A STABLE DECENTRALIZED FILTERING IMPLEMENTATION
FOR JTIDS RELNAV

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ABSTRACT

Eight alternative decentralized linear estimation schemes are surveyed to find the mechanization most appropriate for stable community relative navigation in the JTIDS RelNav tactical scenario. Advantages, disadvantages and caveats are identified for each. The Sequentially Partitioned Algorithm (SPA) is identified as a primary candidate and the "expanded" variant of Surely Locally Unbiased (SLU) filtering as a suggested back-up. This paper presents the discrete-time mechanization equations for SPA and SLU, both in general terms and as specialized for the JTIDS RelNav application. The paper also outlines an analytic proof of asymptotic stability for both SPA and SLU. The computer requirements for SPA and SLU filters are also estimated in terms of core memory requirements and operation counts. Further conclusions await the results of simulation testing in worst case nonlinear situations.

1. INTRODUCTION AND UPDATE TO RELNAV STABILITY

Use of the RelNav Algorithm, applied in a multimember community, is currently configured for one user as depicted with solid lines in Figure 1. For a net of users, RelNav involves the repeated use of multiple Extended Kalman Filters (p.79 of Ref.1) with associated transit and processing time delays. There is also partial information exchange between net members including pertinent approximate statistics indicating heading, position, and time qualities. As discussed in more detail below, there are some indications that the RelNav algorithm exhibits some anomalous stability behavior. The purpose of this paper is to present the results of recent theoretical research that may be applicable to resolving these problems. The goal of the study is to identify stable estimation algorithms appropriate for the JTIDS RelNav application.

Historical Summary of JTIDS Stability Results

Several investigators (2-5) have demonstrated that the JTIDS RelNav algorithms can be unstable. As stated in the Appendix, p.31 of (2), "A Time Division Multiple Access (TDMA) communications system with interacting elements synchronizing and determining position location is inherently unstable. . . However, it has been maintained that through proper filtering and control, the instabilities could be controlled and the advantages of the increased navigation coverage realized." . . . "What subsequently evolved [as claimed in (2) , but not exactly what is currently mechanized in JTIDS RelNav] is an algorithm that is basically stable." (The modifications include several changes from a previous philosophy, notably, establishing "a hierarchy based on data quality that prevents [accuracy degradation due to] interaction" by "allowing elements [net members] to use only sources with better data quality than their own "as discussed further below.)

Over the years steps have been taken (6) to remedy the tendency towards instabilities such as:

- Specification of a user hierarchy for data exchange within the JTIDS grid,
- Specification of criteria for data exchange as contained in transmitted P-messages, based on information pertaining to statistics of data quality,
- Designation of heirarchy levels and functions (e.g., Navigation Controller, Time Reference, Geodetic Reference, Primary and Secondary Users),
- Specification of which users can perform Round Trip Timing (RTT).

There are indications, however, that some situations still exist where the user accuracy provided by the JTIDS grid can still go unstable (3,7,20). Ref. 7, however, suggests that user instability may still occur due to:

- Unobservable rotation of the reference grid,
- Bad user geometry (i.e., bad GDOP),
- Presence of slow moving vehicles,
- Sequential user utilization of degrading information.

Two approaches to decentralized estimation, first conceived in the early 1970's appear to provide the requisite stability properties. The selection and application of these techniques to JTIDS RelNav is the topic of this paper.

Overview

After eliminating several other candidate decentralized filtering approaches (depicted in two left columns of Table 1) from further consideration for JTIDS, both the so-called Sequential Partitioned Algorithm (SPA) and the Surely Locally Unbiased (SLU)

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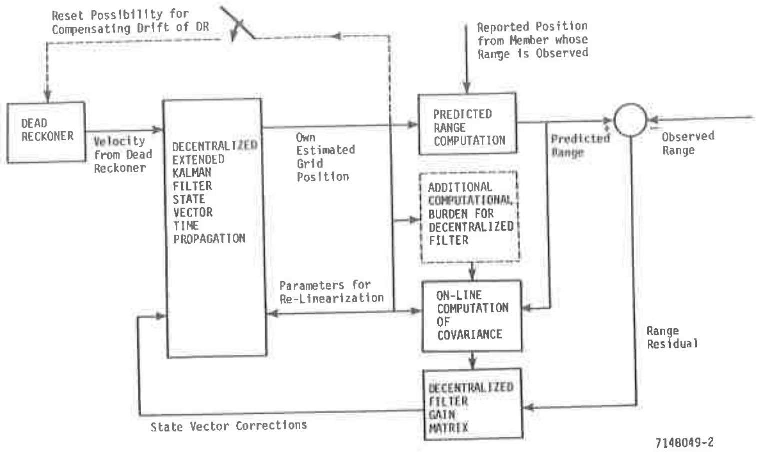

FIGURE 1: Information Flow For Mechanization Of Recommended Decentralized Filtering Candidates

TABLE 1

SUMMARY STATUS OF ALTERNATIVE DECENTRALIZED FILTERING APPROACHES FOR JTIDS RELNAV

PRIMARY INVESTIGATORS	EXHAUSTIVE OPEN LITERATURE DESCRIPTIONS*	COMPATIBLE WITH JTIDS RELNAV	REASONS FOR INCOMPATIBILITY PROVIDED IN	OVERVIEW COMMENTS
J. D. Pearson	Ref. 21 (1970)	NO	Section 3	Historically significant treatment of general structural framework and indication of problem areas or barriers to be overcome in the future.
M. Shah	Ref. 14[+] (1971)	YES		• Accommodates time-varying models • Allows analytic proof of stability (Section 5) for the linear case • "Biased" (but indications are that this aspect can be adequately handled) • Anticipated computer burden qualified (section 5) • Accommodates EKF approach to nonlinear filtering (Section 6)
C. Sanders E.C. Tacker T.D. Linton R.Y.-S. Ling	Ref. 8[≠] (1973) Ref. 9 (1973) Ref. 10 (1976) Ref. 11 (1978) Ref. 12 (1979)	Unmodified SLU: NO ROM: NO ESLU: MAYBE ESLU°: MAYBE Other: 7 Variations: NO	Section 4 Section 4 Section 4	• Accommodates time-varying models • Allows analytical proof of stability (Section 5) for the linear case • Exploits measurement structure when possible for computational savings • Unmodified SLU requires a special SVD matrix factorization • Unmodified SLU is "unbiased" • Anticipated computer burden quantified (Section 5) • Accommodates EKF approach to nonlinear filtering (Section 6)
M.K. Sundareshan	Ref. 26 (1977)	NO	Section 3 (following Eq. 3-13)	• No subsystem interconnections[§] allowed in the measurement model • Crucial matrix C^P_{ij} not explicitly defined
M.F. Hassan G. Salut M.G. Singh A. Titli	Ref. 29 (1976) Ref. 28 (1978) Ref. 13 (1978)	NO	Section 3	• No subsystem interconnections[§] allowed in the measurement model
D.D. Siljak M.B. Vukcevic	Ref. 30 (1978)	NO	Section 3 (following Eq. 3-13) Appendix A Section A.3	• No subsystem interconnections[§] allowed in the measurement model
E. Verriest B. Friedlander M. Morf	Ref. 31 (1979)	NO	Section 3 (following Eq. 3-13) Appendix A Section A.3	• Assumes a central node (contrary to JTIDS tactical policy) that assembles local estimates into global estimates
J. L. Speyer T. S. Chang	Ref. 32 (1979) Ref. 33 (1980)	NO	Section 3 (following Eq. 3-13)	• Framework is only for both estimation and LQG feedback regulation control (where control in the LQG sense and not in the sense of C^2 is of no apparent interest in RelNav)

[*] No known concrete applications or defense related prior considerations of these recent theoretical developments.

[+] Designated as the Sequentially Partitioned Algorithm (SPA) in Ref. 13.

[≠] Designated as the Surely Locally Unbiased (SLU) filter (with 10 modified variations of the original formulation)
including: ROM filter - Reduced Order Model
ESLU filter - Expanded Surely Locally Unbiased
ESLU° filter - (same as above) but without explicitly modeling communications noise (assumed negligible

[§] A transformation (detailed in Appendix A) exists for converting general output measurement coupled subsystems into "output decoupled" subsystems but technique only applicable to time-invariant structure (inconsistent with JTIDS RelNav) and must be applied to aggregate system (with an associated harsh computer burden as discussed in Appendix A.3).

decentralized filtering approaches--that have recently evolved (8-14) along with, but independent of, the JTIDS RelNav application--are investigated for direct relevance to JTIDS in Section 4. Since all previous results and mechanization equations for the SLU approach had only been developed for a continuous-time formulation, the discrete-time mechanization equations (derived in Ref.15) that are consistent with the intended application are presented in Table 2. The modest computational impact of introducing these results from decentralized filtering theory into the current RelNav configuration can be depicted as just the one additional block in Figure 1, representing a software accommodation. A particular viewpoint is adopted herein as decentralized estimation approaches are sought which are compatible with the JTIDS RelNav objectives and constraints. However, general advantages/disadvantages and caveats are provided for all the decentralized estimation algorithms reviewed.

A notable feature of both the SLU and SPA decentralized filtering approaches is that asymptotic stability of the estimation algorithms is provided. This stability aspect had been previously demonstrated analytically for only the continuous-time SLU formulation (10). Stability of the discrete-time SLU and SPA formulations is analogously demonstrated in (15) and outlined here in Section 5, by utilizing an independent previously established stability framework (10,15) that is supplemented with new results from (15,17-19).

The computational burden associated with both SLU and SPA filtering mechanizations is investigated in Section 5 for compatibility with RelNav. Finally, further conclusions are drawn in Section 6 as to which decentralized filtering approach appears to be most appropriate for JTIDS RelNav based on a summary of the theoretical considerations examined within this study.

2. APPROPRIATE DISCRETE-TIME MODELS FOR DECENTRALIZED FILTERING

Consider the following collection $\{S_i,\ i=1,2,\ldots,N\}$ of N interconnected dynamical subsystems (as in 8-14):

$$S_i:\ \dot{x}_i(t) = F_i(t)\overset{(n_i\times 1)}{x_i(t)}+L_{ii}(t)u_i(t)+w_i{}'(t) \tag{2-1}$$

having discrete measurements available to the i^{th} subsystem S_i of the form:

$$\overset{(q_i\times 1)}{z_i(t_k)}=H_i(t_k)\underline{x}(t_k)+v_i(t_k) \tag{2-2a}$$

$$=[\bar{H}_i(t_k)\ \vdots\ \hat{H}_i(t_k)]\begin{bmatrix} P_{x,i} \\ \hline L_i(t_k)\end{bmatrix}\underline{x}(t_k)+v_i(t_k) \tag{2-2b}$$

$$=[\bar{H}_i(t_k)x_i(t_k)+\hat{H}_i(t_k)u_i(t_k)]+v_i(t_k) \tag{2-2c}$$

$$=\bar{H}_i(t_k)x_i(t_k)+\hat{H}_i(t_k)L_i(t_k)\underline{x}(t_k)+v_i(t_k) \tag{2-2d}$$

where $P_{x,i}$ is the projection operator from R^n

$$n = \sum_{i=1}^{N} n_i \tag{2-3}$$

to R^{n_i} and where the vector-valued interaction input is represented [by using the historically standard notation of weighting matrices L_{ij} popularized in (21) on p.122] by

$$\overset{(p_i\times 1)}{u_i(t)} \triangleq \overset{(p_i\times m)}{L_i(t)}\underline{x}(t)=\Big[\overset{(p_i\times 1)}{L_{i1}(t)}\ \vdots\ \overset{(p_i\times 1)}{L_{i2}(t)}\ \vdots\ \cdots\ \vdots\ L_{i,i-1}(t)\ \vdots\ \overset{(p_i\times n_i)}{0}\ \vdots\ L_{i,i+1}(t)\ \vdots\ \cdots\Big]\underline{x}(t) \tag{2-4a}$$

$$=\sum_{\substack{j=1\\ j\neq i}}^{N} \overset{(p_i\times n_j)}{L_{ij}(t)}\ \ \overset{(n_j\times 1)}{x_j(t)} \tag{2-4b}$$

Notice that $u_i(t)$ has no direct component of $x_i(t)$.

The process and measurement noises $w_i(t)$ and $v_i(t)$ are assumed [as in (8-14)] to be independent, zero mean, white Gaussian noises having associated covariance matrices $Q_i'(t)$ and $R_i(t)$, respectively, and uncorrelated with the Gaussian initial condition

$$x_i(0)\sim N(\underline{0},\mathring{P}_i) \tag{2-5}$$

Also $w_i(t)$ and $v_i(t)$ are assumed to be uncorrelated with the noises and initial conditions of other subsystems (viz., $w_j(t)$, $v_j(t)$, and $x_j(0)$ for $j\neq i$).

It is fairly routine to obtain the exact discrete-time representation of Eq.2-1 [p.171 of (19)] as

$$S_i:\ x_i(k+1)=\Phi_{ii}(k+1,k)x_i(k)+\mathring{L}_{ii}(k)u_i(k)+\mathring{w}_i(k) \tag{2-6}$$

with

$$\mathring{w}_i(k) \triangleq \int_{k\Delta}^{(k+1)\Delta}\Phi_{ii}((k+1)\Delta,s)w_i'(s)\ ds \tag{2-7}$$

$$\mathring{Q}(k)\triangleq E\left[\mathring{w}_i(k)\mathring{w}_i^T(k)\right]=\int_{k\Delta}^{(k+1)\Delta}\Phi_{ii}((k+1)\Delta,s)\ Q_i'(s)\ \Phi_{ii}^T((k+1)\Delta,s)ds \tag{2-8}$$

where $\Phi_{ii}(\cdot,\cdot)$ is the appropriate component of the transition matrix associated with $F_i(t)$. While not acknowledged in (8-12), it is also straightforward to establish [as in (15)] that the aggregate global solution of Eq.2-1 ($i=1,\ldots,N$) can be quite different from the i^{th} local subsystem's solution of Eq. 2-1, except for special cases such as

$$L_{\ell\ell}(t) = 0 \quad \text{for } \ell = 1,2,\ldots,N \tag{2-9}$$

as is fortunately satisfied in JTIDS RelNav [cf., Eqs.4-9,4-18].

3. FACTORS IN RESTRICTING CANDIDATES TO ONLY THE SPA AND SLU APPROACHES

The implicit motivation underlying all hierarchical approaches in systems theory is the pervading idea that it is generally easier to handle several low order subsystems than one aggregate system of high order (22). The fundamental idea is to decompose the large system into subsystems and then manipulate the smaller subsystems in such a way that the objectives of the overall system are met.

In the case of decentralized large-scale system applications that deal exclusively with the specification of adequate deterministic control inputs, the objective is to cause the aggregate of local subsystem control solutions to also be the global solution. This objective is frequently accomplished by coordination. For applications of decentralized control, a few of the more common approaches for implementing coordination (22) are:

- The Prediction Principle (23),
- The Balance Principle (23,24),
- Use of Penalty Functions (25).

In stark contrast to the decentralized control problem, the decentralized estimation problem has, in general, no mechanism for enforcing interconnection constraints, since no control is involved. Historically, the mathematical structure of both decentralized control and estimation was examined in exacting detail by Pearson in (21) where the following observations were made:

- Significant computational simplifications accrue in the control problem with quadratic cost function when all the subsystems are linear [pp.152-153 of (21)],
- Techniques exist for analytically proving proper coordination via iteration between aggregates of linear subsystems via a contraction mapping argument [pp.182-188 of (21)], but this approach is appropriate only over a very brief time interval $[t_o, t_1]$,
- An approach to decentralized filtering simplifies the computational coordination requirements when a suboptimal rule is used [p.182 of (21)].

During a critical examination on pp.533-534 of (13) (as motivation for offering an alternate decentralized filtering technique) it is noted that to solve an implementation of Pearson's decentralized filter "in practice would require knowledge of the sequence of observations over the entire time interval k=0 to k=N_0 which is inconsistent with the tenents of sequential estimation since these are unavailable a priori." While possibly "useful for parameter estimation (i.e., identification), Pearson's decentralized estimation approach is not of much significance for state estimation."

The approach pioneered by Sanders in (8)--of restricting the local subsystem's filter to be of the so-called Surely Locally Unbiased (SLU) class--appears to follow through on the predictions of (21), where a suboptimal rule was called for to simplify the computational burden. Similarly, Shah's Sequential Partioned Algorithm (SPA) approach to decentralized filtering reported in (13,14) also utilizes a simplifying suboptimal rule.

For completeness, it is noted that another relatively recent approach (26) exists for implementing decentralized estimation. However, this approach is only applicable to subsystems of the following restrictive form:

$$S_i:\ \overset{(n_i x1)}{\dot{x}_i}(t)=F_i\ x_i(t)+w'_i(t)+\sum_{j=1}^{N} L_{ii}\ L_{ij}\ x_j(t) \qquad (3\text{-}1)$$

where the

$L_{ii}\ L_{ij}$ are time-invariant for $i,j=1,2,\ldots,N$ (3-2)

and with all local measurement structures having no interconnection effects as modeled by

$$z_i(t_k)=\bar{H}_i\, x_i(t_k)+v_i(t_k) \qquad (3\text{-}3)$$

with contributions due to other subsystems $x_j(t_k)$ ($j\neq i$) absent in the above. Consequently, the associated augmented system has the following form:

$$\overset{(nx1)}{\underline{\dot{x}}}(t)=\{\mathrm{diag}(F_1,F_2,\ldots,F_N)+C\}\underline{x}(t_k)+\underline{w}'(t) \qquad (3\text{-}4)$$

and

$$\underline{z}(t_k)=\mathrm{diag}(\bar{H}_1,\bar{H}_2,\ldots,\bar{H}_N)\underline{x}(t_k)+v(t_k) \qquad (3\text{-}5)$$

where

$$\underline{z}(t_k)\triangleq\left[z_1^T(t_k),\ z_2^T(t_k),\ldots,z_N^T(t_k)\right]^T \qquad (3\text{-}6)$$

$$v(t_k)\triangleq\left[v_1^T(t),\ v_2^T(t),\ldots,v_N^T(t)\right]^T \qquad (3\text{-}7)$$

and, as defined in (26,27) the composite interconnection matrix is:

$$C\triangleq\left[L_{ii}\ L_{ij}\right] \quad \text{for } i,j=1,2,\ldots,N \qquad (3\text{-}8)$$

It is asserted in Theorem 4 of (26) that the composite interconnection matrix being factorizable as

$$C=PS \qquad (3\text{-}9)$$

where

$$P\triangleq\mathrm{diag}(P_1,\ P_2,\ldots,P_N) \qquad (3\text{-}10)$$

and the P_i are the positive definite solutions of the algebraic Riccati equation

$$\underline{0}=F_iP_i+P_iF_i^T+Q_i-P_i\bar{H}_i^TR_i^{-1}\bar{H}_iP_i \qquad (3\text{-}11)$$

and

S = any arbitrary skew-symmetric matrix

is necessary and sufficient for the global optimal estimate to consist of the aggregate of local optimal estimates of the following form

$$\dot{\hat{x}}_i(t)=(F_i-K_i\bar{H}_i)\hat{x}_i(t)+K_iz_i(t)-\sum_{j=1}^{N}K_iC_{ij}^P\hat{x}_j(t) \qquad (3\text{-}12)$$

where

$$K_i\triangleq P_i\bar{H}_i^TR_i^{-1} \qquad (3\text{-}13)$$

An unfortunate oversight is that C_{ij}^P, the "appropriate perturbation of C_{ij}", is not explicitly defined in (26). A similar discussion in (27) for decentralized estimators indicated that the matrices premultiplying x_j under the summation sign in Eq.3-11 are obtained by calculations of full dimension n (as in Eq.2-3); an unacceptable computational burden for decentralized estimation within JTIDS RelNav.

This limited perspective is enough to dismiss the approach of (26) from further consideration for the JTIDS RelNav application for the following reasons:

- Eq.3-3 allows no interconnections, but linearized measurement structure for JTIDS contains components of both source and user subsystem states (1). (A specific transformation for reducing a time-invariant system with interconnections within the measurement structure into a block diagonal (uncoupled) form [as required in Eq.3-5 to apply the approach of (26)] is called "output decentralization". The impracticability of its use in the JTIDS RelNav application is discussed in Appendix A.)

- Problem formulation of Eqs.3-1 to 3-8 is for time-invariant steady-state operation only. Significance of the required factorization of Eq.3-9 is questionable in a non-stationary realistic JTIDS RelNav application.
- Computational burden of implied full n-dimensional calculations not practicable for JTIDS RelNav user terminals.

A decentralized (possibly parallel-processing) algorithm for implementing the n-dimensional global exact Kalman filter in a hierarchical manner has been studied in (28). However, the processing hierarchy is dictated by an internal system structure rather than by the JTIDS RelNav ICD protocol hierarchy (6).

Other approaches to decentralized filtering [such as (29,30)] have been surveyed, but the common requirement of having to perform a transformation to achieve "output decentralization" is objected to as incompatible with the JTIDS RelNav application [see Appendix A for details and (30) for numerical examples]. The approach of (31) is rejected outright for RelNav due to the untenable requirement of having a central processing node that would be vulnerable in the tactical environment of JITDS.

The approach of (32) strictly pertains to decentralized LQG estimation and control of a K-node system (where the K-nodes refer to K subsystems) where local filters share their information with all the other nodes. LQG refers to Linear Quadratic Gaussian applications involving linear systems with Gaussian measurement and process noises with a single quadratic performance index (cost function) for control. For the JTIDS application, there is:

- No strong interest in the feedback control aspect,
- No constant number K of subsystems,
- No sharing of information between all of the subsystems,

so the relevance of the results of (32) to JTIDS RelNav is presently perceived to be marginal despite recent simplifications in the associated filter implementation as presented in (33).

For the above reasons the relatively refined and apparently structurally compatible approaches of only SLU (8-12) and SPA (13,14) to decentralized filtering are further considered for the JTIDS RelNav application.

4. RECASTING JTIDS RELNAV STRUCTURES AND METHODOLOGY INTO DECENTRALIZED FILTERING PARAMETERS

The states modeled within the 15 state filter utilized in the Advanced Development Model (ADM) algorithm provided by Singer Corporation (Kearfott Division) are designated in Section 4.4.1.2.2 of (34) and correspond to similar states discussed in (35). The basic form of JTIDS measurements derived from the L-band P-messages that are updated via retransmission within specifically allocated time slots of the Time Division Multiple Access (TDMA) spread spectrum communication system is now described. In both active (allowing Round-Trip-Timing RTT messages for active clock synchronization as initiated by the user) and passive (user silent) modes, the observations for RelNav Time-of-Arrival (TOA) data utilization take the following general form (34)

$$\underline{z} = \begin{bmatrix} \mathrm{TOA}_i \\ \mathrm{TOA}_j \\ \mathrm{TOA}_k \end{bmatrix} \qquad (4\text{-}1)$$

where the subscripts i,j,k designate different sources of transmitted P-messages (and three sources are nominally required for a user position fix via triangulation). Specification of the specific sources that are actually used in slots i,j, and k of Eq.4-1 is currently provided by the logic of the Source Selection Subfunction (SSS).

TOA Observation Matrix [p.4-93 of (34)]

The JTIDS RelNav linearized observation matrix is of the form

$$h_i^T = \left[\alpha_i \quad \beta_i \quad 0 \quad 0 \quad \gamma_i \quad 1 \;\vdots\; 0_{(1x9)}\right] \qquad (4\text{-}2)$$

where the computed elements α , β , γ have the physical interpretation of being related to direction cosines between the user and the source. [Derivation of the form of Eq.4-2 is provided on p.79 of (1)].

RelNav Incompatibilities of Pure SLU

If three TOA measurements from different sources are represented (in a manner corresponding to Eq.4-1) as being simulataneously processed, then the corresponding observation matrix would be

$$\overset{(3x15)}{H} = \left[\begin{array}{cccccc|c} \alpha_1 & \beta_1 & 0 & 0 & \gamma_1 & 1 & \\ \alpha_2 & \beta_2 & 0 & 0 & \gamma_2 & 1 & O_{(3x9)} \\ \alpha_3 & \beta_3 & 0 & 0 & \gamma_3 & 1 & \end{array}\right] \qquad (4\text{-}3)$$

with

$$\alpha_i = \frac{\hat{P}_u^- - P_{us}}{R_{c_i}} \;;\; \beta_i = \frac{\hat{P}_v^- - P_{vs}}{R_{c_i}} \;;\; \gamma_i = \frac{\hat{P}_w^- - P_{ws}}{R_{c_i}} \qquad (4\text{-}4)$$

In the SLU filter notation of Eq.2-2d, from the point of view of a receiver on the user designated (without loss of generality) as subsystem i=4, the observation matrix of Eq.4-3 can be further decomposed into

$$\overset{(3\times15)}{\bar{H}_4} = \left[\begin{array}{cccccc:c} -\frac{\hat{P}_u^-}{R_{c_1}} & -\frac{\hat{P}_v^-}{R_{c_1}} & 0 & 0 & -\frac{\hat{P}_w^-}{R_{c_1}} & 1 & \\ \hdashline -\frac{\hat{P}_u^-}{R_{c_2}} & -\frac{\hat{P}_v^-}{R_{c_2}} & 0 & 0 & -\frac{\hat{P}_w^-}{R_{c_2}} & 1 & 0_{(3\times9)} \\ \hdashline -\frac{\hat{P}_u^-}{R_{c_3}} & -\frac{\hat{P}_v^-}{R_{c_3}} & 0 & 0 & -\frac{\hat{P}_w^-}{R_{c_3}} & 1 & \end{array}\right] \tag{4-5}$$

$$\overset{(3\times60)}{\hat{H}_4 L_4} = \left[\begin{array}{c:c:c:c} -\frac{x_1}{R_{c_1}}\ -\frac{x_2}{R_{c_1}}\ 0\ 0\ -\frac{x_5}{R_{c_1}}\ 0_{(1\times10)} & 0_{(1\times15)} & 0_{(1\times15)} & 0_{(1\times15)} \\ \hdashline 0_{1\times15} & -\frac{x_1}{R_{c_2}}\ -\frac{x_2}{R_{c_2}}\ 0\ 0\ -\frac{x_5}{R_{c_2}}\ 0_{(1\times10)} & 0_{(1\times15)} & 0_{(1\times15)} \\ \hdashline 0_{1\times15} & 0_{(1\times15)} & -\frac{x_1}{R_{c_3}}\ -\frac{x_2}{R_{c_3}}\ 0\ 0\ -\frac{x_5}{R_{c_3}}\ 0_{(1\times10)} & 0_{(1\times15)} \end{array}\right] \tag{4-6}$$

One further decomposition of Eq.4-6 into the basic building blocks identified in Eqs.2-2d, 2-4a, and 2-1 is possible as

$$\overset{(3\times3)}{\hat{H}_4(t_k)} = \begin{bmatrix} -\frac{1}{R_{c_1}} & 0 & 0 \\ 0 & -\frac{1}{R_{c_2}} & 0 \\ 0 & 0 & -\frac{1}{R_{c_3}} \end{bmatrix} \tag{4-7}$$

$$L_4(t_k) = \left[L_{41}(t_k) \,\vdots\, L_{42}(t_k) \,\vdots\, L_{43}(t_k) \,\vdots\, 0_{(3\times15)}\right] \tag{4-8a}$$

$$= \left[\begin{array}{ccccc:c:ccccc:c:ccccc:c:c} x_1 & x_2 & 0 & 0 & x_5 & & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0_{(3\times10)} & x_1 & x_2 & 0 & 0 & x_5 & 0_{(3\times10)} & 0 & 0 & 0 & 0 & 0 & 0_{(3\times10)} & 0_{(3\times15)} \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & & x_1 & x_2 & 0 & 0 & x_5 & & \end{array}\right] \tag{4-8b}$$

$$L_{44}(t) = \left[0_{(15\times3)}\right] \tag{4-9}$$

(and similarly for i=1,2,3) so that Eq.2-1 is of the simplified degenerate form:

$$S_4 : \dot{x}_4(t) = F_4(t)x_4(t) + w'_4(t) \tag{4-10}$$

representing no interconnections entering directly through the dynamics of the subsystems comprising the user model of the JTIDS RelNav application.

The RelNav subsystem interconnections are apparently only via the measurements

$$z_4(t_k) = \bar{H}_4(t_k)x_4(t_k) + \hat{H}_4(t_k)L_4(t_k)\underline{x}(t_k) + v_4(t_k) \tag{4-11}$$

In seeking to exploit the inherent JTIDS RelNav structure in the manner characteristic of the unmodified SLU filtering algorithm summarized in Table 2, the following test is applied

$$\text{rank } (H_4) = 3 (= p_4 \text{ as defined in Eq.2-4}) \tag{4-12}$$

but the row dimension of H_4 is also $q_4 = 3$ so

$$p_4 \not< q_4 \tag{4-13}$$

(i.e., since $q_4 = p_4 = 3$ there is no strict inequality as necessary for SLU to achieve actual nondegenerate exploitation of the measurement structure). Lacking an opportunity to avail the filter of an expedient exploitation of the measurement structure, a direct unmodified SLU filtering mechanization appears to be inappropriate for JTIDS RelNav.

Caveat: In (9) comment occurs prior to Eq.7 that the rank condition being satisfied "roughly speaking" means "that at least one element of the state vector of each user is measured and that each of the interactions to each user has a direct effect on the local measurement obtained by the user." Similarly on p.17 of (8), the above loose interpretation is reinforced in stating that the rank condition "means intuitively that the user can observe all the interactions to his unit." Both the above two intuitive structural requirements are adhered to in the JTIDS RelNav application. While the equivalence of Condition 1 and 1' have been validated in (15), the verification of the "loose interpretations" implying compliance with Conditions 1 and 1' cannot be verified and are not satisfied by JTIDS RelNav as demonstrated in Eq.4-13.

RelNav Possibilities of Expanded Surely Locally Unbiased (ESLU) Filtering

It is theoretically possible to improve the performance of each local filter of an SLU implementation [pp.41,42 of (12)] merely by allowing each subsystem (e.g., RelNav user) access to additional information as transmitted by other subsystems (RelNav users) over a communications channel (noisy or noiseless) and consequently to also use an expanded local system while continuing to maintain the usual SLU constraints (as implicitly contained in the implementation equations of Table 2). On p. 42 of (12) the precise definition of an expanded local subsystem is provided via recourse to Eqs.2-1, 2-4b.

The so-called Expanded Surely Locally Unbiased (ESLU) version of SLU is only computationally satisfactory as a true SLU filter if the associated direct sum of

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_N = \text{diag}\{H_1, H_2, \ldots H_N\} \tag{4-14}$$

(where the $H_i(k)$ are defined in Eq.2-2a) along with other obvious replacements [discussed in (12) and on p.4-19 of (15)] still satisfy the SLU conditions. For the JTIDS RelNav application, the observation matrix consisting of the direct sum has the following form:

$$H = \text{diag}\Bigg\{\underbrace{\left[\begin{array}{cccccc:c:cccccc:c:c} \alpha_1 & \beta_1 & 0 & 0 & \gamma_1 & 1 & & \alpha_1' & \beta_1' & 0 & 0 & \gamma_1' & 1 & & \\ \alpha_2 & \beta_2 & 0 & 0 & \gamma_2 & 1 & 0_{(3\times9)} & \alpha_2' & \beta_2' & 0 & 0 & \gamma_2' & 1 & 0_{(3\times9)} & \\ \alpha_3 & \beta_3 & 0 & 0 & \gamma_3 & 1 & & \alpha_3' & \beta_3' & 0 & 0 & \gamma_3' & 1 & & \cdots \end{array}\right]}_{N}\Bigg\} \tag{4-15}$$

While ESLU is also summarized in Table 2, its final significance for JTIDS RelNav is yet to be conclusively determined.

RelNav Implementation of an SPA Filter

Returning now to consider the details of SPA implementation for the specific JTIDS RelNav structure as identified in Eqs.4-3 to 4-9, there are only two equations that differ from a standard Kalman filter mechanization (as noted in Table 3). The form of the two equations, respectively, is

TABLE 2: SUMMARY OF DISCRETE-TIME MECHANIZATION EQUATIONS OF SURELY LOCALLY UNBIASED (SLU) FILTERING

Order of Calculations	Mechanization Equations* for the General Case	Mechanization Equations Specialized to JTIDS RelNav	
		"Unmodified" SLU	"Expanded" version of SLU
Step 1	SVD algorithm performs the following factorization: $\underset{(q_i\times n_i)}{\bar{H}_i} = \underset{(q_i\times q_i)}{U_i}\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_{p_i} \\ \hline & 0 & \end{bmatrix}\underset{(p_i\times p_i)}{V_i^T}$ Upon making the following assignments: $z_{i2} \triangleq U_i \cdot \begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_{q_i}\end{bmatrix}, \quad z_{i1} \triangleq V_i^T$ hence+ $z_{i2}^{-1}(k)\bar{H}_i(k)z_{i1}(k) = \begin{bmatrix} I_{p_i} \\ \hline 0 \end{bmatrix}$ for the condition $p_i < q_i$ (as confirmed/denied from the factorization)	SVD performs same factorization on H_i but $p_i = q_i = 3$	SVD performs factorization on $\hat{H}$ as obtained from the decomposition (as in Eq. 4-14) of $H \triangleq H_i \oplus H_1 \oplus H_2 \oplus H_3 = \mathrm{diag}\{H_i, H_1, H_2, H_3\}$ (unknown but) likely that $p_i < q_i$
Step 2	$\dot{L}_{ii} \longrightarrow \dot{L}_{ii}(k)z_{i1}(k) \triangleq \dot{L}_{ii}'(k)$	Not necessary	Same as general case
Step 3	$L_i \longrightarrow z_{i1}^{-1}(k)L_i(k) \triangleq L_i'(k)$	Not necessary	Same as general case
Not necessary in practice	$v_i(k) \longrightarrow z_{i2}^{-1}(k)v_i(k) \triangleq v_i'(k)$	Not necessary	Same as general case
Step 4	$\bar{H}_i(k) \longrightarrow z_{i2}^{-1}(k)\bar{H}_i(k) \triangleq \bar{H}_i'(k)$ $\hat{H}_i(k) \longrightarrow \begin{bmatrix} I_{p_i} \\ \hline 0 \end{bmatrix}$	Not necessary $\hat{H}_i(k) \rightarrow I_{p_i}$	Same as general case
Step 5	$z_i(k) \longrightarrow z_{i2}^{-1}(k)z_i(k) \triangleq z_i'(k)$	Not necessary	Same as general case
Step 6	$R_i(k) \longrightarrow z_{i2}^{-1}(k)R_i(k)z_{i2}^{-T}(k) \triangleq R_i'(k)$ Positioning and Assigning as: $R_i'(k) = \begin{bmatrix} R_{11}'(k) & R_{12}'(k) \\ \hline R_{21}'(k) & R_{22}'(k) \end{bmatrix}$ p_i, $q_i - p_i$; $\underset{(q_i\times 1)}{z_i'(k)} = \begin{bmatrix} z_{i1}'(k) \\ \hline z_{i2}'(k) \end{bmatrix}$	Not necessary	Same as general case
Step 7	$P_i(k+1\|k) = [\Phi_{11} - \hat{L}_{11}'\bar{H}_{12}']P_i[\Phi_{11} - \hat{L}_{11}'\bar{H}_{12}']^T + [\hat{Q}_i + \hat{L}_{11}'R_{11}'\hat{L}_{11}'^T]$ $- [(\Phi_{11} - \hat{L}_{11}'\bar{H}_{11}')P_i\bar{H}_{12}'^T - \hat{L}_{11}'R_{12}'](\bar{H}_{12}'P_i\bar{H}_{12}'^T + R_{12}')^{-1}[(\Phi_{11} - \hat{L}_{11}'\bar{H}_{11}')P_i\bar{H}_{12}'^T - \hat{L}_{11}'R_{12}']^T$	Degenerates to Standard Kalman Filter	Same as general case
Step 8	$K_{12}^{*}(k) = [\Phi_{11}(k+1,k)P_i^-(k\|k-1)\bar{H}_{12}'^T - \hat{L}_{11}(\bar{H}_{11}'P_i^-(k\|k-1)\bar{H}_{12}'^T + R_{12}'(k))](\bar{H}_{12}'(k)P_i^-(k\|k-1)\bar{H}_{12}'^T + R_{12}'(k))^{-1}$	Degenerates to Standard Kalman Filter	Same as general case
Step 9	$\hat{x}_i(k+1\|k) = \Phi_{11}(k+1,k)\hat{x}_i(k\|k-1) + \hat{L}_{11}(k)[z_{11}'(k) - \bar{H}_{11}'(k)\hat{x}_i(k\|k-1)] + K_{12}(k)[z_{12}'(k) - \bar{H}_{12}'(k)\hat{x}_i(k\|k-1)]$	Degenerates to Standard Kalman Filter	Same as general case

* This descrete-time formulation (derived in Ref. 15 and previously unavailable as acknowledged on p. 82 of Ref. 12) was necessary before computational burden of Table 4 could be quantified.

\+ This factorization was theoretically guaranteed to be possible in Refs. 8-12, but accomplishment of this task in the general case by SVD first identified here and in Ref. 15. Validated SVD software is available for this task (Refs. 38,39).

TABLE 3: SUMMARY OF DISCRETE-TIME MECHANIZATION EQUATIONS OF SEQUENTIALLY PARTITIONED ALGORITHM (SPA)

Order of Calculations	Mechanization Equations for the General Case*	Mechanization Equations Specialized to JTIDS RelNav
Step 1	$\bar{x}_i(k+1\|k) = \Phi_{ii}(k+1,k)\bar{x}_i(k\|k)$	Same as general case and Standard Kalman Filter
Step 2	$Q_{ii}^*(k) \triangleq E[w_i^*(k)w_i^{*T}(k)] = Q_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Phi_{ij}(k+1,k)P_j(k\|k)\Phi_{ij}^T(k+1,k)$	$w_i^*(k) \triangleq w_i(k)$ $Q_{ii}^*(k) \triangleq E[w_i^*(k)w_i^{*T}(k)] = E[w_i(k)w_i^T(k)] = Q_{ii}(k)$
Step 3	$P_{ii}'(k+1\|k) = \Phi_{ii}(k+1,k)P_{ii}^*(k\|k)\Phi_{ii}^T(k+1,k) + Q_{ii}^*(k)$	Same as general case and Standard Kalman Filter
Step 4	$R_{ii}^*(k) \triangleq E[v_i^*(k)v_i^{*T}(k)] = R_{ii}(k) - \bar{H}_i(k)\Big[\sum_{\substack{j=1 \\ j\neq i}}^{N} L_{ij}(k)P_j(k\|k-1)L_{ij}^T(k)\Big]\bar{H}_i^T(k)$	Same as general case (herein lies the distinction between SPA and Standard Kalman Filter)
Step 5	$\bar{K}_i(k+1) = P_{ii}(k+1\|k)\bar{H}_i^T(k+1)[\bar{H}_i(k+1)P_{ii}(k+1\|k)\bar{H}_i^T(k+1) + R_{ii}^*(k+1)]^{-1}$	Same as general case and Standard Kalman Filter
Step 6	$\hat{x}_i(k+1\|k+1) = \hat{x}_i(k+1\|k) + \bar{K}_i[z_{ii}(k+1) - \bar{H}_i(k+1)\hat{x}_i(k+1\|k) - \bar{H}_i(k+1)\sum_{\substack{j=1 \\ j\neq i}}^{N} L_{ij}(k+1)\Phi_{ij}(k+1,k)\hat{x}_j(k\|k)]$	$\hat{x}_i(k+1\|k+1) = \hat{x}_i(k+1\|k) + \bar{K}_i(k+1)[z_{ii}(k+1) - \bar{H}_i(k+1)\hat{x}_i(k+1\|k)]$
Step 7	$P_{ii}(k+1\|k+1) = [I - \bar{K}_i(k+1)\bar{H}_i(k+1)]P_{ii}(k+1\|k)[I - \bar{K}_i(k+1)\bar{H}_i(k+1)]^T + \bar{K}_i(k+1)R_{ii}^*(k+1)\bar{K}_i^T(k+1)$	Same as general case and Joseph's form for Standard Kalman Filter

* Only the case of two subsystems is considered in Ref. 13, while generalization to N is presented here and in Ref. 15

TABLE 4

SUMMARY OF MECHANIZATION EQUATIONS AND QUANTIFICATION OF COMPUTATIONAL BURDEN * OF SURELY LOCALLY UNBIASED (SLU) FILTERING

Order of Calculating	Total Number of ADDS	Total Number of Multiples	(Estimated) Logic Time
Step 1	See SVD in Table	See SVD in Table	See SVD in Table
Step 2	$n_ip_i^2 - n_ip_i$	$n_ip_i^2$	$10+6n_ip_i^2+21n_ip_i+16n_i$
Step 3	$n_ip_i^2 - n_ip_i$	$n_ip_i^2$	$10+6n_ip_i^2+21n_ip_i+16p_i$
Not necessary in pratice	$q_i^2 - q_i$	q_i^2	$10+6q_i^2+37q_i$
Step 4	$n_iq_i^2 - n_iq_i$	$n_iq_i^2$	$10+6n_iq_i^2+21n_iq_i^2+16q_i$
Step 5	$q_i^2 - q_i$	q_i^2	$10+6q_i^2+37q_i$
Step 6	$2q_i^3 - 2q_i^2$	$2q_i^3$	$20+12q_i^3+42q_i^2+32q_i$
Step 7	$2n_i^3 - n_i^2(3q_i-p_i) - n_i^2 + 12n_i(q_i-p_i)^2 + n_iq_ip_i - 3n_i(q_i-p_i) - n_ip_i + (q_i-p_i)^3$	$2n_i^3 + n_i^2(3q_i-p_i) + n_i \cdot (q_i-p_i)(2q_i-p_i) + n_ip_i^2 + (q_i-p_i)^3$	$282+12n_i^3+125n_i^2+160n_i+6n_i^2(3q_i-p_i)+12n_i(q_i-p_i)^2$ $+6n_i(q_i-p_i)p_i+6n_ip_i^2+89n_i(q_i-p_i)+21n_ip_i+7.5(q_i-p_i)^4$ $+165.5(q_i-p_i)^2+108(q_i-p_i)+\mathrm{MUL}\{0.5(q_i-p_i)^2$ $+2.5(q_i-p_i)+6\}+\mathrm{DIV}\{2(q_i-p_i)^2+(q_i-p_i)\} + 41(q_i-p_i)^3$
Step 8	$2n_i^2(q_i-p_i) + 2n_i \cdot (q_i-p_i)q_i - 3n_i(q_i-p_i) + (q_i-p_i)^3$	$2n_i^2(q_i-p_i) + 2n_i(q_i-p_i) \cdot q_i + (q_i-p_i)^3$	$151+12n_i^2(q_i-p_i)+12n_i(q_i-p_i)q_i+89n_i(q_i-p_i)+64n_i$ $+41(q_i-p_i)^3+139.5(q_i-p_i)^2+26(q_i-p_i)q_i+108(q_i-p_i)$ $+16p_i+7.5(q_i-p_i)^4+[3+25(q_i-p_i)+0.5(q_i-p_i)^2]\mathrm{MUL}$ $+[(q_i-p_i)+2(q_i-p_i)^2]\mathrm{DIV}$
Step 9	$n_i^2 - n_i + 2n_iq_i$	$n_i^2 + 2n_iq_i$	$158+6n_i^2+121n_i+12n_iq_i+42q_i+4\mathrm{MUL}$

*Compiled based on precedent convention, and guidelines set forth in Refs. 36, 37 for such calculations where in the above SLU specialization:

$n_i \triangleq$ dimension of the subsystems local SLU filter (defined in Eq. 2-1).
$p_i \triangleq$ rank of component of observation matrix $\hat{H}_i$ (defined in Eq. 2-2)
$q_i \triangleq$ dimension of subsystems local measurements (defined in Eq. 2-2a).
MUL$\triangleq$unit logic time required for a multiply by an element retrieved from core memory (by indirect addressing) rather than by an element from the Arithmetic Logic Unit (ALU).
DIV$\triangleq$unit logic time required for a division by an element retrieved from core memory (by indirect addressing) rather than by an element from the ALU.
\+ If $p_i=q_i$, no SVD is required and mechanization should revert to standard Kalman filter.

TABLE 5

SUMMARY OF MECHANIZATION EQUATIONS AND QUANTIFICATION OF COMPUTATIONAL BURDEN* OF SEQUENTIALLY PARTITIONED ALGORITHM (SPA) FILTERING AS TAILORED TO JTIDS RELNAV.

ORDER OF CALCULATIONS	TOTAL NUMBER OF ADDS	TOTAL NUMBER OF MULTIPLES	(ESTIMATED) LOGIC TIME
STEP 1	$n_i^2 - n_i$	n_i^2	$10+6n_i^2+37n_i$
STEP 2	$(2n_i^3-n_i^2)N$	$(2n_i^3)N$	$(20+12n_i^3+42n_i^2+16n_i)N$ $+27+5n_i+\mathrm{MUL}$
	none	none	none
STEP 3	$4n_i^3-3n_i^2$	$4n_i^3$	$24n_i^3+89n_i^2+64n_i+67+\mathrm{MUL}$
STEP 4	$q_i^3+(n_i^2q_i+n_iq_i^2-n_iq_i-q_i^2)N$	$q_i^3+(n_i^2q_i+n_iq_i^2)N$	$(20+6n_i^2q_i+6n_iq_i^2+21n_iq_i+21q_i^2$ $+16n_i+16q_i)N+37+6q_i^3+21q_i^2+21q_i$ $+\mathrm{MUL}$
STEP 5	$n_i^2+2n_iq_i^2-2n_iq_i+q_i^3$	$n_i^2q_i+2n_iq_i^2+q_i^2$	$67+6n_i^2q_i+42n_iq_i+32n_i+12n_iq_i^2+7.5q_i^4+41q_i^3+165n_i^2+108q_i+(q_i+2q_i^2)\mathrm{DIV}+(0.5q_i^2+2.5q_i+1)\mathrm{MUL}$
STEP 6	$2n_iq_i+q_i^2+(n_i^2q_i-n_iq_i+n_i^2-n_i)N$	$n_i+2n_iq_i+(n_i^2+n_i^2q_i)N$	$84+12q_i^2+12n_iq_i+42n_i+116q_i+3 \cdot \mathrm{MUL}+(20+6n_i^2q_i+6n_i^2+21n_iq_i+53n_i)N$
STEP 7	$2n_i^3-n_i^2$	$2n_i^3$	$47+12n_i^3+47n_i^2+22n_i+\mathrm{MUL}$

*Compiled based on precedent, convention, and guidelines set forth in Refs. 36, 37, for such calculations, where in the above SPA specialization:

$n_i \triangleq$dimension of the subsystems local SPA filter (defined in Eq. 2-1)

$q_i \triangleq$dimension of subsystems local measurements (defined in Eq. 2-2a)

MUL$\triangleq$unit logic time required for a multiply by an element retrieved from cove memory (by indirect addressing) rather than by an element from the Arithmetic Logic Unit(ALU)

DIV$\triangleq$unit logic time required for a division by an element retrieved from cove memory (by indirect addressing) rather than by an element from the ALU.

N$\triangleq$number of individual subsystems contributing covariance information (for JTIDS RelNav, N=1)

$$R^{*}_{44}(k) = R_4(k) + \hat{H}_4(k)\left\{\sum_{\substack{j=1\\j\neq 4}}^{4} L_{4j}(k)\begin{bmatrix} c(t_1) & 0 & 0 \\ 0 & c(t_2) & 0 \\ 0 & 0 & c(t_3)\end{bmatrix} L^{T}_{4j}(k)\right\}\hat{H}_4(k) \tag{4-16}$$

$$\hat{x}_4(k+1|k+1) = \hat{x}_4(k+1|k) + \bar{K}_4(k+1)\left[z_{44}(k+1) - \bar{H}_4(k+1)\hat{x}_4(k+1|k) - \hat{H}_4(k+1)\sum_{\substack{j=1\\j\neq 4}}^{4} L_{4j}(k+1)\Phi_{4j}(k+1,k)\hat{x}_j(k|k)\right] \tag{4-17a}$$

$$= \hat{x}_4(k+1|k) + \bar{K}_4(k+1)\left[z_{44}(k+1) - \bar{H}_4(k+1)\hat{x}_4(k+1|k)\right] \tag{4-17b}$$

where in the above the following variables are defined on p.4-165 and p.4-161 of (34): R(i,i), $P_{ii}(0)$, $c(t_i)$. Eq.4-17a simplifies nicely since

$$\bar{\Phi}_{ij}(k+1,k) = 0 \tag{4-18}$$

for the JTIDS RelNav application (i.e., no interconnection in the linear dynamics, just interconnection in the linearized measurements). There is nothing to interfere with the SPA filter being used on one TOA measurement at a time since no numerical difficulties ensue for single measurement incorporation.

5. CONSIDERATIONS OF COMPUTATIONAL BURDEN AND STABILITY

The SLU computer memory allotment required may be obtained in two steps. By first reading from Table V, p.750 of (36), the memory required for the standard Kalman filter is

$$5n_i^2 + 3n_i + 2n_iq_i + q_i^2 + q_i + 1 \tag{5-1}$$

Second, accounting for the additional occurrence of such characteristic SLU terms as L_{ii}, Z_{i1}, Z_{i2}, Z_{i1}^{-1}, Z_{i2}^{-1}, the corresponding appropriate dimensions are then added to Eq.5-1 to yield

$$5n_i^2+3n_i+2n_iq_i+3q_i^2+q_i+n_ip_i+3p_i^2+1 \tag{5-2}$$

The number of adds, multiplies, and logic time requirements [as compiled in (15) using the techniques of (36,37)] are summarized in Table 4 for an SLU filter. This information can be used to establish the filter cycle time required for processing a measurement once the add, multiply, and logic times are provided for the intended host machine.

The SPA computer memory allotment required may also be obtained by the same two step procedure mentioned above. The SPA filter, as tailored for the JTIDS RelNav application, uses additional intermediate scratch calculations of dimension $(n_i \times n_i)$, $(q_i \times q_i)$, and $(q_i \times n_i)$ beyond those encountered in a conventional Kalman filter so the total memory requirement is

$$6n_i^2 + 3n_i + 3n_iq_i + 2q_i^2 + q_i + 1 \tag{5-3}$$

The number of adds, multiplies, and logic time requirements are summarized in Table 5 for an SPA filter.

The proof of stability for the discrete-time formulations of SLU and SPA [presented in detail in (15)] proceeds in a manner analogous to the continuous-time proof provided in (10) by also utilizing the stability framework of (16). The crux of the proof involves demonstrating that

$$V[\hat{x}(k|k-1),k] = \hat{x}^T(k|k-1)\ P^{-1}(k|k)\hat{x}(k|k-1) \tag{5-4}$$

is a valid Lyapunov function by demonstrating that it is positive definite and that the first variation along the trajectory of

$$\hat{x}(k+1|1) = [\Phi(k+1,k) - \Phi(k+1,k)K(k)H(k)]x(k|k-1) \tag{5-5}$$

is negative definite. The inner product matrix in Eq.5-4 being the inverse of the matrix that evolves from the matrix Riccati equations (encountered in both SPA and SLU filtering which can be demonstrated to be uniformly bounded above and below under the condition of uniform complete observability of the subsystem [without also requiring uniform complete controllability or even controllability due to weakened hypothesis provided by (16)].

However, recent results [(17), Appendix C of (18) p.244 of (19)] indicated a minor error in the upper and lower bounds appearing in (40,41) [as utilized in (16,10)]. This error is corrected in (15) and shown to not adversely affect the stability conclusions for SPA and SLU as long as all observation matrices $H_i(k)$ are of full rank (a new restriction) as well as $R_i^{-1}(k)$ being bounded and $P_i(0)>0$. By this approach, only asymptotic stability is established for SLU and SPA in contradistinction to exponential asymptotic stability.

6. CONCLUSIONS

Several alternative approaches to decentralized filtering (summarized in the two left-most columns of Table 1) were derived between 1970-80. Interest in decentralized filtering formulations here stems mainly from the advertised promise (10,11) of providing a stable filtering algorithm, a topic of some concern within JTIDS RelNav where instabilities are occasionally displayed. A remedy via a decentralized filtering approach appears especially attractive since implementation requires only somewhat minor changes in the current software processing algorithm associated with the filtering function. As a result of the issues reviewed in this paper, the field of eight candidates has been narrowed down as indicated in the center column of Table 1. The corresponding sections of this paper are identified (in the second column from the right) in Table 1 where the detailed reasons and analytic justification for each dismissal is provided. The Sequentially Partitioned Algorithm (SPA) filter is recommended as appropriate for further testing as a prime candidate for a JTIDS RelNav implementation, while it is also recommended that the Surely Locally Unbiased (SLU) variation termed "Expanded" SLU (ESLU°) be considered as a backup.

The discrete-time mechanization equations, as needed for a real-time implementation are provided in Tables 3 and 2, respectively, for both SPA and SLU (presented as a first time for a discrete-time formulation of SLU, with slight modification for ESLU indicated in Table 2 and Section 4). Parametric tables enabling a convenient quantification of the real-time onboard computational burden in terms of memory required and algorithm cycle times anticipated are presented in Section 5 and Tables 4 and 5 for the SLU and SPA filters, respectively.

The structure of both SPA and SLU filtering is so convenient that an analytic proof (in contradistinction to an empirical proof or proof by limited simulation only) is possible within an existing independently derived stability framework. A sketch of the stability proof for SPA and SLU filtering is provided in Section 5. This guarantee of the stability of the decentralized SPA and SLU linear estimation mechanization is of utmost interest in the JTIDS RelNav application and is a necessary first step for stability in the actual nonlinear filtering scenario of JTIDS RelNav, where repreated relinearization is encountered. The SPA and SLU decentralized frameworks also routinely accommodate the Extended Kalman Filtering actually expected for the JTIDS RelNav [p.245 of (9)]. Further conclusions await the results of simulation testing in worst case nonlinear situations.

Another potential application of decentralized filtering theory distinctly different from JTIDS RelNav is now considered. C-4 Trident SSBNs (i.e., ballistic missile carrying, nuclear powered submarines denoted as Ships Submersible Ballistic Nuclear) represent an unrelated strategic navigation system application that could possibly benefit by pursuing a coordinated decentralized filtering strategy for all of its existing localized filters (all found within the SSBN's navigation room and paged into the computers as required for processing a nav "fix" or "reset" modification). The following filters are utilized within Trident SSBNs:

* 14 state ESGM Reset filter,

* 15 state SINS (Ships Inertial Navigation System) Correction Filter,

* (proposed) Velocity Measuring Sonar (VMS) filter,

* (future possibility) Gradiometer filter,

* two 7 state STAR (STAtistical Reset) filters as a back-up, so that if the ESGM fails, the system can revert back to the C-3 Poseidon navigation configuration of SINS/SINS with Master SINS Selection.

A decentralized filtering framework is especially appealing in the SSBN application because:

* Error equations of localized redundant navigation system are purely linear,

* Current simultaneous SSBN filters have some redundant models (e.g., ESGM modeled in ESGM Reset filter, while contributions of six of the ESGM states are also modeled in the SINS Correction Filter); elimination of redundant states will reduce the computer burden,

* Current thrust in C-4 Trident is to reduce delays while paging in programs from the Magnetic Tape Unit (MTU) and to alleviate the computational burden experienced by the three Univac CP-890/UYK computers (i.e., ESGM computer, Central Navigation Computer [CNC], and Back-up Computer) used in navigation,

* No constraints to prevent use of a central processing node for collating and coordinating the results of several local filters (possibly using distributed processing with microprocessors) into an equivalent global or "centralized" Kalman estimate.

Other approaches to decentralized filtering publicized too late to be considered here, are (44,45).

APPENDIX A: SALIENT FEATURES OF A CANONICAL TRANSFORMATION TO ACHIEVE OUTPUT DECENTRALIZATION

A.1 Introduction

Several approaches to decentralized filtering, such as (26,29,30), require that the original system be of the "output decentralized" form as a condition for applicability. An approach is available that can be applied to some large-scale systems having a general measurement structure of the form:

$$z_i(k)=\bar{H}_i x_i(k)+\hat{H}_i \sum_{\substack{j=1\\ j\neq i}}^{N} L_{ij} x_j(k)+v_i(k) \quad (\text{for } i=1,\ldots,N) \qquad (A.1-1)$$

so that so-called "output decentralization" of the form

$$z_j(k) = H_j\tilde{x}_j(k) + \tilde{v}_j(k) \qquad (A.1-2)$$

(for j=1,...,K) is achieved, where each subsystem has access to (and responsibility for) measurements only for that subsystem. In output decentralization, the parameter K is not constrained to be identical to N and the subsystem state and noise groupings are different, in general, from those in Eq. A.1-1 as denoted, respectively, by $\tilde{x}_j$ and $\tilde{v}_j$.

"Output decentralization," if achievable, is attained via a single transformation that must be applied to the entire system aggregate. Only after the output decoupling transformation has been applied are distinct individual subsystems revealed. Conditions for applicability of the output decentralization transformation, its mechanization, its general convenience to apply, and its apparent lack of compatibility with JTIDS RelNav constraints are discussed in detail in Appendix B of (15) as highlighted here.

A.2 "A Priori" and "A Posteriori" System Structures Associated with Output Decentralizing Transformation

The requisite decentralizing transformation was originally presented in (42) and discussed in more detail (as Chapter 7) in (43) but only for the case of "control input decoupling" or "control decentralization" without a consideration of measurement or process noise. "Output decoupling" results were left only as an implicit afterthought (via the concept of duality between control and measurement structures). For ease in accessibility and use of the same consistent notation throughout, the specific transformation for achieving "output measurement decoupling" is explicitly presented in Appendix B of (15) following the approach of (30) but going further to show the effect of the transformation on the measurement and process noises that are naturally encountered in filtering applications.

It is now discussed how a large-scale linear constant (i.e., time-invariant) system can be transformed into a number of interconnected subsystems (with local outputs only) to result in output decentralization. For convenience, only the continuous-time case is considered to expedite the presentation, yet expose the computational burden to be encountered as one of several reasons, discussed further in Section A.3, why circumvention of this entire approach is strongly recommended for the JTIDS RelNav application. A more lengthy discrete-time formulation of output decentralization would provide results analagous to those described here.

A general aggregated time-invariant version of the system of (2-1) is of the form:

$$\dot{\underline{x}}(t) = \overline{F}\underline{x}(t) + \underline{w}'(t) \qquad (A.2\text{-}1)$$

while the measurements (from Eq.2-1 and consistent with Eq.A.1-1) are of the form

$$z_i(t_k) = \bar{H}_i x_i(t_k) + \hat{H}_i \sum_{\substack{j=1 \\ j\neq i}}^{N} L_{ij} x_j(t_k) + v_i(t_k) \qquad (A.2\text{-}2)$$

or equivalently,

$$\overset{(p\mathrm{x}1)}{\underline{z}}(t_k) = H\underline{x}(t_k) + \underline{v}(t_k) \qquad (A.2\text{-}3)$$

where

$$\overset{(n\mathrm{x}1)}{x} \triangleq \left[x_1^T, x_2^T, \ldots, x_N^T\right]^T$$

$$\overset{(p\mathrm{x}1)}{z} \triangleq \left[z_1^T, z_2^T, \ldots z_N^T\right]^T$$

$$\overset{(p\mathrm{x}1)}{v} \triangleq \left[v_1^T, v_2^T, \ldots, v_N^T\right]^T$$

$$H \triangleq \begin{bmatrix} \bar{H}_1 & \hat{H}_1 L_{12} & \hat{H}_1 L_{13} \cdots & \hat{H}_1 L_{1N} \\ \hat{H}_2 L_{21} & \bar{H}_2 & \hat{H}_2 L_{23} \cdots & \hat{H}_2 L_{2N} \\ \hat{H}_N L_{N1} & & \cdots & \bar{H}_N \end{bmatrix} \equiv \left[H_1 \vdots H_2 \vdots \cdots \vdots H_N\right]$$

with

$$p = \sum_{i=1}^{N} p_i \qquad (A.2\text{-}4)$$

$$n = \sum_{i=1}^{N} n_i \qquad (A.2\text{-}5)$$

First, the system of Eqs. A.2-1 and A.2-3 is decomposed into N dynamic subsystems (such as can be associated with N members of a JTIDS RelNav Net) of the form

$$\dot{x}_i = F_i \overset{(n_i\mathrm{x}1)}{x_i} + \sum_{\substack{j=1 \\ j\neq i}}^{N} F_{ij} x_j + w_i' \qquad (A.2\text{-}6)$$

$$(\text{for } i=1,\ldots,N)$$

with aggregated measurements

$$z = \sum_{j=1}^{N} \overset{(p\mathrm{x}n_j)}{H_j} x_j + \underline{v} \qquad (A.2\text{-}7)$$

where

$$(H_j, F_j) \text{ are observable pairs } (\text{for } j=1,\ldots,N) \qquad (A.2\text{-}8)$$

or specifically,

$$\text{rank}\left[H_j^T \vdots F_j^T H_j^T \vdots F_j^{2T} H_j^T \vdots \cdots \vdots F^{(n_j-1)T} H_j^T\right] = n_j \qquad (A.2\text{-}9)$$

By utilizing a linear transformation of the form

$$\underline{x}_j = T_j \underline{\bar{x}}_j \qquad (A.2\text{-}10)$$

[where T_j is defined in (15,30)], the following form results

$$\dot{\underline{\bar{x}}}_j = \bar{F}_j \underline{\bar{x}}_j + \sum_{\substack{k=1 \\ k\neq j}}^{N} \bar{F}_{jk} \underline{\bar{x}}_k + \bar{w}_j \text{ for } j=1,\ldots,N \qquad (A.2\text{-}11)$$

$$z = \sum_{j=1}^{N} \bar{C}_j \underline{\bar{x}}_j + v \qquad (A.2\text{-}12)$$

Upon grouping together all of the $(\underline{x}_i)_j$ for each of the elements from j=1,...,N which contribute via the superposition of linearity to the same output entry $\overset{(1\mathrm{x}1)}{z_i}$ (and rearranging the subsystems in a corresponding manner) results in a final representation of the same system of Eqs. A.2-1 and A.2-3 as explicitly composed of p interconnected output-decentralized subsystems. The final result of the above described reshuffling is achieved by defining a non-singular transformation P in such a way as to render

$$x' \triangleq \left[x_1'^T, x_2'^T, \ldots, x_p'^T\right]^T = P\left[\bar{x}_1^T, \bar{x}_2^T, \ldots, \bar{x}_N^T\right]^T \qquad (A.2\text{-}13)$$

by using a permutation matrix P of the following block form

$$\overset{(n\mathrm{x}n)}{P} = \left[P_1^T \vdots P_2^T \vdots \cdots \vdots P_p^T\right]^T \qquad (A.2\text{-}14)$$

where the i^{th} block matrix P_i is defined in (30) [and corrected in Eq. B.2-26 of (15)].

Applying the above discussed successive transformations of T and then P upon the original system of Eqs. A.2-1 and A.2-3 yields

$$\dot{x}' = Ax' + w'' \qquad (A.2\text{-}15)$$

$$z = Cx' + v \qquad (A.2\text{-}16)$$

where

$$w'' \triangleq P\left[\bar{w}_1^T, \bar{w}_2^T, \ldots, \bar{w}_N^T\right]^T \qquad (A.2\text{-}17)$$

Exploiting the available structural simplifications described in (15,30), yields the objective of the final output decentralized form

$$\dot{x}_i' = A_i x_i' + \sum_{\substack{j=1 \\ j\neq i}}^{p} A_{ij} x_j' + w_i'' \qquad (A.2\text{-}18)$$

$$z_i = c_i x_i' + v_i \qquad (A.2\text{-}19)$$

for i=1,...,p (where p is the dimension of the aggregate non-redundant output measurements of the original system of Eq. A.2-3). Note that while the output is now represented in a completely decentralized manner in Eq. A.2-19, the p scalar measurement noises of the different subsystems are correlated (a type of coupling), in general, unless the original system of Eq. A.2-3 also has uncorrelated measurement noise components, as indicated by a block diagonal measurement noise covariance matrix of the following form

$$R = \begin{bmatrix} R_1 & & & \bigcirc \\ & R_2 & & \\ & & \ddots & \\ \bigcirc & & & R_p \end{bmatrix} \tag{A.2-20}$$

(which is trivially achievable if R is strictly diagonal, but not very likely otherwise). Restrictions relating to the presence of measurement and process noise as encountered here while considering the salient features of output decentralizing transformations were not considered in (30,42,43).

A.3 Drawbacks Limiting Applicability of Output Decentralization to JTIDS RelNav

While "output decentralization" could be acceptable for many applications, its usefulness for JTIDS RelNav appears to be limited since requirements for its application appear to violate certain JTIDS RelNav guidelines or operational constraints as now summarized. Apparent restrictions to applying "output decentralization":

- Decentralized transformation only strictly applicable to time-invariant linear systems (where the matrices F, and $\bar{H}_i$, $\hat{H}_i$, L_{ij} of Eqs. A.2-1 and A.2-2, respectively, are constant);
- Original grouping of N subsystems must consist entirely of "observable pairs" (H_j,F_j) otherwise computational problems with corresponding transformation matrices T_j are encountered;
- Test of "observability" requirements being met (Eq. A.2-9) is a difficult computational burden for real-time verification;
- Formation of transformation T_j (Eq. A.2-10) appears to be a formidable computational burden for each j=1,...,N;
- While transformations using T_j can be applied at the subsystem level in a decentralized fashion (consistent with JTIDS RelNav operational policy of not requiring a central computing facility), reshuffling of the aggregate of states using the permutation matrix P (Eq. A.2-14) constitutes a computational burden that is apparently only compatible with a central computing facility (a requirement which is in conflict with JTIDS RelNav tactical operating guidelines);
- Reshuffled "output decentralized" representation of Eqs. A.2-18 and A.2-19 will not necessarily correspond to the distinct users in a JTIDS RelNav net (a consequence that can be inconvenient when a desirable physical association frequently provides insight for error overrides).

While the restriction of item 1 above in requiring a time-invariant system is obviously violated in JTIDS RelNav operations where repeated relinearizations are time-varying in general, the RelNav application could still be accommodated, in principle, by repeated application of the T and P transformations discussed in the above items 4 to 6 after any measurement update anywhere within the net. However, the drawback to doing this is that it represents a multifold increase in what is already perceived as so substantial a computational burden that it appears impracticable for the JTIDS RelNav application to be required to perform these tedious transformations even once.

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