Technical Notes and Correspondence

An Invalid Norm Appearing in Control and Estimation

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Abstract—This correspondence calls attention to a so-called norm that has appeared and reappeared in the estimation and modern control theory literature. By means of a simple counterexample, the properties of a valid norm should satisfy are shown to be violated by this bogus norm.

Occasionally analytical misconceptions will arise and even propagate in the literature (e.g., [1] discusses a flaw that was found to be inherent in a very common test for matrix nonnegative definiteness). It is the purpose of this note to call attention to the fairly common error of designating a matrix norm to be the result of finding the minimum of the standard matrix row-sum norm and the standard column-sum norm. This mistake recently occurred in [2, eq. (6)]; however, the oversight did not detract from the validity of the final results. Historically, this error also occurred as far back as 1966 (3, sec. 3, ch. 2) in the development of a convergence criterion to signal termination in a computational procedure for numerically evaluating the matrix exponential. Reference [3] served as a model for many packaged computer programs that were subsequently developed by industry and universities for systems analysis and synthesis by modern control techniques. Because of the importance of [3] as an analytic and computational milestone, the effect of this minor misdesignation of a matrix norm in [3] may have had far reaching consequences, which could be deleterious if this so-called norm is used in an application where use of a valid norm is crucial.

THE COUNTEREXAMPLE

Consider the following two well-known norms of the square matrix $A$:

$$
\|A\|_1 \triangleq \max_j \left\{ \sum_i |a_{ij}| \right\} \quad \text{(column-sum norm)}
$$

$$
\|A\|_2 \triangleq \max_i \left\{ \sum_j |a_{ij}| \right\} \quad \text{(row-sum norm).}
$$

It will now be shown that the following function of $A$ (as used in [2] and [3])

$$
\|A\|_3 \triangleq \min (\|A\|_1, \|A\|_2)
$$

is not a norm by demonstrating that it violates the triangle inequality\footnote{Since a pseudonorm is only allowed to violate the norm property of being zero if and only if the argument is the null vector, $\|\vec{0}\|$ is not a pseudonorm either.} that any valid norm must satisfy (4, p. 22)). Consider the following square matrices that serve as the counterexample

$$
B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}; \quad B + C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.
$$

Now proceeding with the following intermediate calculations

$$
\|B\|_1 = \max \{2, 0\} = 2; \quad \|B\|_2 = \max \{1, 1\} = 1;
$$

$$
\|B\|_3 = \min \{2, 1\} = 1
$$

$$
\|C\|_1 = \max \{1, 1\} = 1; \quad \|C\|_2 = \max \{2, 0\} = 2;
$$

$$
\|C\|_3 = \min \{1, 2\} = 1
$$

$$
\|B + C\|_1 = \max \{3, 1\} = 3; \quad \|B + C\|_2 = \max \{3, 1\} = 3;
$$

$$
\|B + C\|_3 = \min \{3, 3\} = 3
$$

yields the following result which violates the well-known triangle inequality:

$$
3 = \|B + C\|_3 > \|B\|_3 + \|C\|_3 = 2.
$$

Even though it is asserted in [3, p. 93] that the function $\|\cdot\|_3$ satisfies the following additional property of norms (that allows further structure classification as a Banach algebra in a complete normed space [5, p. 103]):

$$
\|BC\|_3 < \|B\|_3 \cdot \|C\|_3,
$$

the same counterexample that demonstrates that the function $\|\cdot\|_3$ violates the triangle inequality also serves to demonstrate that (9) is also violated since

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Output Feedback Decoupling of Multivariable Systems
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Abstract—Necessary and sufficient conditions for output feedback decoupling of linear time-invariant multivariable systems into single-input—single-output subsystems are derived in the frequency domain. The conditions imply a new constructive method for determining a constant gain output feedback matrix and some of its elements can be freely chosen to reassign the closed-loop poles.

I. INTRODUCTION

The decoupling of linear time-invariant multivariable systems using a constant gain output feedback has been discussed in a number of papers. Falb and Wolovich [1] and Howe and Sennott [2] derived necessary and sufficient conditions in time domain for systems having equal number of inputs and outputs. Wang and Davison [3] and Wolovich [4] investigated the same problem in frequency domain, and more recently, El-Bagoury and Bayoumi [5] employed a frequency domain approach to obtain the desired closed-loop transfer function matrix for systems having unequal number of inputs and outputs.

In this correspondence we will treat the same problem as in [5] in frequency domain and develop necessary and sufficient conditions for decoupling which imply a new flexible method to specify the output feedback gain matrix, where some of its elements can be freely chosen to reassign closed-loop poles of the resulting decoupled systems.

II. PROBLEM

Consider a system characterized by a full-rank, strictly proper transfer function matrix, i.e.,

\[ y(s) = T(s)u(s) \]  

(1)

where \( y(s) \) (p-vector) and \( u(s) \) (m-vector, \( m < p \)) are the Laplace transform of the output and the input. We define the linear output feedback law as

\[ u(s) = Ky(s) + Gc(s) \]  

(2)

where \( K \) is a constant \( m \times p \) matrix, \( G \) is a constant \( m \times m \) nonsingular matrix, and \( c(s) \) is an external \( m \)-vector input. Substituting (2) in (1), we have the closed-loop transfer function matrix \( T_d(s) \) as

\[ T_d(s) = (I_p - T(s)K)^{-1}T(s)I_m - KT(s)^{-1}G \]  

(3)

where \( I_p \) denotes an \( n \times n \) identity matrix.

Then, our objective is to choose \( K \) and \( G \) so that \( T_d(s) \) has nonzero elements only in its main diagonal blocks, where each block is corresponding to a single input—output subsystem, \( p_i > 1 \) (\( i = 1, 2, \ldots, m \)), and \( \sum_{i=1}^{m} p_i = p \).

Before proceeding further, we should note some properties of a pseudoinverse matrix which can be found in detail in Rao and Mitra [6].

Lemma 1: The pseudoinverse matrix \( A^+ \) of a \( p \times m \), full-rank matrix \( A \) has the following properties:

a) if \( p > m \); \( A^+ = (A'A)^{-1}A' \)

(4)

b) if \( p < m \); \( A^+ = (AA')^{-1}A \)

(5)

where a prime denotes a transpose of a matrix or a vector.

Lemma 2: If \( A \) is a \( p \times m \) (\( p > m \)) full-rank matrix and \( B = \) an \( m \times n \) nonsingular matrix, we have

\[ (AB)^+ = B^{-1}A^+ \]  

(6)

We now partition \( T(s) \) into \( m \) blocks, each of which has \( p_i \) rows.