilon\ TK^2 \int_0^T E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2\ ds$$

(15)
$$= 2/\varepsilon\ TK^2 \int_0^T \sup_{0 \leq u \leq T} E|x_{n-1}(u,\omega) - x_{n-2}(u,\omega)|^2\ ds$$

$$= 2/\varepsilon\ T^2K^2 \sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2.$$

So that a bound for the first term on the right hand side of eq.11 is

12

$$P\{\omega| \sup_{0 \leq t \leq T} |\int_0^t [f(x_{n-1}(s,\omega)) - f(x_{n-2}(s,\omega))]ds|^2 > \varepsilon\ \} \leq$$
$$2/\varepsilon\ T^2K^2 \sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2.$$

For the second term on the right hand side of eq.11 it is true that

$$P\{\omega| \sup_{0 \leq t \leq T} |(I) \int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))]d\beta(s,\omega)|^2 >$$
$$\varepsilon/2\}$$

---

(14) The above is obtained by use of Fubini's theorem which is justified by the absolute integrability of $E \int_0^t |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2\ ds$.

(15) The above is obtained since for all s $\ni$ $0 \leq s \leq t$, $0 \leq E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2 \leq \sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2$ $\int_0^T E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2\ ds \leq$ $\int_0^T \sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2\ ds = T \sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2$.

(16)
$$= P\{\omega | \sup_{0 \leq t \leq T} |(I) \int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))] d\beta(s,\omega)| > \sqrt{\varepsilon}/2\}$$

(17)
$$\leq \frac{1}{(\sqrt{\varepsilon}/2)^2} \cdot E|(I) \int_0^T [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))] d\beta(s,\omega)|^2$$

(18)
$$\leq 2/\varepsilon \int_0^T E|\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))|^2 \, ds$$

(19)
$$\leq 2/\varepsilon \int_0^T E \, K^2 |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2 \, ds$$

---

(16) The above is obtained by the same reasoning as in (7).

(17) The above is obtained since the stochastic integral is a continuous martingale (Varadhan, 1968, p. 129) and, as such, satisfies the continuous martingale inequality, $P\{\omega | \sup_{0 \leq t \leq T} |n(t,\omega)| \geq \ell\} \leq \frac{1}{\ell^2} \cdot E|n(T,\omega)|^2$, where $n(t,\omega)$ is the martingale.

(18) The above is obtained by the property of the Ito integral used in (3).

(19) The above is obtained since for each $s \in [0,t] \subset [0,T]$

$$|\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))|^2 \leq K^2 |x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2 \Longrightarrow$$

$$E|\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))|^2 \leq K^2 \, E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2$$ for each $s \in [0,t] \subset [0,T] \Longrightarrow$

$$\int_0^T E|\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))|^2 \, ds \leq K^2 \int_0^T E|x_{n-1}(s,\omega) - x_{n-2}(s,\omega)|^2 \, ds.$$

(20)

$$\leq 2/\varepsilon\, K^2 \int_0^T \sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2\, ds$$

$$= 2/\varepsilon\, K^2 T \sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2.$$

So that a bound for the second term on the right hand side of eq. 11 is

13

$$P\{\omega | \sup_{0 \leq t \leq T} |(I) \int_0^t [\sigma(x_{n-1}(s,\omega)) - \sigma(x_{n-2}(s,\omega))] d\beta(s,\omega)|^2 > \varepsilon/2\} \leq K^2 T \sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2.$$

Using eq. 13 and eq. 12, the bound in eq. 11 can be extended to

14

$$P\{\omega | |R_n(\omega)| > \varepsilon\} \leq 2/\varepsilon\, T(T+1)K^2 \sup_{0 \leq t \leq T} E|x_{n-1}(t,\omega) - x_{n-2}(t,\omega)|^2.$$

Now using the expression developed in eq. 10, the bound in eq. 14 can be extended to

15

$$P\{\omega | |R_n(\omega)| > \varepsilon\} \leq 2/\varepsilon\, T(T+1)K^2 \frac{[2(T^2+T)K^2]^{n-2}}{(n-2)!} \cdot \sup_{0 \leq u \leq T} E|x_1(u,\omega) - x_0(u,\omega)|^2$$

for $\varepsilon > 0$. In particular, for $\varepsilon = \varepsilon_n > 0$, the above expression becomes

16

$$P\{\omega | |R_n(\omega)| > \varepsilon_n\} \leq 2/\varepsilon\, T(T+1)K^2 \frac{[2(T^2+T)K^2]^{n-2}}{(n-2)!} \cdot \sup_{0 \leq u \leq T} E|x_1(u,\omega) - x_0(u,\omega)|.$$

And in particular, for $\varepsilon_n = \frac{1}{(n-2)^2} > 0$, the above expression

---

(20) The above is obtained by the same reasoning as in (15).

becomes

$$P\{\omega \mid |R_n(\omega)| > \frac{1}{(n-2)^2}\} \leq 2T(T+1)K^2(n-2)^2 \frac{[2(T^2+T)K^2]^{n-2}}{(n-2)!} \cdot \sup_{0 \leq u \leq T} E|x_1(u,\omega) - x_0(u,\omega)|.$$

In the above, the expression $\sup_{0 \leq u \leq T} E|x_1(u,\omega) - x_0(u,\omega)|^2$ is ever present. This expression will now be investigated. From the fact that $x_0(t,\omega) = \alpha(\omega)$ for $t \in [0,T]$ and several of the arguments used earlier in the proof, it is true that

$$E|x_1(t,\omega) - x_0(t,\omega)|^2 = E|\int_0^t f(x_0(s,\omega))ds + (I)\int_0^t \sigma(x_0(s,\omega))d\beta(s,\omega) + \alpha(\omega) - x_0(t,\omega)|^2$$

$$= E|\int_0^t f(\alpha(\omega))ds + (I)\int_0^t \sigma(\alpha(\omega))d\beta(s,\omega) + \alpha(\omega) - \alpha(\omega)|^2$$

$$\leq 2E|\int_0^t f(\alpha(\omega))ds|^2 + 2E\ (I)\int_0^t \sigma(\alpha(\omega))d\beta(s,\omega)|^2$$

$$\leq 2E[t\int_0^t |f(\alpha(\omega))|^2\, ds] + 2\int_0^t E|\sigma(\alpha(\omega))|^2\, ds$$

$$= 2t\int_0^t E|f(\alpha(\omega))|^2\, ds + 2t\, E|\sigma(\alpha(\omega))|^2$$

$$= 2t\, E|f(\alpha(\omega))|^2 + 2t\, E|\sigma(\alpha(\omega))|^2$$

$$\leq 2t^2\, E|f(\alpha(\omega))|^2 + 2T\, E|\sigma(\alpha(\omega))|^2 < \infty$$

for each $t \in [0,T]$. Therefore the

$\sup_{0 \leq u \leq T} E|x_1(u,\omega) - x_0(u,\omega)|^2 \leq 2T^2\, E|f(\alpha(\omega))|^2 + 2T\, E|\sigma(\alpha(\omega))|^2 < \infty$, so $\sup_{0 \leq u \leq T} E|x_1(u,\omega) - x_0(u,\omega)|^2$ is a finite constant. Denote $\sup_{0 \leq u \leq T} E|x_1(u,\omega) - x_0(u,\omega)|^2$ by K_1 and $[2(T^2+T)K^2]$ by K_2, respectively, for a more compact notation. The expression in eq. 16 now becomes

$$P\{\omega \mid |R_n(\omega)| > \frac{1}{(n-2)^2}\} \leq 2T(T+1)K^2 \frac{(n-2)^2 K_2^{n-2}}{(n-2)!} \cdot K_1 \;\forall\, n \in I^+. \qquad 17$$

Now, the series $\sum_{n=3}^{\infty} \frac{1}{(n-2)^2}$ converges as a result of being a p-series with $p = 2$.

Also the series $2T(T+1)K^2 \cdot K_1 \sum_{n=3}^{\infty} \frac{(n-2)^2 K_2^{n-2}}{(n-2)!}$ converges as a result of the following application of the ratio test.

$$\limsup_{n\to\infty} \left| \frac{\dfrac{(n+1-2)^2 K_2^{n+1-2}}{(n+1-2)!}}{\dfrac{(n-2)^2 K_2^{n-2}}{(n-2)!}} \right| = \lim_{n\to\infty} \left| \frac{(\frac{1}{n} - \frac{1}{n^2}) K_2}{1 - \frac{4}{n} + \frac{4}{n^2}} \right| = 0 < 1.$$

The fact that the ratio test yields a number less than 1 garantees that the series under examination converges absolutely.

The convergence of the above two series having non-negative terms and the expression in eq.17 satisfy an equivalent formulation of probability one convergence given by Cramer and Leadbetter (1967, p. 42 expression 3.5.5), that makes use of the Borel-Cantelli lemma to state that there exists a random variable $x(t,\omega) \ni x_n(t,\omega) \to x(t,\omega)$ as $n \to \infty$ with probability one.

The uniqueness of the solution will now be established. Suppose that there exists two solutions, $x(t,\omega)$ and $y(t,\omega)$, of the stochastic integral equation on $[0,T]$. For $x(t,\omega)$ and $y(t,\omega)$ to be solutions it must be true that

$x(t,\omega) = \int_0^t f(x(s,\omega))ds + (I) \int_0^t \sigma(x(s,\omega))d\beta(s,\omega) + \alpha(\omega)$ and

$y(t,\omega) = \int_0^t f(y(t,\omega))ds + (I) \int_0^t \sigma(y(s,\omega))d\beta(s,\omega) + \alpha(\omega)$.

Define $k(t) \triangleq E|x(t,\omega) - y(t,\omega)|^2$, which is a function of t. For each $t \in [0,T]$, by repeating the steps in the derivation of eq. 5, it is true that $0 \leqq E|x(t,\omega) - y(t,\omega)|^2 \leqq 2t \int_0^t K^2E|x(s,\omega) - y(s,\omega)|^2 ds + 2K^2 \int_0^t E|x(s,\omega) - y(s,\omega)|^2 ds = 2(t+1)K^2 \int_0^t E|x(s,\omega) - y(s,\omega)|^2 ds \leqq 2(T+1)K^2 \int_0^t E|x(s,\omega) - y(s,\omega)|^2 ds$. Using the shorter notation, we have that $0 \leqq k(t) \leqq 2(T+1)K^2 \int_0^t k(s)ds < \infty$. Now iterating once, we obtain $0 \leqq k(t) \leqq 2(T+1)K^2 \int_0^t k(s)ds \leqq 2^2(T+1)^2 K^4 \int_0^t \int_0^s k(\omega)du\, ds$; therefore, by iterating n-times we have $0 \leqq k(t) \leqq 2^n(t+1)^n K^{2n} \int_0^t \cdots \int_0^{s_{n-1}} k(s_{n-1})ds_{n-1}\, ds_{n-2} \cdots = 2^n(t+1)^n K^{2n} [\frac{t}{n!}]$ or $0 \leqq k(t) \leqq$

$$[2(T+1)K^2]^{n-1} \int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-2}} k(s_{n-1})ds_{n-1} \leqq$$

$$[2(T+1)K^2]^{n-1} \int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-2}} \sup_{0 \leqq u \leqq T} k(u)ds_{n-1}.$$

As before, evaluating the iterated integral reduces the above expression to

$$0 \leq k(t) \leq \frac{[2T(T+1)K^2]^{n-1}}{(n-1)!} \sup_{0 \leq u \leq T} k(u).$$

Now since $\sup_{0 \leq u \leq T} k(u)$ is finite because $E|x(t,\omega)|^2 \leq 2E|\alpha(\omega)|^2 + 2 \int_0^t E|f(x(u,\omega))|^2 \, du < \infty$ and $E|y(t,\omega)|^2 \leq 2E|\alpha(\omega)|^2 + 2 \int_0^t E|f(y(u,\omega))|^2 \, du < \infty$, we have that $E|x(t,\omega) - y(t,\omega)|^2 \leq 2E|x(t,\omega)|^2 + 2E|y(t,\omega)|^2 < \infty$ for each $t \in [0,T]$; therefore, $\sup_{0 \leq u \leq T} |x(u,\omega) - y(u,\omega))|^2 < \infty$.

Now, $0 \leq k(t) \leq \frac{[2T(T+1)K^2]^{n-1}}{(n-1);} \sup_{0 \leq u \leq T} k(u)$ is valid for every $n \in I^+$. The right hand side of the above expression goes to zero. The convergence of the right hand side to zero can be seen from the fact that $\sum_{n=1}^{\infty} \frac{[2T(T+1)K^2]^{n-1}}{(n-1)!}$ is a convergent series converging to $e^{2T(T+1)K^2}$, and as such the n^{th} term goes to zero. Therefore $0 \leq k(t) \leq 0$ or $k(t) = 0$ for each $t \in [0,T]$ (i.e., $0 = E|x(t,\omega) - y(t,\omega)|^2 = \int_\Omega |x(t,\omega) - y(t,\omega)|^2 \, dP$). Therefore, $x(t,\omega) = y(t,\omega)$ a.e. [P] for all $t \in [0,T]$.

By using the device of the R_n's, uniform convergence was established. Because the R_n's satisfied the equivalent formulation of probability one convergence, we have that $x_n(t,\omega) \to x(t,\omega)$, for some $x(t,\omega)$, with probability one. These two facts together imply that $x_n(t,\omega) \to x(t,\omega)$

uniformly with probability one. The solution of the iteration equation $x_n(t,\omega) = \alpha(\omega) + \int_0^t f(x_{n-1}(s,\omega))ds +$ (I) $\int_0^t \sigma(x_{n-1}(s,\omega)d\beta(s,\omega)$ is almost everywhere [P] continuous in t $\forall$ n since the Ito integral is a.e. [P] continuous in t as shown in Doob (1953, p. 445). Since a sequence of a.e. [P] continuous function converges uniformly almost everywhere to a continuous a.e. [P] function, $x(t,\omega)$ is continuous almost everywhere [P].

A proof of the Markov property of the solutions on [0,T] is given in Jazwinski (1970, p. 110). Actually, the solution is a strong Markov process with homogeneous transition probabilities; the proof is in Varadhan (1968, p. 141).

Since T was arbitrary, the above theorem applies for any finite T, so the existence of a global solution with the above mentioned properties has been shown. This establishes all the claims of the theorem.

The previous proof was for scalar random processes and variables. This proof can be generalized to apply to vector random processes and variables (the components being random processes or random variables). In footnote (17), an inequality for scalar martingale processes was used in establishing a bound. To generalize the proof to vector random processes, a bound is needed for a vector of martingales.

That the concept of an inequality for a vector of martingales is reasonable can be seen from the following derivation. There is a scalar martingale inequality. This inequality is, for $\ell > 0$, $P\{\omega: \sup_{0 \leqq t \leqq T} |n(t,\omega)| \geqq \ell\} \leqq \frac{1}{\ell^2} E|n(T,\omega)|^2$, where $\{n(t,\omega), \{\mathcal{B}_t\}, t \in T\}$ is a martingale. Applying this scalar martingale inequality to two martingale processes $n_1(t,\omega)$ and $n_2(t,\omega)$, a vector matringale inequality will be derived for the 2-vector case which will be seen to immediately generalize by executing analogous steps in the proof for the m-vector case. For $\ell \geqq 0$, applying the scalar martingale inequality twice yields

$$P\{\omega: \sup_{0 \leqq t \leqq T} |n_1(t,\omega)| \geqq \ell/2\} \leqq \frac{4}{\ell^2} E|n_1(T,\omega)|^2$$

$$P\{\omega: \sup_{0 \leqq t \leqq T} |n_2(t,\omega)| \geqq \ell/2\} \leqq \frac{4}{\ell^2} E|n_2(T,\omega)|^2$$

Now let $A \stackrel{\Delta}{=} \{\omega: \sup_{0 \leqq t \leqq T} |n_1(t,\omega)| \geqq \ell/2\}$

and $B \stackrel{\Delta}{=} \{\omega: \sup_{0 \leqq t \leqq T} |n_2(t,\omega)| \geqq \ell/2\}$;

then $A^c = \{\omega: \sup_{0 \leqq t \leqq T} |n_1(t,\omega)| < \ell/2\}$

and $B^c = \{\omega: \sup_{0 \leqq t \leqq T} |n_2(t,\omega)| < \ell/2\}$.

<u>Claim</u>: $\{\omega: \sup_{0 \leqq t \leqq T} [|n_2(t,\omega)| + |n_1(t,\omega)|] < \ell\} \supseteq$

$\{\omega: \sup_{0 \leqq t \leqq T} |n_1(t,\omega)| < \ell/2\} \cap \{\omega: \sup |n_2(t,\omega)| < \ell/2\}$.

Proof: For arbitrary $\bar{\omega} \in A^c \cap B^c$, $\Rightarrow$ $\bar{\omega} \in A^c$ and $\bar{\omega} \in B^c$

$$\sup_{0\leq t\leq T} |n_2(t,\bar{\omega})| < \ell/2$$

$$\oplus \sup_{0\leq t\leq T} |n_1(t,\bar{\omega})| < \ell/2$$

$$\sup_{0\leq t\leq T} |n_1(t,\bar{\omega})| + \sup_{0\leq t\leq T} |n_2(t,\bar{\omega})| < \ell/2 + \ell/2 = \ell$$

Applying a property of supremums we have that

$\sup_{0\leq t\leq T} [|n_1(t,\bar{\omega})| + |n_2(t,\bar{\omega})|] \leq \sup_{0\leq t\leq T} |n_1(t,\bar{\omega})| + \sup_{0\leq t\leq T} |n_2(t,\bar{\omega})| < \ell$. Therefore, $\bar{\omega} \in \{\omega: \sup_{0\leq t\leq T} [|n_1(t,\omega)| + |n_2(t,\omega)|] < \ell\}$, hence $\{\omega: \sup_{0\leq t\leq T} [|n_1(t,\omega)| + |n_2(t,\omega)| < \ell]\} \supseteq A^c \cap B^c$.

Taking complements of both sides of the above claim and applying DeMorgan's law yields the following:

$\{\omega: \sup_{0\leq t\leq T} [|n_1(t,\omega)| + |n_2(t,\omega)|] \geq \ell\} \subseteq \{\omega: \sup_{0\leq t\leq T} |n_1(t,\omega)| \geq \ell/2\} \quad \{\omega: \sup_{0\leq t\leq T} |n_2(t,\omega)| \geq \ell/2\}$.

Since P is a positive measure, the monotone property of positive measures and the subadditivity of positive measures (A and B are not disjoint) imply that the following is true.

$P\{\omega: \sup_{0\leq t\leq T} [|n_1(t,\omega)| + |n_2(t,\omega)|] \geq \ell\} \leq P(A \cup B) \leq P(A) + P(B) \leq \frac{4}{\ell^2} E|n_1(T,\omega)|^2 + \frac{4}{\ell^2} E|n_2(T,\omega)|^2 = \frac{4}{\ell^2} (E|n_1(T,\omega)|^2 + E|n_2(T,\omega)|^2)$.

This result may be given a vector interpretation.

Let $\tilde{n}(t,\omega) \triangleq \begin{bmatrix} n_1(t,\omega) \\ n_2(t,\omega) \end{bmatrix}$.

Let $||\tilde{n}(t,\omega)|| \triangleq |n_1(t,\omega)| + |n_2(t,\omega)|$

Claim: $||\cdot||$, as defined above, is a norm.

Proof:

1) $||\lambda\tilde{x}|| = \left|\left| \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix} \right|\right| = |\lambda x_1| + |\lambda x_2| = |\lambda|(|x_1| + |x_2|) = |\lambda| \cdot ||\tilde{x}||, \quad \lambda \varepsilon \quad .$

2) $||\tilde{x} + \tilde{y}|| = \left|\left| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right|\right| = \left|\left| \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} \right|\right| =$

$|x_1 + y_1| + |x_2 + y_2| \leqq |x_1| + |y_1| + |x_2| + |y_2| =$

$(|x_1| + |x_2|) + (|y_1| + |y_2|) = ||\tilde{x}|| \quad ||\tilde{y}||.$

3) $||\tilde{x}|| = |x_1| + |x_2| \geqq 0 \quad \tilde{x}.$

Now, $0 = ||\tilde{x}|| = |x_1| + |x_2|$

$$\left.\begin{array}{l} 0 \leqq |x_1| \leqq |x_1| + |x_2| = 0 \Longrightarrow |x_1| = 0 \Longrightarrow x_1 = \theta \\ 0 \leqq |x_2| \leqq |x_1| + |x_2| = 0 \Longrightarrow |x_2| = 0 \Longrightarrow x_2 = \theta \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{l} x_1 = \theta_1 \\ x_2 = \theta_2 \end{array}\right\} \Longrightarrow \tilde{x} = \theta.$$

Therefore $||\cdot||$, as defined, is a norm.

Also notice that $E|n_1(T,\omega)|^2 + E|n_2(T,\omega)|^2 =$

$$E\ [n_1(T,\omega), n_2(T,\omega)] \begin{bmatrix} n_1(T,\omega) \\ n_2(T,\omega) \end{bmatrix} = E[\tilde{n}^T(T,\omega)\tilde{n}(T,\omega)].$$

The 2-vector result is the following inequality:

$P\{\omega: \sup\limits_{0 \leqq t \leqq T} ||\tilde{n}(t,\omega)|| \geqq \ell\} \leqq \frac{4}{\ell^2} \cdot E[\tilde{n}^T(T,\omega)\tilde{n}(T,\omega)].$ This result immediately generalizes to the m-vector martingale inequality.

$$P\{\omega: \sup_{0 \leq t \leq T} ||\tilde{n}(t,\omega)|| \geq \ell\} \leq \left(\frac{m}{\ell}\right)^2 E[\tilde{n}^T(T,\omega)\tilde{n}(T,\omega), \text{ where}$$

$$||\tilde{n}(t,\omega)|| \triangleq \sum_{j=1}^{m} |n_1(t,\omega)| \text{ and } \tilde{n}(t,\omega) = \begin{bmatrix} n_1(t,\omega) \\ n_2(t,\omega) \\ \vdots \\ n_m(t,\omega) \end{bmatrix}.$$

This proof has shown how the techniques of measure theory are useful in a thorough investigation of some of the properties of stochastic integral equations. Even more of the techniques of measure theory are employed in obtaining useful results in the area of nonlinear filtering.

CHAPTER IV

A STOCHASTIC IDENTIFICATION AND MODELING PROBLEM

In a 1967 article, Mathematical Problems of Modeling Stochastic Nonlinear Dynamic Systems, Section VIII: "Modeling the Real World," Mortensen has presented an example of a deterministic linear system that is altered to take into account the random forces in the environment by the addition of noise terms to the deterministic model. Mortensen then shows that the resulting stochastic differential equations yield paradoxical results. Mortensen suggests that since the ultimate objective of setting up a mathematical model is to obtain a predicted output of the model that is an acceptable approximation to the actually observed output of the physical system one is attempting to model, the safest approach is to "throw away the deterministic model and remodel the whole problem, with the objective being to get the statistics of the output of a Monte Carlo computer simulation to agree with the statistics of the observed data from the physical system."

It is with Mortensen's example and suggestion in mind that this present research approach is taken.

REFERENCES

Anderson, B. D. O., "A System Theory Criterion for Positive Real Matrices," J.S.I.A.M. Control, Vol. 5, pp. 171-182, May 1967.

Aoki, M., Optimization of Stochastic Systems, Academic Press, New York, 1967.

Athans, M., and Falb, P., Optimal Control, McGraw-Hill Book Co., Inc., New York, 1966.

Balakrishnan, A. V., "Matched Filters for Multiple Processes," I.R.E. Trans. on Information Theory, Vol. IT-7, January 1961.

Bartle, R. G., The Elements of Integration, John Wiley & Sons, Inc., New York, 1966.

Berberian, S., Measure and Integration, Publication of the Dept. of Math., State University of Iowa, Iowa City, Iowa, 1962.

Bucy, R. S., and Joseph, P. D., Filtering for Stochastic Processes with Applications to Guidance, Interscience Publishers, A Division of John Wiley & Sons, Inc., New York, 1968.

Churchill, R. V., Operational Mathematics, Second Edition, McGraw-Hill Book Co., Inc., New York, 1958.

Cramer, C., and Jensen, D. "Fundamentals of Multivariate Analysis Part I: Inference About Means," Journal of Quality Technology, 1969.

Cramér, H., and Leadbetter, M. R., Stationary and Related Stochastic Processes, John Wiley & Sons, Inc., New York, 1967.

Davis, M. C., "Factoring the Spectral Matrix," I.E.E.E. Trans. on Automatic Control, Vol. AC-12, pp. 296-305, October 1963.

DeRusso, P. M., Roy, R. J., and Close, C. M., State Variables for Engineers, John Wiley & Sons, Inc., New York, 1967.

Dettman, J. W., Applied Complex Variables, The Macmillan Company, New York, 1965.

Doob, J. L., Stochastic Processes, John Wiley & Sons, Inc., New York, 1953.

Dynkin, E. B., Theory of Markov Processes, Translated by D. E. Brown, Edited by T. Kövåry, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1961.

Feller, W., An Introduction to Probability Theory and Its Applications, Vol. II, John Wiley & Sons, Inc., New York, 1966.

Frost, P. A., "Nonlinear Estimation in Continuous Time Systems," Ph.D. Thesis, Standford University, 1968.

Gantmacher, F. R., The Theory of Matrices, Vol. 1, Chelsea Publishing Co., New York, 1959.

Gilbert, E., "Controllability and Observability in Multivariable Control Systems," J.S.I.A.M. Control, Ser. A., Vol. 2, No. 1, pp. 128-151, 1963.

Goldberg, R. R., Methods of Real Analysis, Blaisdell Publishing, Waltham, Mass., 1964.

Greever, J., Theory and Examples of Point-Set Topology, Brooks/Cole Publishing Co., Belmont, California, 1967.

Gupta, S. C., Transform and State Variable Methods in Linear Systems, John Wiley & Sons, Inc., New York, 1966.

Halmos, P. R., Measure Theory, D. Van Nostrand Company, Inc., Princeton, N.J., 1950.

Ito, K., and McKean, H. P., Diffusion Processes and Their Sample Paths, Academic Press, Inc., New York, 1965.

Jazwinski, A. H., Stochastic Processes and Filtering Theory, Academic Press, New York, 1970.

Johnson, N., and Leone, F. C., Statistics and Experimental Design, Vol. I, John Wiley & Sons, Inc., New York, 1964.

Kailath, T., "An Innovations Approach to Least-Squares Estimation Part I: Linear Filtering in Additive White Noise," I.E.E.E. Trans. on Automatic Control, Vol. AC-12, pp. 646-655, December 1968.

Kailath, T., and Frost, P., "An Innovations Approach to Least-Squares Estimation Part II: Linear Smoothing in Additive White Noise," I.E.E.E. Trans. on Automatic Control, Vol. AC-12, pp. 655-660, December 1968.

Kalman, R. E., "Mathematical Description of Linear Systems," J.S.I.A.M. Control, Ser. A., Vol. 1, No. 2, 1963.

Kalman, R. E., "Irreducible Realizations and the Degree of a Rational Matrix," J. Soc. Indust. Appl. Math., Vol. 13, No. 2, June 1965.

Karlin, S., A First Course in Stochastic Processes, Academic Press, New York, 1966.

Khinchin, A. I., "Teoriya Korrelyatsii Statsionarykh Sluchaynykh Protsessov" (Theory of Correlation of Stationary Random Processes), Uspekhi Matem. Nauk, No. 5, 1938.

Kras, G., and Pötzl, H., "Limiting Conditions on the Correlation Properties of Random Signals," I.R.E. Trans. on Circuit Theory, Vol. CT-3, pp. 282-285, December 1956.

Kreyszig, E., Advanced Engineering Mathematics, John Wiley & Sons, Inc., New York, 1962.

Kuo, B. C., Analysis and Synthesis of Sampled-Data Control Systems, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.

Kuo, B. C., Linear Networks and Systems, McGraw-Hill Book Co., Inc., New York, 1967.

Laning, J. H., and Battin, R. H., Random Processes in Automatic Control, McGraw-Hill Book Co., Inc., New York, 1956.

Lee, E. B., and Markus, L., Foundations of Optimal Control Theory, John Wiley & Sons, Inc., New York, 1967.

Levin, M., "Generation of a Sampled Gaussian Time Series Having a Specified Correlation Function," I.R.E. Trans. on Information Theory, Vol. IT-2, pp. 545-548, December 1960.

Loève, M., Probability Theory, Third Edition, D. Van Nostrand Company, Inc., Princeton, N.J., 1963.

Meditch, J. S., Stochastic Optimal Linear Estimation and Control, McGraw-Hill Book Co., Inc., New York, 1969.

Miller, R. S., Multidimensional Gaussian Distributions, John Wiley & Sons, Inc., New York, 1964.

Mortensen, R. E., "Mathematical Problems of Modeling Stochastic Nonlinear Dynamic Systems," N.A.S.A. Contract Report, No. NA52-4553, T.R.W. Group, Redondo Beach, California.

Munroe, M. E., Introduction to Measure and Integration, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959.

Newcomb, R. W., Linear Multiport Synthesis, McGraw-Hill Book Co., Inc., New York, 1966.

O'Donnell, J. J., "Investigations of Problems in the Optimal Control of Linear Multidimensional Systems," Ph.D. Thesis, Carnegie Institute of Technology, Pittsburgh, Pa., 1963.

Papoulis, A., Probability, Random Variables, and Stochastic Processes, McGraw-Hill Book Co., Inc., New York, 1965.

Parthasarathy, K. R., Probability Measures on Metric Spaces, Academic Press, New York, 1967.

Parzen, E., Stochastic Processes, Holden-Day, Inc., San Francisco, 1962.

Pervin, W. J., Foundations of General Topology, Academic Press, New York, 1964.

Rozanov, Y. A., Stationary Random Processes, Translated by A. Feinstein, Holden-Day, Inc., San Francisco, 1963.

Rudin, W., Real and Complex Analysis, McGraw-Hill Book Co., Inc., New York, 1966.

Saks, S., Theory of the Integral, Translated by L. C. Young, G. E. Stechert & Co., New York, 1937.

Schwarz, R. J., and Friedland, B., Linear Systems, McGraw-Hill Book Co., Inc., New York, 1965.

Solodovnikov, V. V., Introduction to the Statistical Dynamics of Automatic Control Systems, Translation Edited by J. B. Thomas and L. A. Zadeh, Dover Publications, Inc., New York, 1960.

Varadhan, S. R. S., Stochastic Processes, Courant Institute of Mathematical Sciences, New York University, 1968.

Walljasper, R. B., "The Investigations of Noise Covariance and Its Effect on Optimal Linear Stochastic Control Systems," Ph.D. Thesis, University of Iowa, Iowa City, Iowa, 1970.

White, A. J., Real Analysis: An Introduction, Addison-Wesley Publishing Company, Reading, Mass., 1968.

Wonham, W. M., "Random Differential Equations in Control Theory," in Probabalistic Methods in Applied Mathematics, Edited by A. T. Bharucha-Reid, Vol. 2, Academic Press, New York, 1970.

Wozencraft, J. M., and Jacobs, I. M., Principles of Communication Engineering, John Wiley & Sons, Inc., New York, 1965.

Wylie, C. R., Advanced Engineering Mathematics, Third Edition, McGraw-Hill Book Co., Inc., New York, 1966.

Youla, D. C., "The Theory and Design of Multiple Channel Matched Filters," Atlantic Res. Corp., Alexandria, Va., June 25, 1959.

Youla, D. C., "On the Factorization of Rational Matrices," I.R.E. Trans. on Information Theory, Vol. IT-7, pp. 172-189, July 1961.