ASEN 5050
SPACEFLIGHT DYNAMICS
Lambert's Problem

Prof. Jeffrey S. Parker
University of Colorado – Boulder

10/8/2013

Announcements

- Homework #5 due Thursday
- Quiz #10 tomorrow

- Mid-term to be handed out on Thursday, Oct 17th. It will be due on Tuesday, Oct 22nd. CAETE due date is Tuesday, Oct 29th. Each person will have the same amount of time for the test – CAETE students just have more flexibility to schedule when they work on the test. Open book open note, no working with others.

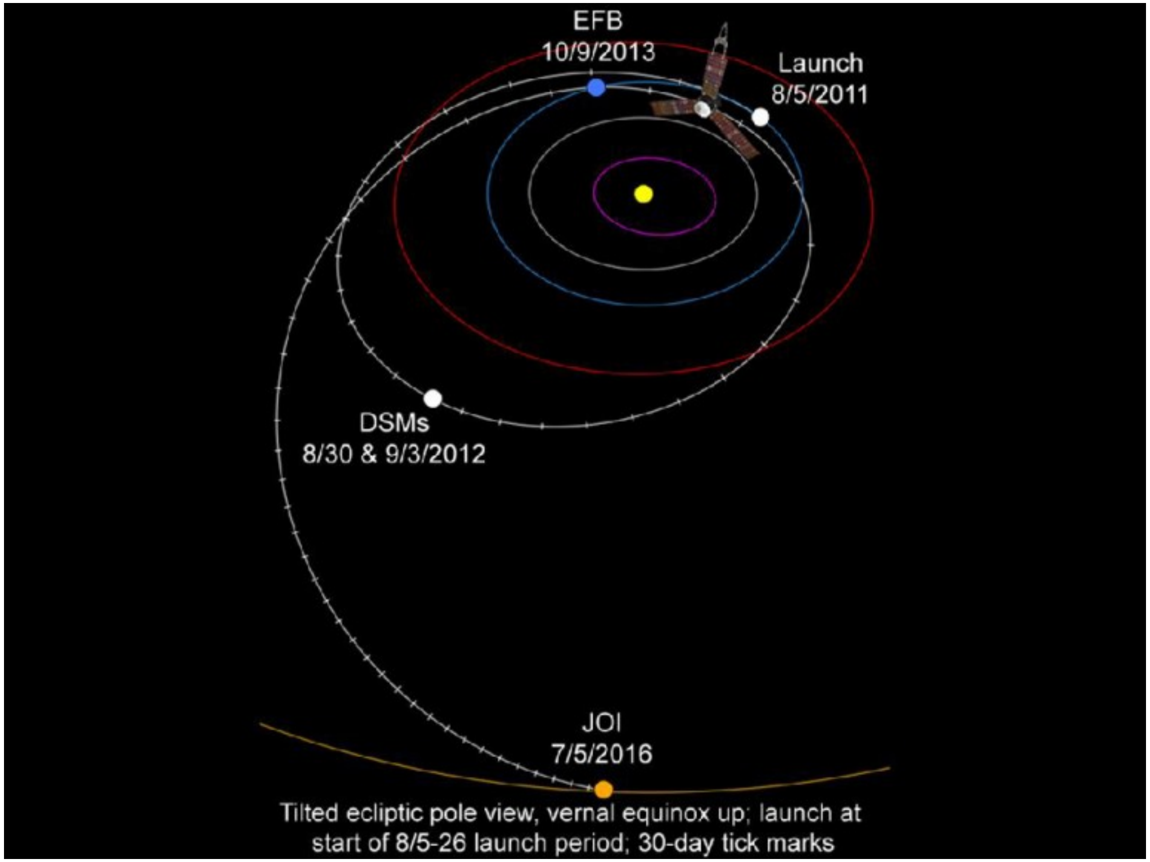
- Reading: Chapter 7

- Space News:
  - Juno's Earth flyby is tomorrow!

Credit: NASA/JPL-Caltech

Juno's Earth Flyby

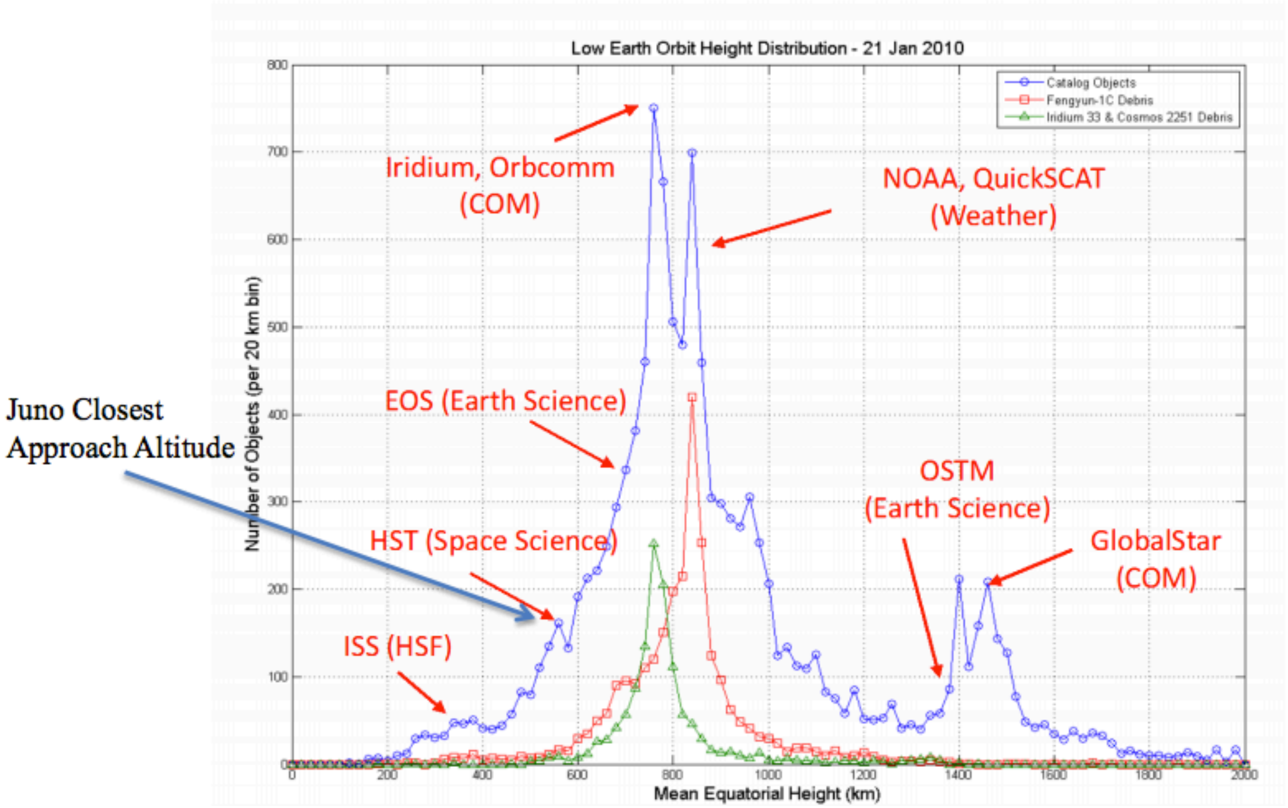
- Earth flyby provides 7.3 km/s of ΔV!
- Altitude ~559 km, Inclination ~47.1 deg



Credit: NASA/JPL-Caltech

Juno’s Earth Flyby

- Collision Assessment
- The Juno Navigation team is deciding right about now whether or not to execute a small divert maneuver to avoid any collisions.



Credit: NASA/JPL-Caltech, Bordi, J., and Bryant, L., “Conjunction Assessment Plans for the Juno Earth Flyby”, Jet Propulsion Laboratory, California Institute of Technology, May 1, 2013

Today

- Go Juno!

- Today:
  - Review Quiz 9
  - Lambert’s Problem

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Lambert’s Problem

- Lambert’s Problem has been formulated for several applications:
  - Orbit determination. Given two observations of a satellite/asteroid/comet at two different times, what is the orbit of the object?
    - Passive object and all observations are in the same orbit.
  - Satellite transfer. How do you construct a transfer orbit that connects one position vector to another position vector at different times?
    - Transfers between any two orbits about the Earth, Sun, or other body.

Lambert's Problem

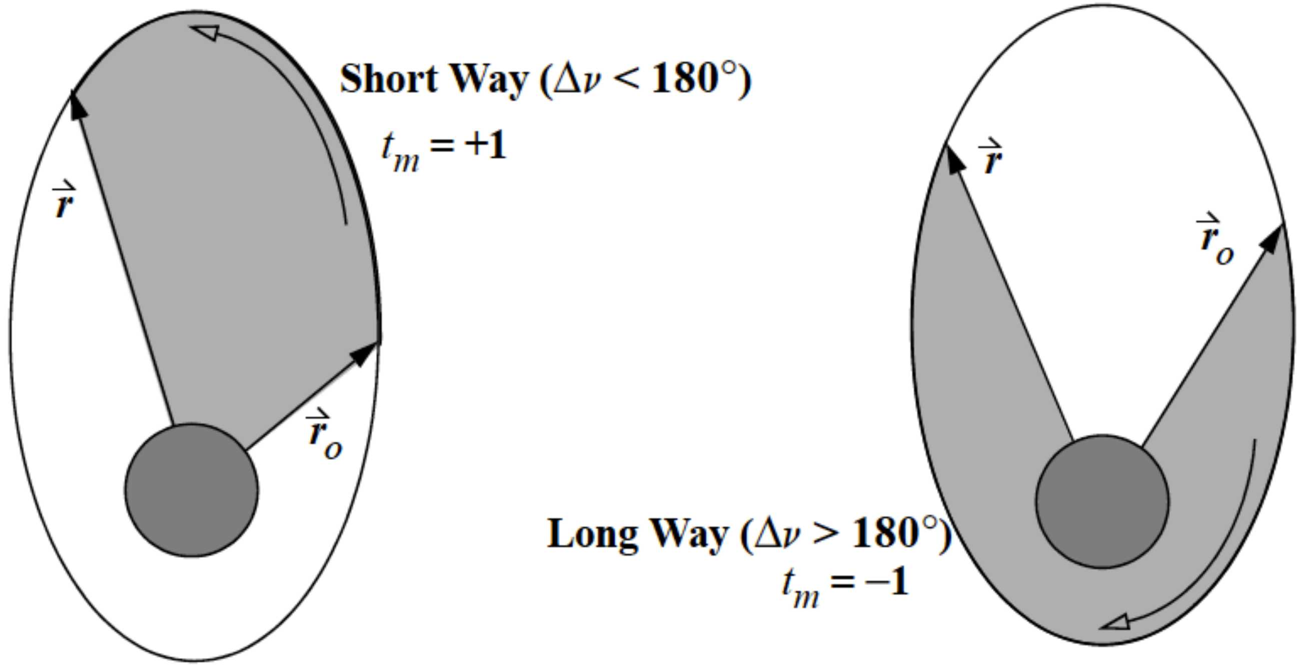


Figure 7-8. **Transfer Methods, t_m, for the Lambert Problem.** Traveling between the two specified points can take the long way or the short way. For the long way, the change in true anomaly exceeds 180°.

Given two positions and the time-of-flight between them, determine the orbit between the two positions.

Orbit Transfer

- We'll consider orbit transfers in general, though the OD problem is always another application.

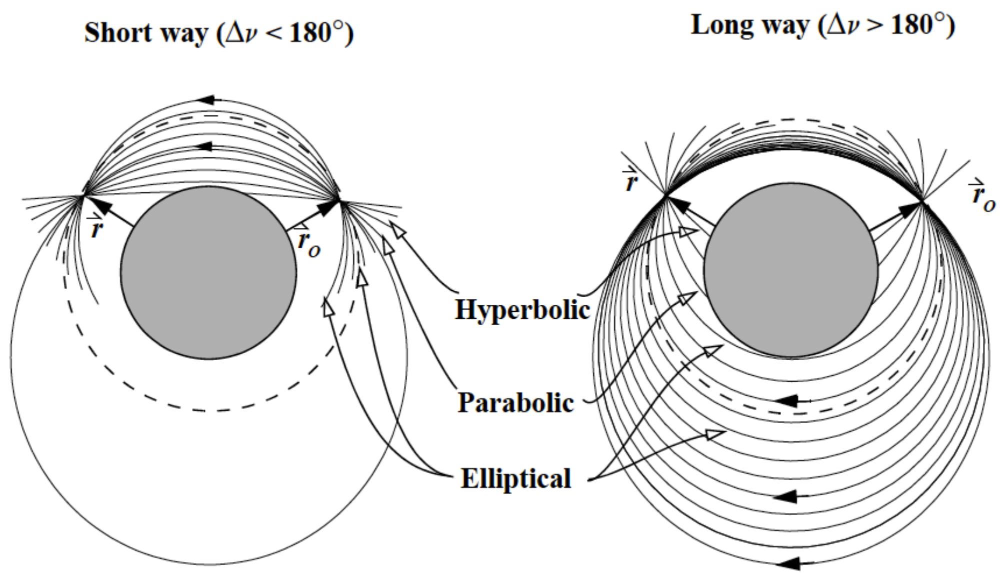
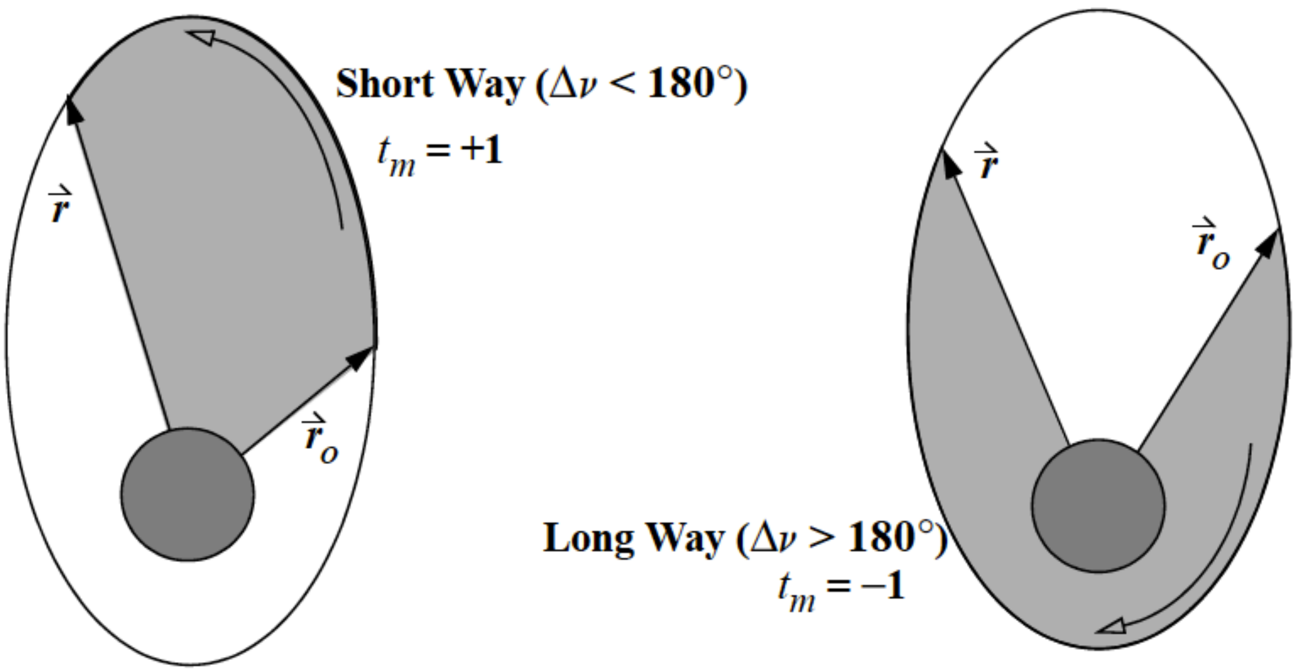


Figure 7-15. Varying Time of Flight for Intercept. As the time of flight increases for the transfer to the target (short way left, long way right), the transfer orbit becomes less eccentric until it reaches a minimum, shown as a dashed line. The eccentricity then begins to increase after this transfer. There is always a minimum eccentricity transfer, and a minimum change in velocity transfer. However, they generally don't occur at the same time. Finally, I've shown the initial and final positions with the same magnitude. This is not an additional requirement.

Orbit Transfer

- Note: there's no need to perform the transfer in < 1 revolution. Multi-rev solutions also exist.



Orbit Transfer

- Consider a transfer from Earth orbit to Mars orbit about the Sun:

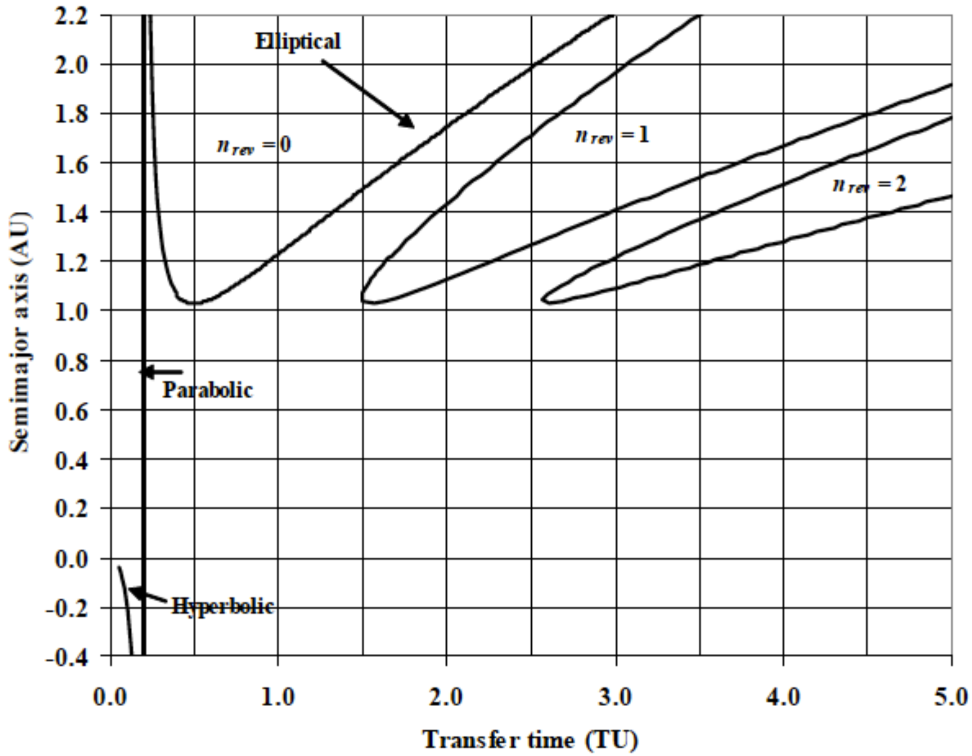


Figure 7-9. **General solution options for the Lambert problem.** This figure shows the various orbits that are possible for transfers considering the Lambert problem. The units are normalized, and the initial separation between the spacecraft is 75°. The interceptor and target positions are 1.0 and 1.524 AU respectively. Long and short way trajectories are possible with the elliptical transfers. Notice the corresponding increase in minimum transfer time as the number of revolutions increases. (Source Thorne, 2007 using his series solution discussed later)

Orbit Transfer

Orbit Transfer	True Anomaly Change
"Short Way"	$\Delta\nu < 180°$
"Long Way"	$\Delta\nu > 180°$
Hohmann Transfer (assuming coplanar)	$\Delta\nu = 180°$
Type I	$0° < \Delta\nu < 180°$
Type II	$180° < \Delta\nu < 360°$
Type III	$360° < \Delta\nu < 540°$
Type IV	$540° < \Delta\nu < 720°$
…	…

Lambert's Problem

- Given: $\vec{R}_0 \;\; \vec{R}_f \;\; t_0 \;\; t_f$
- Find: $\vec{V}_0 \quad \vec{V}_f$
- Numerous solutions available.
  - Some are robust, some are fast, a few are both
  - Some handle parabolic and hyperbolic solutions as well as elliptical solutions
  - All solutions require some sort of iteration or expansion to build a transfer, typically finding the semi-major axis that achieves an orbit with the desired Δt.

Lambert's Problem

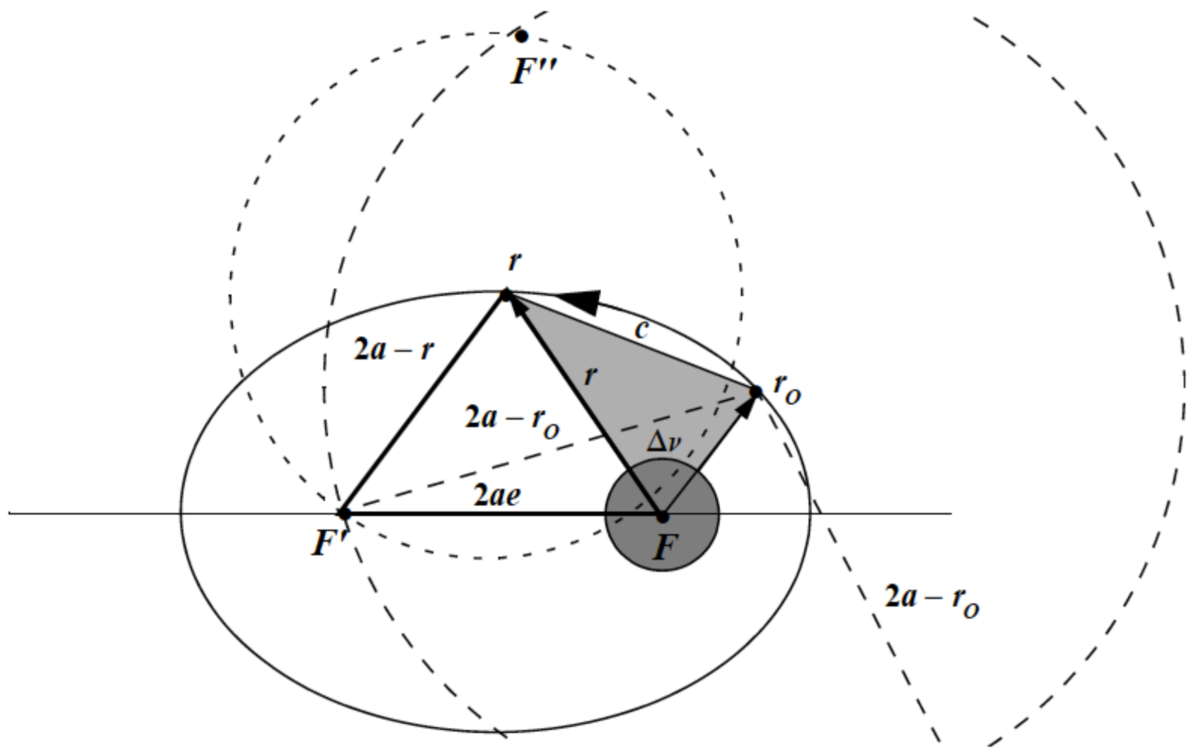


Figure 7-10. Geometry for the Lambert Problem (I). This figure shows how we locate the secondary focus—the intersection of the dashed circles. The chord length, c, is the shortest distance between the two position vectors. The sum of the distances from the foci to any point, r or r_o, is equal to twice the semimajor axis.

Lambert's Problem

Find the chord, c, using the law of cosines and $cos\,\Delta\nu = \dfrac{\vec{r}_0 \cdot \vec{r}}{r_0 r}$

$$c = \sqrt{r_0^2 + r^2 - 2r_0 r\,cos\,\Delta\nu} \qquad \sin(\Delta\nu) = t_m\sqrt{1-\cos^2(\Delta\nu)}$$

Define the semiperimeter, s, as half the sum of the sides of the triangle created by the position vectors and the chord

$$s = \frac{r_0 + r + c}{2}$$

We know the sum of the distances from the foci to any point on the ellipse equals twice the semi-major axis, thus

$$2a = r + (2a - r)$$

Lambert's Problem

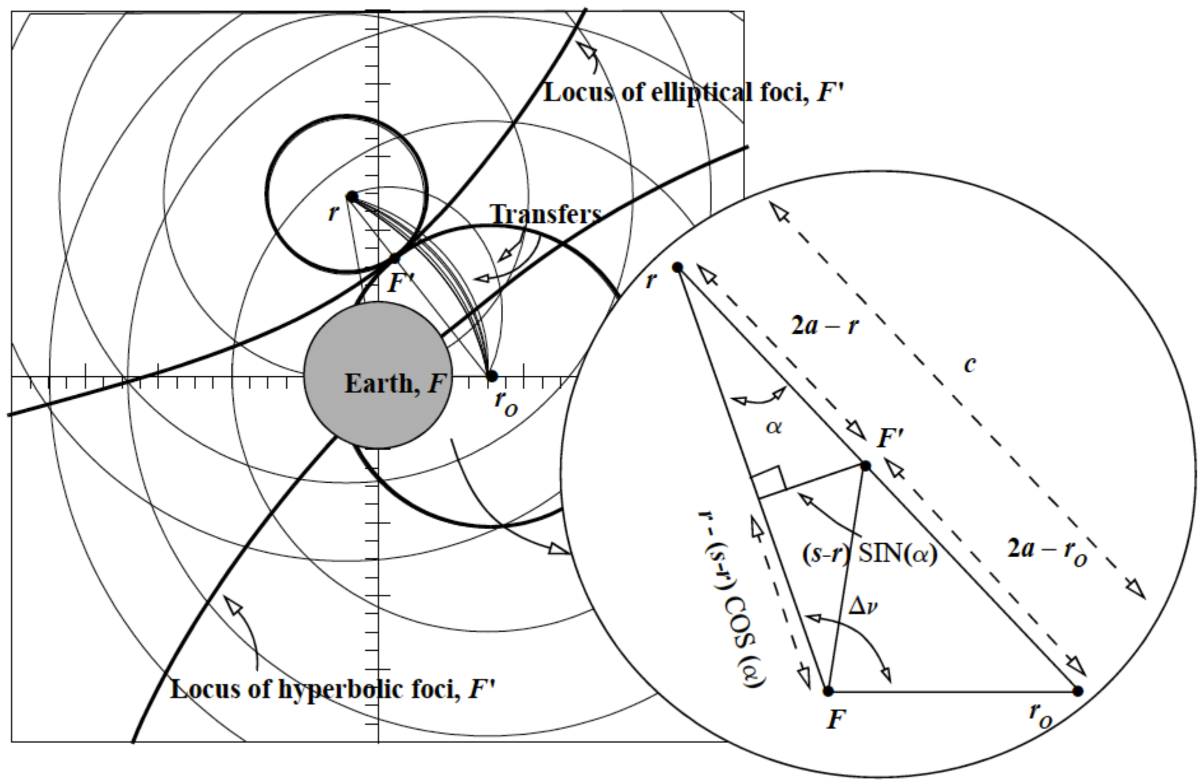


Figure 7-11. Geometry for the Lambert Problem (II). Solving Lambert's problem relies on many geometrical quantities. Be sure to allow for the Earth when viewing representations like this. I've shown transfers between a satellite at 9567.2 km and 15,307.5 km from the Earth's center. We can use the inset figure to find the transfer semimajor axis. The concentric circles are drawn for elliptical values of *a*. When the circles (of the same *a*) touch, half the sum of their radii equals *a* of the transfer, and the intersection is the location of the second focus, *F*'.

Lambert's Problem

The minimum-energy solution: where the chord length equals the sum of the two radii (a single secondary focus)

$$2a - r + 2a - r_0 = c$$

Thus,

$$a_{min} = \frac{s}{2} = \frac{r_0 + r + c}{4}$$

(anything less doesn't have enough energy)

Lambert's Problem

If $\Delta\nu > 180°$, then $\beta_e = -\beta_e$

If $t > t_{min}$, then $\alpha_e = 2\pi - \alpha_e$

Elliptic Orbits

$$\sin\left(\frac{\alpha_e}{2}\right) = \sqrt{\frac{r_0 + r + c}{4a}} = \sqrt{\frac{s}{2a}}$$

$$\sin\left(\frac{\beta_e}{2}\right) = \sqrt{\frac{r_0 + r - c}{4a}} = \sqrt{\frac{s-c}{2a}}$$

Hyperbolic Orbits

$$\sinh\left(\frac{\alpha_h}{2}\right) = \sqrt{\frac{r_0 + r + c}{-4a}} = \sqrt{\frac{s}{-2a}}$$

$$\sinh\left(\frac{\beta_h}{2}\right) = \sqrt{\frac{r_0 + r - c}{-4a}} = \sqrt{\frac{s-c}{-2a}}$$

$$t = \sqrt{\frac{a^3}{\mu}}\left[\alpha_e - \sin(\alpha_e) - (\beta_e - \sin(\beta_e))\right]$$

$$= \sqrt{\frac{-a^3}{\mu}}\left[\sinh(\alpha_h) - \alpha_h - (\sinh(\beta_h) - \beta_h)\right]$$

Lambert's Solution

Lambert's Problem

For minimum energy

$$-- \quad \textit{elliptic orbit}$$

$$-- \quad \alpha_e = \pi$$

$$-- \quad \sin\left(\frac{\beta_e}{2}\right) = \sqrt{\frac{s-c}{s}}$$

$$t_{min} = \sqrt{\frac{a_{min}^3}{\mu}}\left[\pi - \beta_e + \sin(\beta_e)\right]$$

$$\vec{v}_0 = \frac{\sqrt{\mu \ p_{min}}}{r_0 r \sin \Delta\nu}\left\{\vec{r} - \left[1 - \frac{r}{p_{min}}\{1 - \cos \Delta\nu\}\right]\vec{r}_0\right\}$$

Universal Variables

- A very clear, robust, and straightforward solution.
  - There are a few faster solutions, but this one is pretty clean.

- Begin with the general form of Kepler's equation:

$$t_f - t_0 = \Delta t = \sqrt{\frac{a^3}{\mu}}\left[2\pi k + (E_f - e\sin E_f) - (E_0 - e\sin E_0)\right]$$

Transfer Duration

revolutions

Universal Variables

- Simplify

$$t_f - t_0 = \Delta t = \sqrt{\frac{a^3}{\mu}}\Big[2\pi k + (E_f - e\sin E_f) - (E_0 - e\sin E_0)\Big]$$

$$\begin{aligned}
\sqrt{\mu}\Delta t &= \sqrt{a^3}\big[\Delta E + e\left(\sin E_0 - \sin E_f\right)\big] \\
\sqrt{\mu}\Delta t &= \sqrt{a^3}\Delta E + \sqrt{a^3}e\left(\sin E_0 - \sin E_f\right)
\end{aligned}$$

Universal Variables

- Define Universal Variables:

$$\chi = \sqrt{a}\,(E_f - E_0) = \sqrt{a}\,\Delta E$$

$$c_2 = \frac{1 - \cos \Delta E}{\Delta E^2}$$

$$c_3 = \frac{\Delta E - \sin \Delta E}{\Delta E^3}$$

Universal Variables

The quantity $\chi^3 c_3$ is computed and rearranged:

$$\begin{aligned}
\chi^3 c_3 &= \left(\sqrt{a}\Delta E\right)^3 \frac{\Delta E - \sin\Delta E}{\Delta E^3} \\
\chi^3 c_3 &= \sqrt{a^3}\Delta E - \sqrt{a^3}\sin\Delta E \\
\sqrt{a^3}\Delta E &= \chi^3 c_3 + \sqrt{a^3}\sin\Delta E.
\end{aligned}$$

$$\downarrow$$

$$\sqrt{\mu}\Delta t = \sqrt{a^3}\Delta E + \sqrt{a^3}e\left(\sin E_0 - \sin E_f\right)$$

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\sin\Delta E + \sqrt{a^3}\,e\left(\sin E_0 - \sin E_f\right)$$

Universal Variables

- Use the trigonometric identity

$$\sin \Delta E = \sin E_f \cos E_0 - \cos E_f \sin E_0$$

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3} \sin \Delta E + \sqrt{a^3}\, e \left(\sin E_0 - \sin E_f\right)$$

$$\begin{aligned}
\sqrt{\mu}\Delta t &= \chi^3 c_3 + \sqrt{a^3}\big(\sin E_f \cos E_0 - \cos E_f \sin E_0 + e \sin E_0 - e \sin E_f\big) \\
\sqrt{\mu}\Delta t &= \chi^3 c_3 + \sqrt{a^3}\big[\sin E_0 \left(e - \cos E_f\right) - \sin E_f \left(e - \cos E_0\right)\big].
\end{aligned}$$

Universal Variables

- Now we need somewhere to go

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\left[\sin E_0\left(e - \cos E_f\right) - \sin E_f\left(e - \cos E_0\right)\right]$$

- Let's work on converting this to true anomaly, via:

$$\cos\nu = \frac{e - \cos E}{e\cos E - 1}$$

$$\sin\nu = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}$$

Universal Variables

- Multiply

$$\sqrt{\mu}\Delta t \;\; = \;\; \chi^3 c_3 + \sqrt{a^3}\left[\sin E_0\,(e - \cos E_f) - \sin E_f\,(e - \cos E_0)\right]$$

by a convenient factoring expression:

$$\beta = 1 = \frac{\sqrt{1-e^2}(1 - e\cos E_0)(1 - e\cos E_f)}{\sqrt{1-e^2}(1 - e\cos E_0)(1 - e\cos E_f)}$$

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\left[\frac{\sin E_0\,(e - \cos E_f) - \sin E_f\,(e - \cos E_0)}{(1 - e\cos E_0)(1 - e\cos E_f)}\right]\frac{(1 - e\cos E_0)(1 - e\cos E_f)\sqrt{1-e^2}}{\sqrt{1-e^2}}$$

Universal Variables

- Collect into pieces that can be replaced by true anomaly

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\left[\frac{\sin E_0\,(e-\cos E_f) - \sin E_f\,(e-\cos E_0)}{(1-e\cos E_0)(1-e\cos E_f)}\right]\frac{(1-e\cos E_0)(1-e\cos E_f)\sqrt{1-e^2}}{\sqrt{1-e^2}}$$

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\left[\frac{\sqrt{1-e^2}\sin E_0}{(1-e\cos E_0)}\frac{e-\cos E_f}{(1-e\cos E_f)} - \frac{\sqrt{1-e^2}\sin E_f}{(1-e\cos E_f)}\frac{e-\cos E_0}{(1-e\cos E_0)}\right]\frac{(1-e\cos E_0)(1-e\cos E_f)}{\sqrt{1-e^2}}$$

$$\begin{aligned}\cos\nu &= \frac{e-\cos E}{e\cos E - 1}\\ \sin\nu &= \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}\end{aligned}$$

Universal Variables

- Substitute in true anomaly:

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\left[\frac{\sqrt{1-e^2}\sin E_0}{(1-e\cos E_0)}\frac{e-\cos E_f}{(1-e\cos E_f)} - \frac{\sqrt{1-e^2}\sin E_f}{(1-e\cos E_f)}\frac{e-\cos E_0}{(1-e\cos E_0)}\right]\frac{(1-e\cos E_0)(1-e\cos E_f)}{\sqrt{1-e^2}}$$

$$\begin{aligned}\cos\nu &= \frac{e-\cos E}{e\cos E-1}\\ \sin\nu &= \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}\end{aligned}$$

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\left(\sin\nu_0(-\cos\nu_f) - \sin\nu_f(-\cos\nu_0)\right)\left[\frac{(1-e\cos E_0)(1-e\cos E_f)}{\sqrt{1-e^2}}\right]$$

Universal Variables

- Trig identity again:

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\left(\sin\nu_0(-\cos\nu_f) - \sin\nu_f(-\cos\nu_0)\right)\left[\frac{(1-e\cos E_0)(1-e\cos E_f)}{\sqrt{1-e^2}}\right]$$

$$\sin\Delta E = \sin E_f \cos E_0 - \cos E_f \sin E_0$$

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sin(\Delta\nu)\frac{a(1-e\cos E_0)a(1-e\cos E_f)}{\sqrt{a(1-e^2)}}$$

Universal Variables

- Note: $r = a(1 - e\cos E)$

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sin(\Delta\nu)\frac{a(1 - e\cos E_0)a(1 - e\cos E_f)}{\sqrt{a(1-e^2)}}$$

$$\sqrt{\mu}\Delta t \quad = \quad \chi^3 c_3 + \frac{r_0 r_f \sin\Delta\nu}{\sqrt{a(1-e^2)}}$$

$$\sqrt{\mu}\Delta t \quad = \quad \chi^3 c_3 + \frac{r_0 r_f \sin\Delta\nu}{\sqrt{a(1-e^2)}}\frac{\sqrt{1-\cos\Delta\nu}}{\sqrt{1-\cos\Delta\nu}}$$

$$\sqrt{\mu}\Delta t \quad = \quad \chi^3 c_3 + \frac{\sqrt{r_0 r_f}\sin\Delta\nu}{\sqrt{1-\cos\Delta\nu}}\frac{\sqrt{r_0 r_f}\sqrt{1-\cos\Delta\nu}}{\sqrt{a(1-e^2)}}$$

Universal Variables

- Use some substitutions:

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \frac{\sqrt{r_0 r_f}\sin\Delta\nu}{\sqrt{1-\cos\Delta\nu}} \frac{\sqrt{r_0 r_f}\sqrt{1-\cos\Delta\nu}}{\sqrt{a(1-e^2)}}$$

$$A = \frac{\sqrt{r_0 r_f}\sin\Delta\nu}{\sqrt{1-\cos\Delta\nu}}$$

$$y = \frac{r_0 r_f\left(1-\cos\Delta\nu\right)}{a\left(1-e^2\right)}$$

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + A\sqrt{y}$$

$$\Delta t = \frac{\chi^3 c_3 + A\sqrt{y}}{\sqrt{\mu}}$$

Universal Variables

- Summary:

$$
\begin{aligned}
\chi &= \sqrt{\frac{y}{c_2}} \\
A &= \mathrm{DM}\sqrt{r_0 r_f\left(1+\cos\Delta\nu\right)} \\
y &= r_0 + r_f + \frac{A\left(\Delta E^2 c_3 - 1\right)}{\sqrt{c_2}}
\end{aligned}
$$

$$
\mathrm{DM} = \text{Direction of Motion} = \begin{cases} +1 & \text{if } \Delta\nu < \pi \\ \text{-}1 & \text{if } \Delta\nu > \pi \end{cases}
$$

Many texts also replace the quantity ΔE^2 with ψ $\quad (\psi = \Delta E^2)$.

Universal Variables

- It is useful to convert to *f* and *g* series (remember those!?)

$$f = 1 - \frac{y}{r_o} \qquad \dot{g} = 1 - \frac{y}{r} \qquad g = A\sqrt{\frac{y}{\mu}} \qquad \dot{f} = \frac{f\dot{g} - 1}{g} = \frac{\sqrt{\mu}y(-r - r_o + y)}{r_o r A}$$

$$\vec{r} = f\vec{r}_o + g\vec{v}_o$$

$$\vec{v}_o = \frac{\vec{r} - f\vec{r}_o}{g}$$

$$\vec{v} = \frac{f\dot{g} - 1}{g}\vec{r}_o + \dot{g}\frac{\vec{r} - f\vec{r}_o}{g} = \frac{\dot{g}\vec{r} - \vec{r}_o}{g}$$

UV Algorithm

ALGORITHM 58: Lambert—Universal Variables $(\vec{r}_o, \vec{r}, \Delta t, t_m, \Rightarrow \vec{v}_o, \vec{v})$

$$\cos(\Delta\nu) = \frac{\vec{r}_o \cdot \vec{r}}{r_o r}$$

$$\sin(\Delta\nu) = t_m\sqrt{1 - \cos^2(\Delta\nu)}$$

$$A = t_m\sqrt{r r_o(1 + \cos(\Delta\nu))}$$

If $A = 0.0$, we can't calculate the orbit.

$\psi_n = 0.0$, therefore $c_2 = \frac{1}{2}$ and $c_3 = \frac{1}{6}$

$\psi_{up} = 4\pi^2$ and $\psi_{low} = -4\pi$

LOOP

$$y_n = r_o + r + \frac{A(\psi_n c_3 - 1)}{\sqrt{c_2}}$$

IF $A > 0.0$ and $y < 0.0$ THEN

readjust ψ_{low} until $y > 0.0$

$$\chi_n = \sqrt{\frac{y_n}{c_2}}$$

$$\Delta t_n = \frac{\chi_n^3 c_3 + A\sqrt{y_n}}{\sqrt{\mu}}$$

IF $\Delta t_n \le \Delta t$

reset $\psi_{low} \Leftarrow \psi_n$

ELSE

reset $\psi_{up} \Leftarrow \psi_n$

$$\psi_{n+1} = \frac{\psi_{up} + \psi_{low}}{2}$$

Find $\mathbf{c_2 c_3}(\psi_{n+1} \Rightarrow c_2, c_3)$

$\psi_n \Leftarrow \psi_{n+1}$

Check if the first guess is too close

UNTIL $|\Delta t_n - \Delta t| < 1 \times 10^{-6}$

$$f = 1 - \frac{y_n}{r_o} \qquad \dot{g} = 1 - \frac{y_n}{r} \qquad g = A\sqrt{\frac{y_n}{\mu}}$$

$$\vec{v}_o = \frac{\vec{r} - f\vec{r}_o}{g} \qquad \vec{v} = \frac{\dot{g}\vec{r} - \vec{r}_o}{g}$$

Results

- Let's apply the solution to Lambert's Problem to a few problems and see what happens.

- Example Scenario. Build a transfer from Orbit 1 to Orbit 2:
  – Orbit 1: 300 x 1000 km, inclination = 10 deg
  – Orbit 2: 2000 x 5000 km, inclination = 50 deg

Quick Break

- After break we'll cover the Quiz and more details on the Universal Variables algorithm.

Quiz #9

Information

Lambert's Problem: Let's say we have a satellite in orbit about the Earth. We want to perform a maneuver "DV1" at time "t1" to place the satellite on an orbit transfer that will intersect a target orbit, at which point we will execute a second maneuver "DV2" to insert into that orbit, at time "t2". We know the position and velocity of the satellite at t1 and we know the desired position and velocity of the target orbit at t2.

Question 1 (1 point)

If we don't care what the transfer duration is (t2 - t1) and fuel is no constraint, how many different trajectories may we build to satisfy this orbit transfer problem?

- 1, since the transfer uses tangential burns and there's only one way to build a transfer via tangential burns to satisfy the problem.
- 2, since the transfer uses tangential burns and you can build a prograde and a retrograde solution (since fuel is no constraint).
- 2*n, where "n" is the maximum number of full revolutions about the Earth you are willing to take during the transfer; since transfer duration is unconstrained, n is technically infinity!
- Infinite, since there's a unique transfer (typically involving non-tangential burns) for each (t1,t2) combination.

Quiz #9

Question 2 (1 point)

The motion of a satellite in any given two-body Keplerian orbit is constrained to be in a plane (easily defined via the orbit-normal vector / angular momentum vector!). Lambert's Problem includes orbit transfers from one plane to another. If orbit #1 is in one plane and orbit #2 is in another plane, what is the plane of the transfer orbit?

- The same plane as Orbit #1.
- The same plane as Orbit #2.
- The average of Orbit #1's and #2's planes, computed by taking the average of the angular momentum vectors of each orbit.
- The plane that includes the following three points in space: (1) the Earth, (2) the position in space of the start of the orbit transfer, and (3) the position in space of the end of the orbit transfer.

Quiz #9

Question 3 (1 point)

Extending on the previous question, when is a transfer orbit undefined? That is, when might there be an ambiguity in the solution to the problem?

- If Orbit #1 and Orbit #2 have the same orbital planes.
- If Orbit #1 and Orbit #2 have opposite values of inclination.
- If the initial position vector is on the opposite side of the Earth as the final position vector and they are perfectly lined up along a line.
- There is never any ambiguity to the orbit transfer.

A few details on the Universal Variables algorithm

Universal Variables

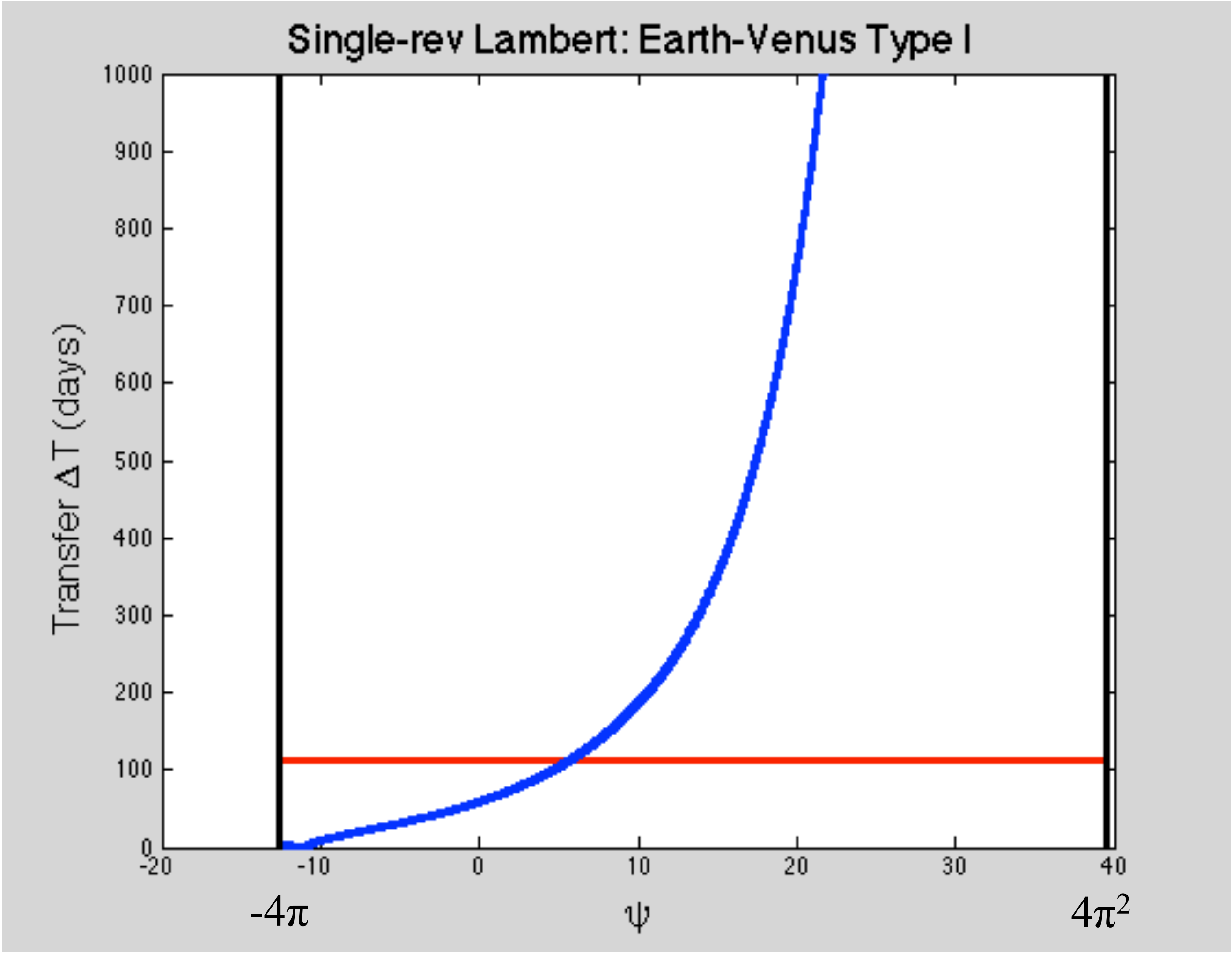
- Let's first consider our Universal Variables Lambert Solver.

- Given: R_0, R_f, ΔT

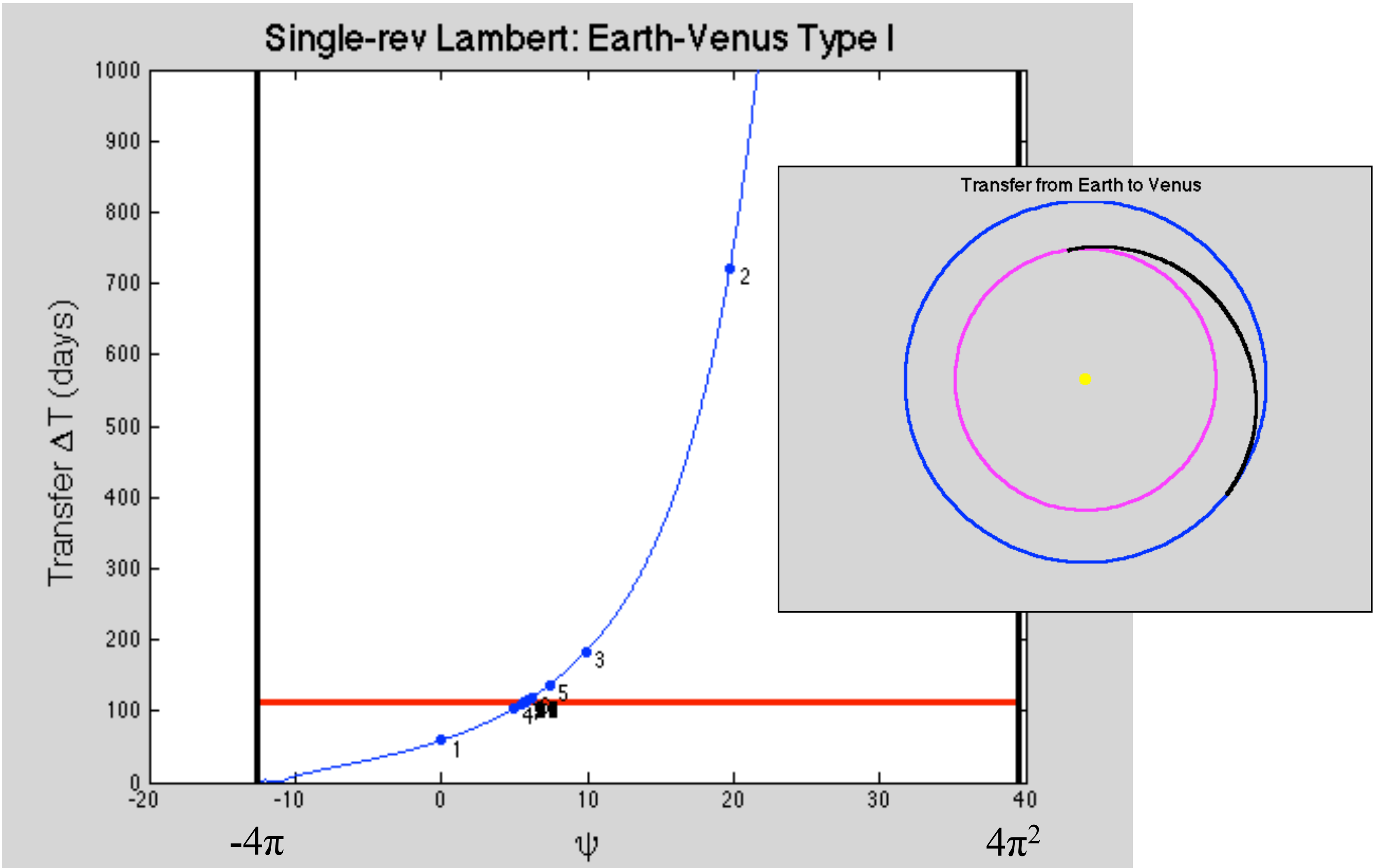
- Find the value of ψ that yields a minimum-energy transfer with the proper transfer duration.

- Applied to building a Type I transfer

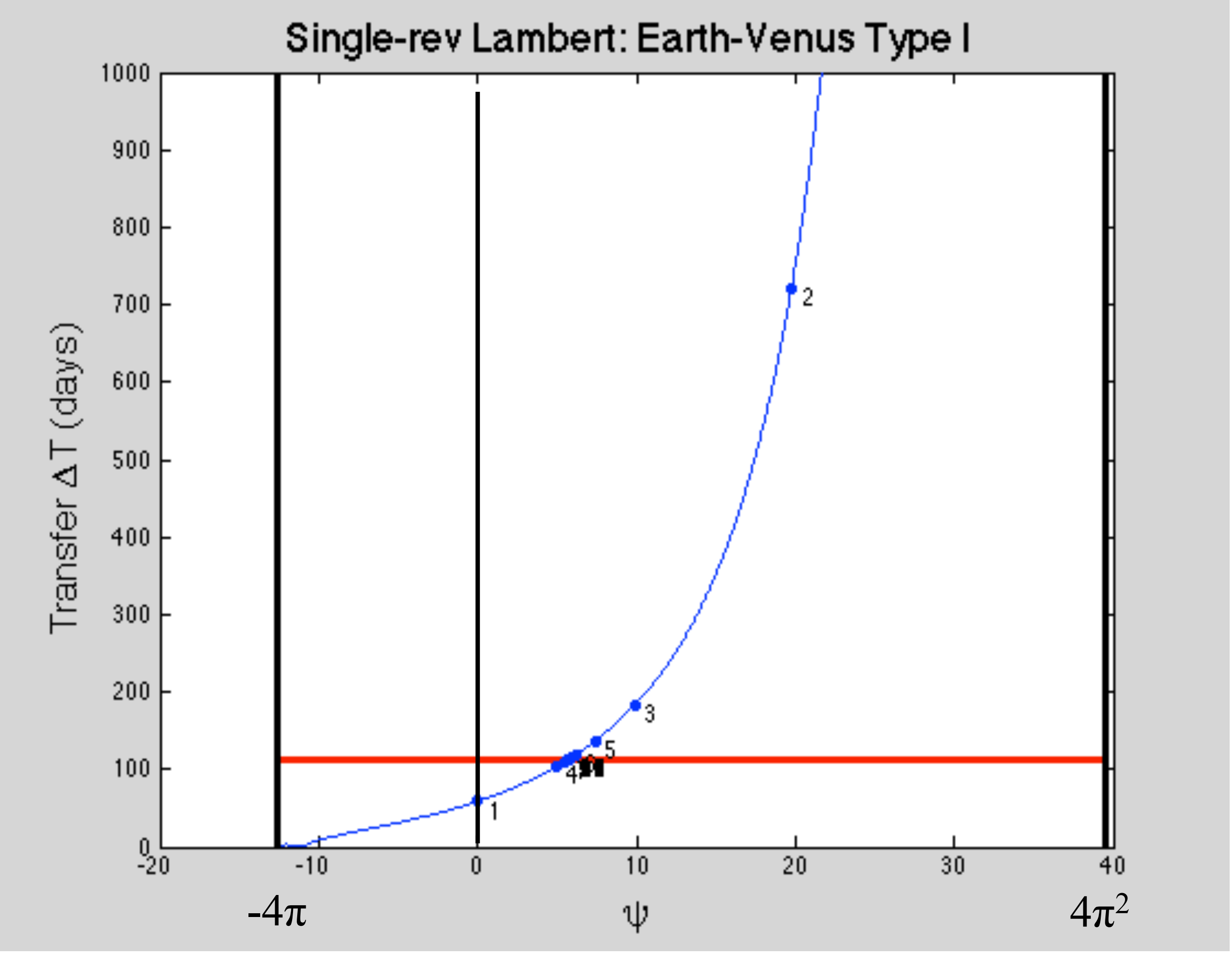
Single-Rev Earth-Venus Type I



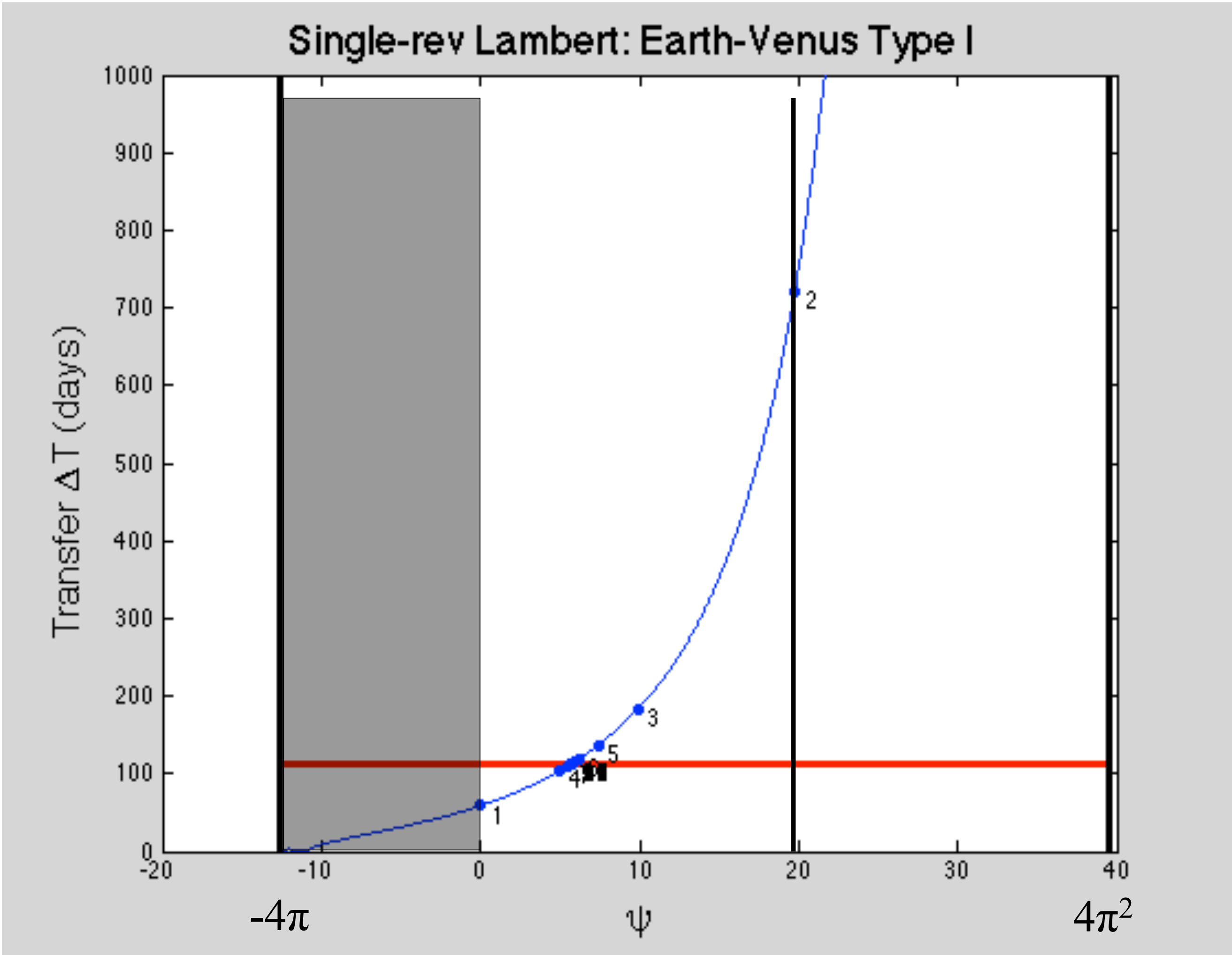
Single-Rev Earth-Venus Type I



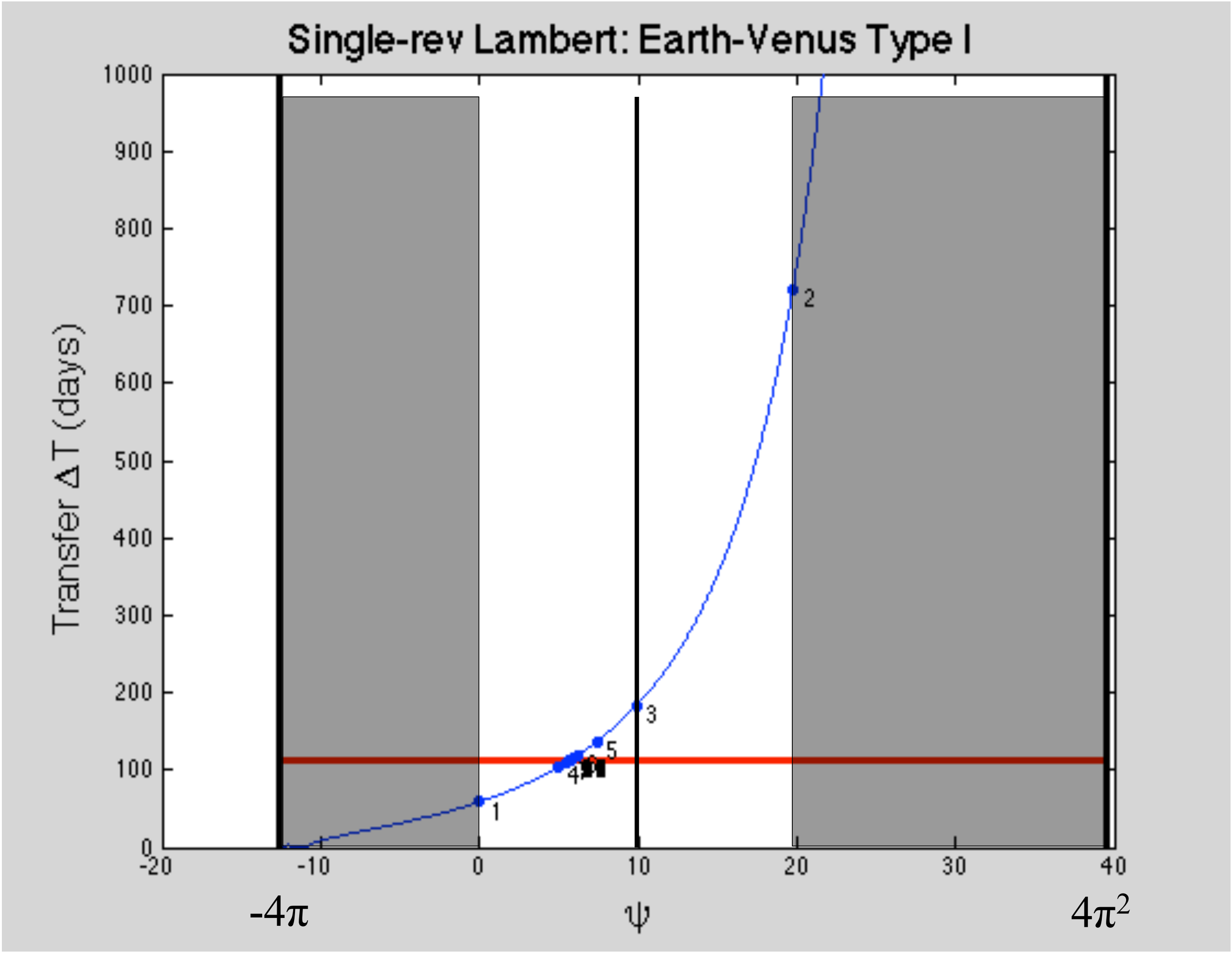
Note: Bisection method



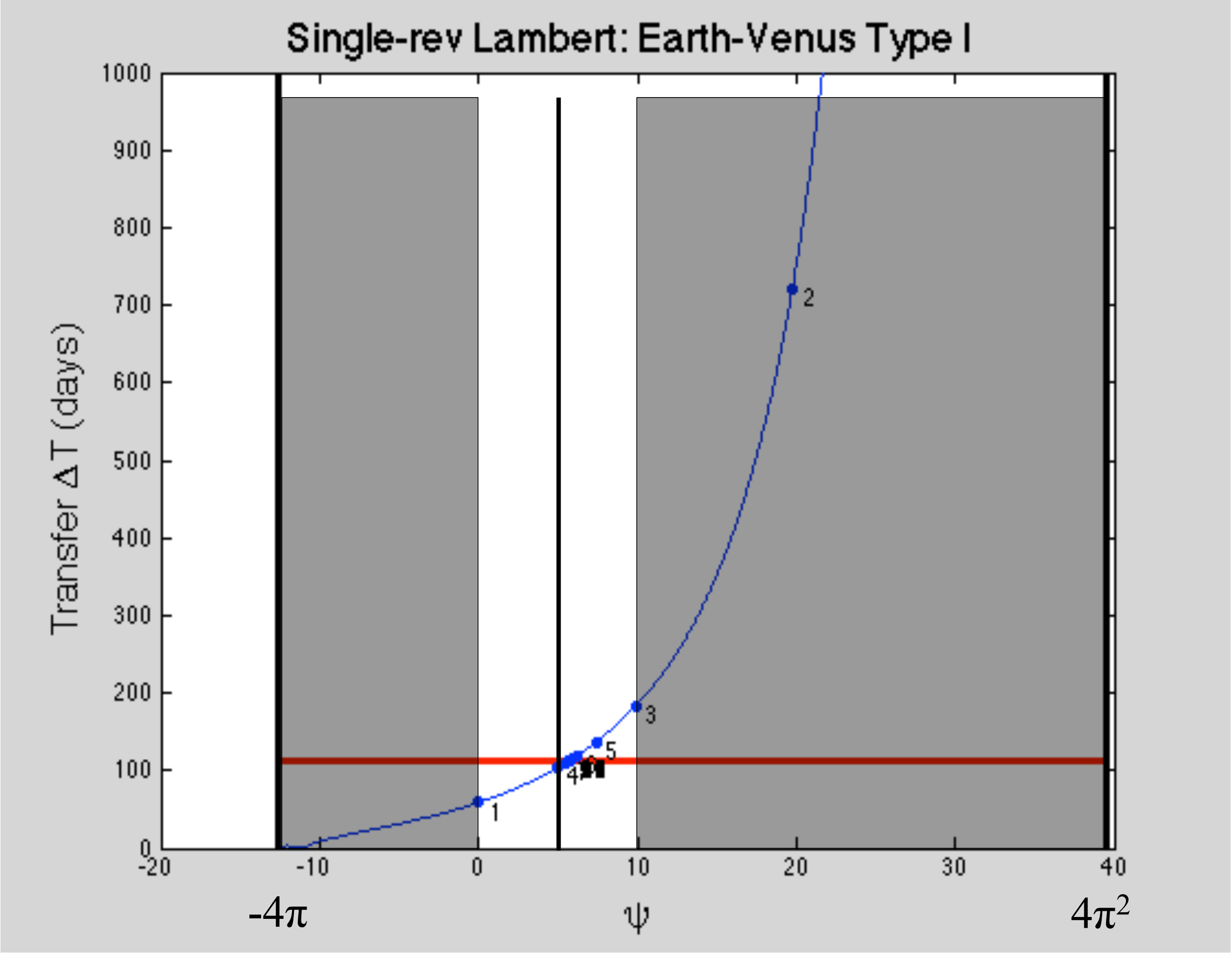
Note: Bisection method



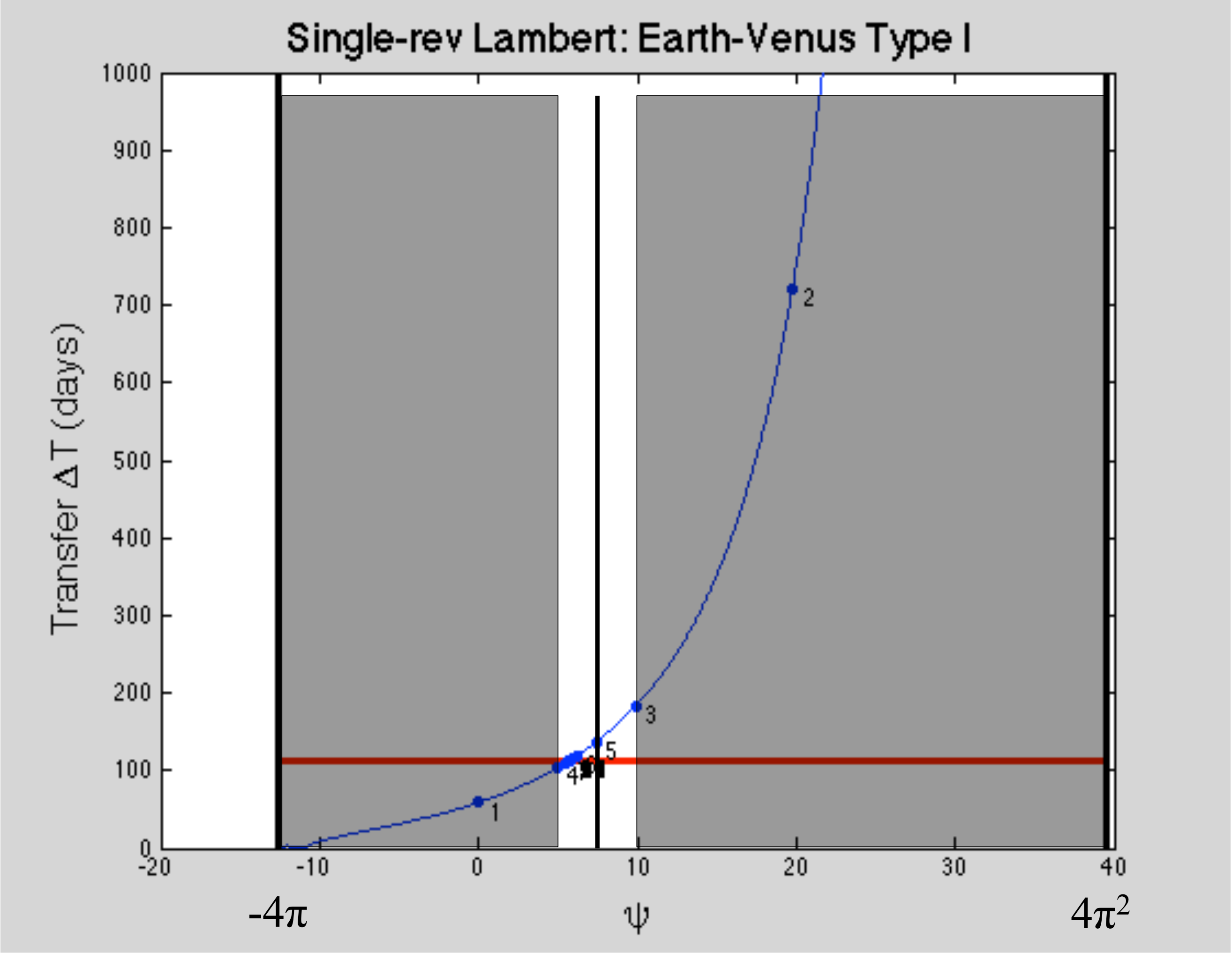
Note: Bisection method



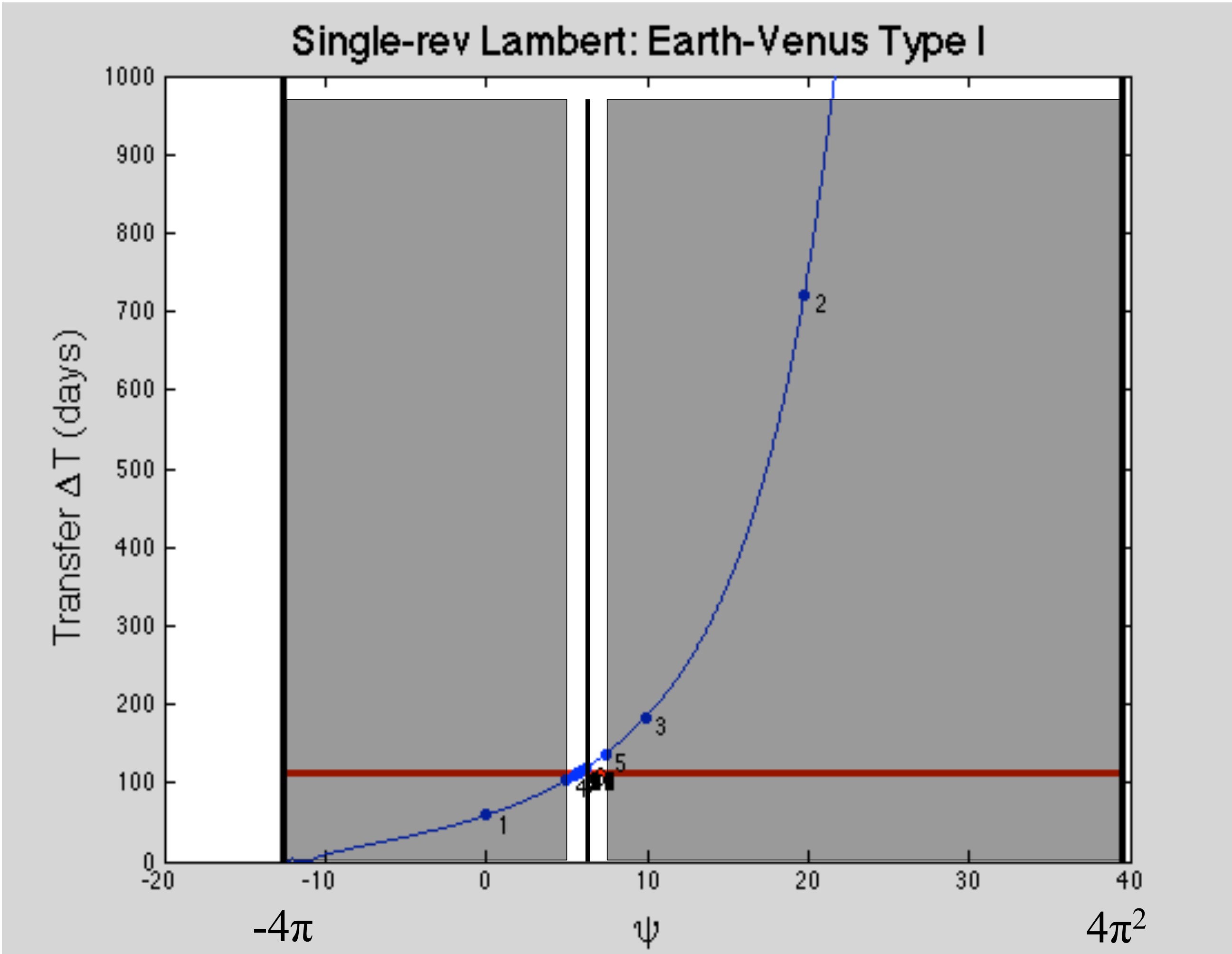
Note: Bisection method



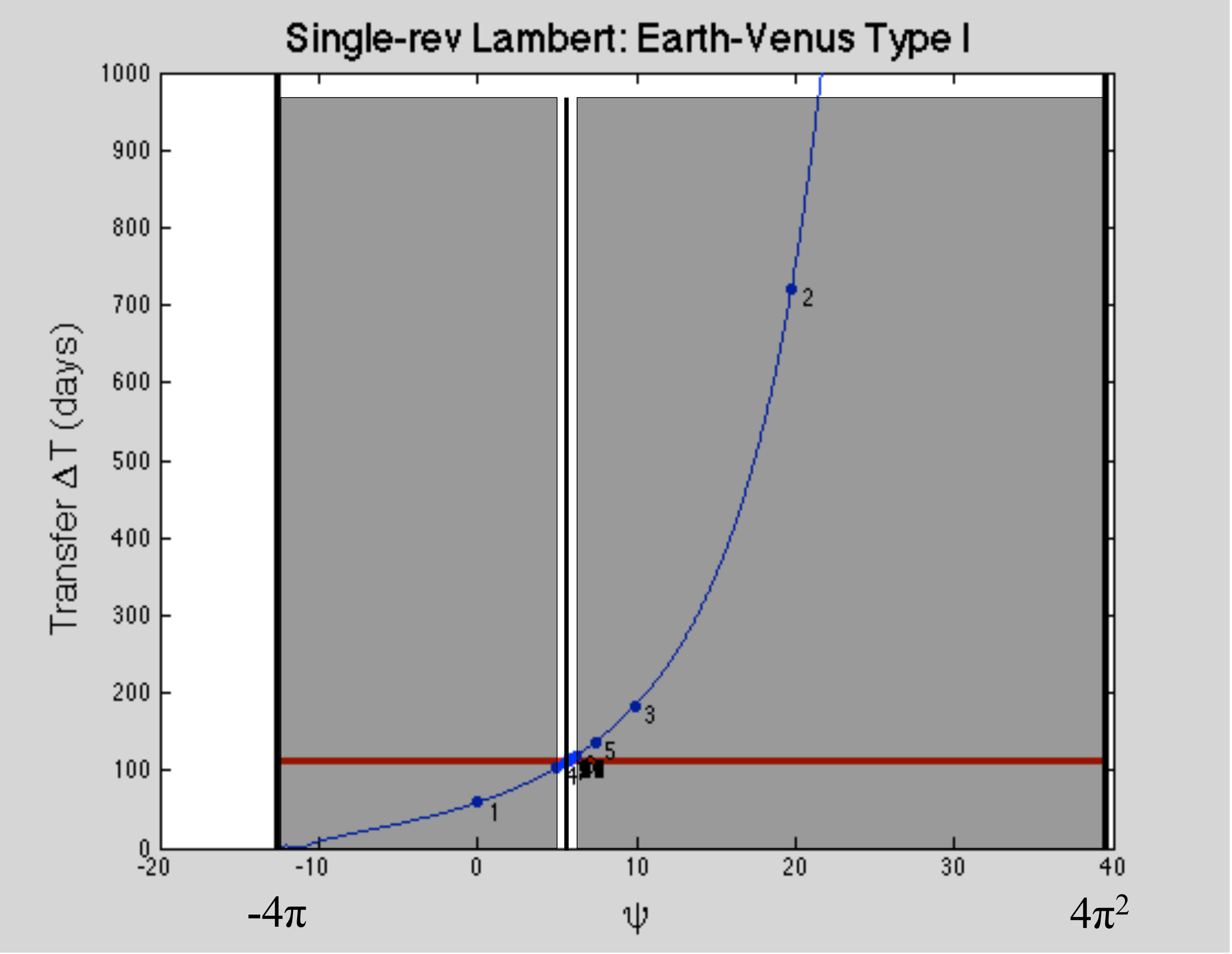
Note: Bisection method



Note: Bisection method



Note: Bisection method

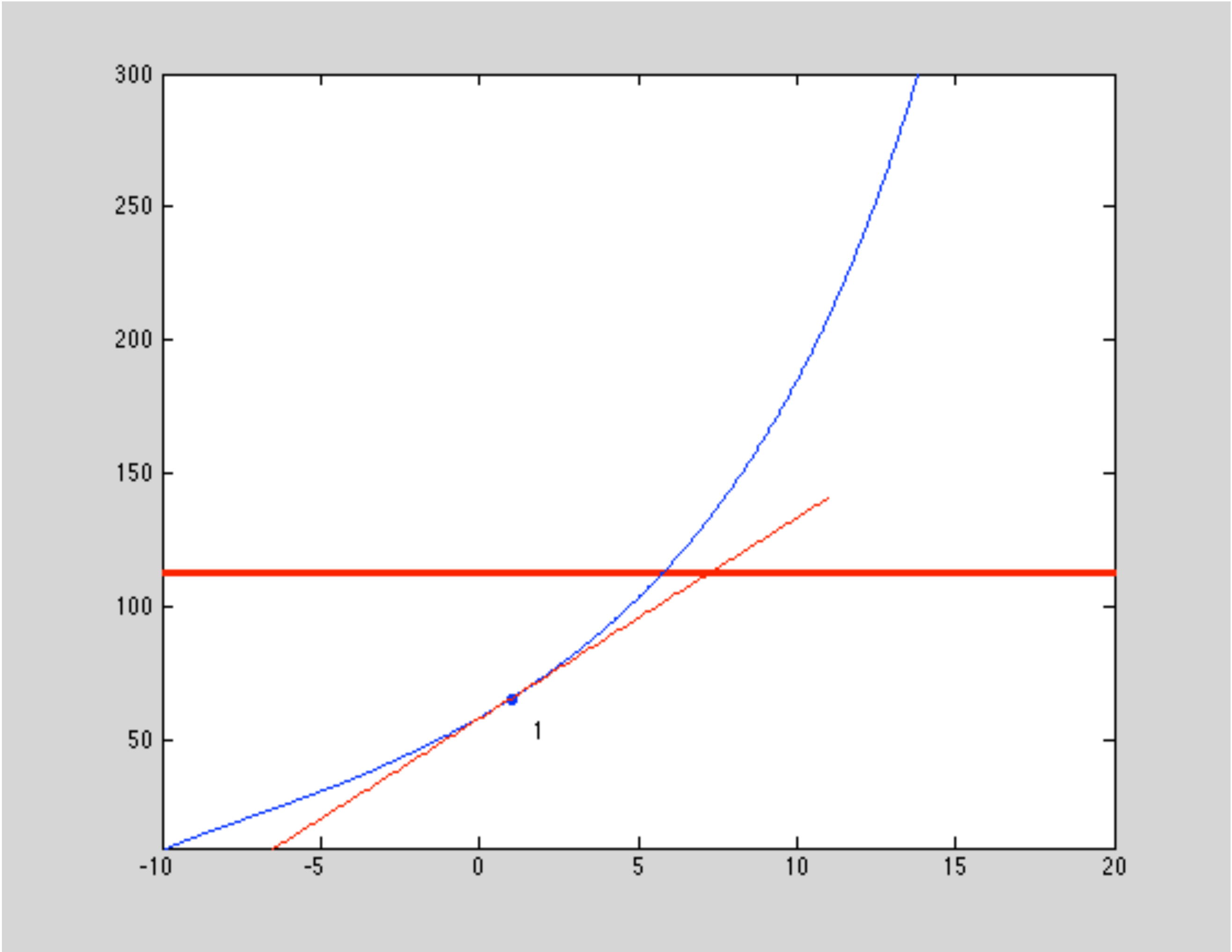


Note: Bisection method

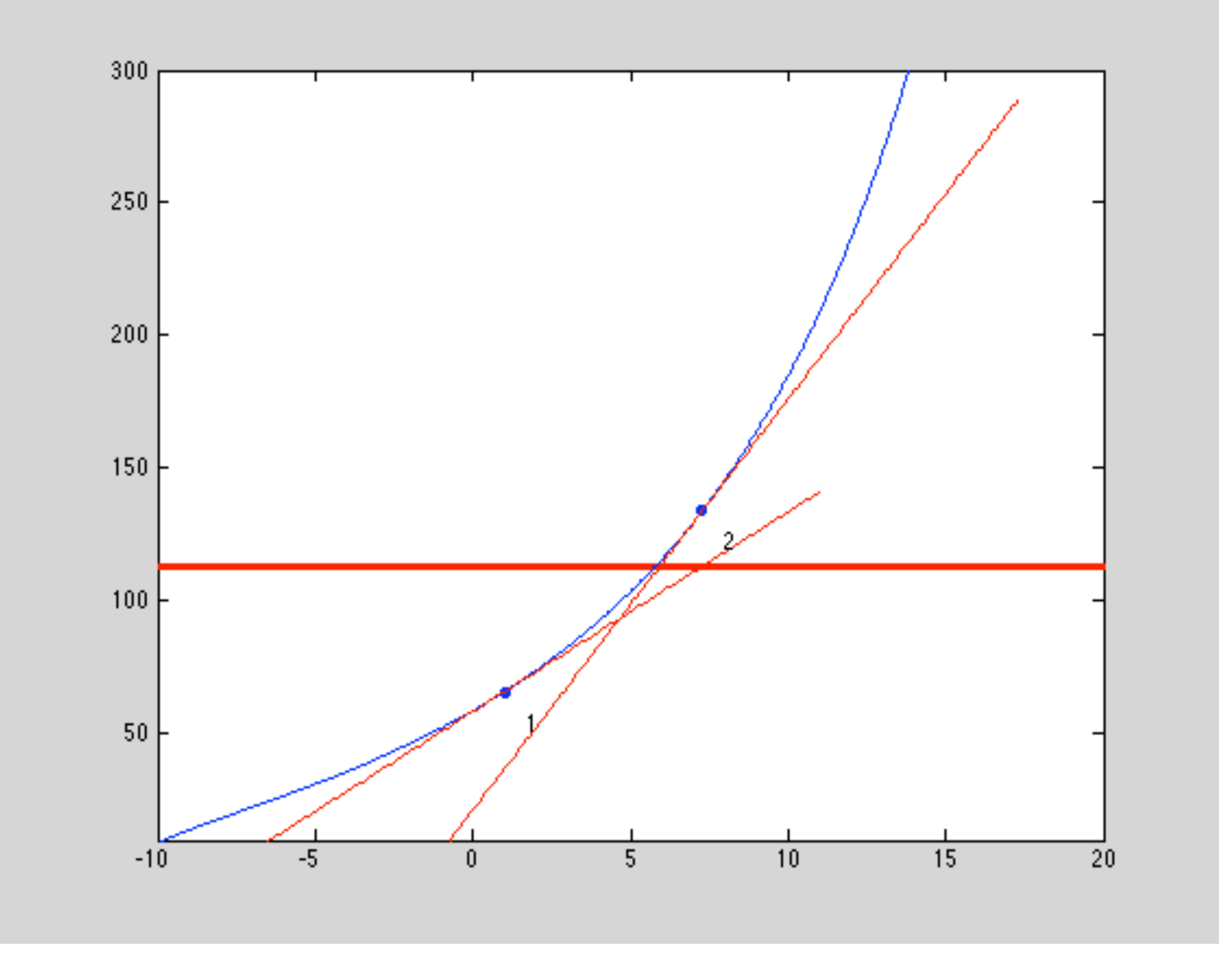
- Time history of bisection method:
- Requires 42 steps to hit a tolerance of 10^{-5} seconds!

```
Step 01:   Tol:  4690082.968075 sec     Psi Window: 5.20e+01
Step 02:   Tol: 52520154.898378 sec     Psi Window: 3.95e+01
Step 03:   Tol:  5970033.357845 sec     Psi Window: 1.97e+01
Step 04:   Tol:   889981.083213 sec     Psi Window: 9.87e+00
Step 05:   Tol:  1995796.491269 sec     Psi Window: 4.93e+00
Step 06:   Tol:   442783.480215 sec     Psi Window: 2.47e+00
Step 07:   Tol:   248507.049603 sec     Psi Window: 1.23e+00
Step 08:   Tol:    90606.211352 sec     Psi Window: 6.17e-01
Step 09:   Tol:    80543.974820 sec     Psi Window: 3.08e-01
Step 10:   Tol:     4627.884702 sec     Psi Window: 1.54e-01
Step 11:   Tol:    38058.242592 sec     Psi Window: 7.71e-02
Step 12:   Tol:    16740.304353 sec     Psi Window: 3.86e-02
Step 13:   Tol:     6062.500709 sec     Psi Window: 1.93e-02
Step 14:   Tol:      718.881918 sec     Psi Window: 9.64e-03
Step 15:   Tol:     1954.107764 sec     Psi Window: 4.82e-03
Step 16:   Tol:      617.514535 sec     Psi Window: 2.41e-03
Step 17:   Tol:       50.708286 sec     Psi Window: 1.20e-03
Step 18:   Tol:      283.396975 sec     Psi Window: 6.02e-04
Step 19:   Tol:      116.342807 sec     Psi Window: 3.01e-04
Step 20:   Tol:       32.816876 sec     Psi Window: 1.51e-04
Step 21:   Tol:        8.945801 sec     Psi Window: 7.53e-05
Step 22:   Tol:       11.935513 sec     Psi Window: 3.76e-05
Step 23:   Tol:        1.494850 sec     Psi Window: 1.88e-05
Step 24:   Tol:        3.725477 sec     Psi Window: 9.41e-06
Step 25:   Tol:        1.115314 sec     Psi Window: 4.71e-06
Step 26:   Tol:        0.189768 sec     Psi Window: 2.35e-06
Step 27:   Tol:        0.462773 sec     Psi Window: 1.18e-06
Step 28:   Tol:        0.136503 sec     Psi Window: 5.88e-07
Step 29:   Tol:        0.026633 sec     Psi Window: 2.94e-07
Step 30:   Tol:        0.054935 sec     Psi Window: 1.47e-07
Step 31:   Tol:        0.014151 sec     Psi Window: 7.35e-08
Step 32:   Tol:        0.006241 sec     Psi Window: 3.68e-08
Step 33:   Tol:        0.003955 sec     Psi Window: 1.84e-08
Step 34:   Tol:        0.001143 sec     Psi Window: 9.19e-09
Step 35:   Tol:        0.001406 sec     Psi Window: 4.60e-09
Step 36:   Tol:        0.000132 sec     Psi Window: 2.30e-09
Step 37:   Tol:        0.000506 sec     Psi Window: 1.15e-09
Step 38:   Tol:        0.000187 sec     Psi Window: 5.74e-10
Step 39:   Tol:        0.000028 sec     Psi Window: 2.87e-10
Step 40:   Tol:        0.000052 sec     Psi Window: 1.44e-10
Step 41:   Tol:        0.000012 sec     Psi Window: 7.18e-11
Step 42:   Tol:        0.000008 sec     Psi Window: 3.59e-11
```

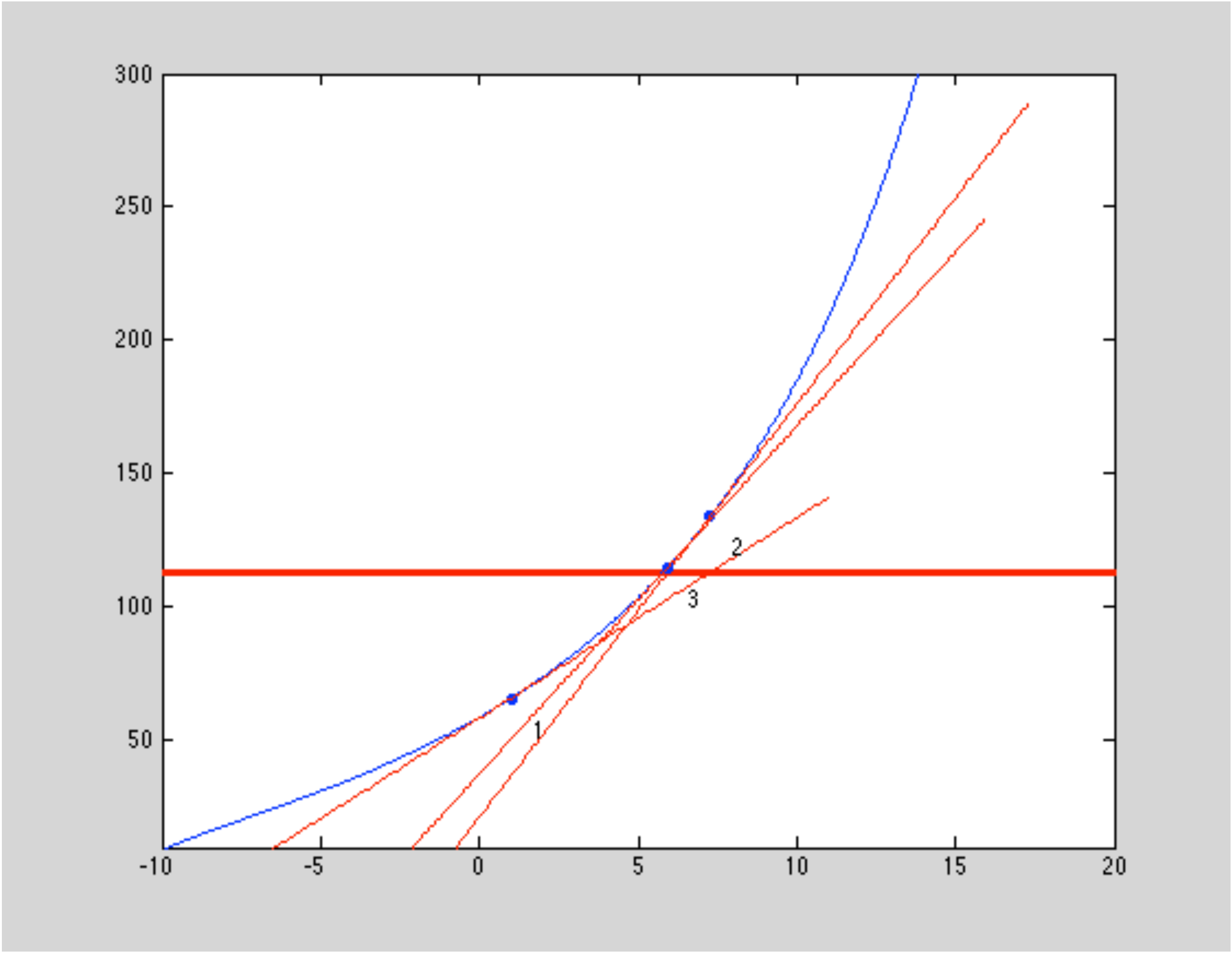
Note: Newton Raphson method



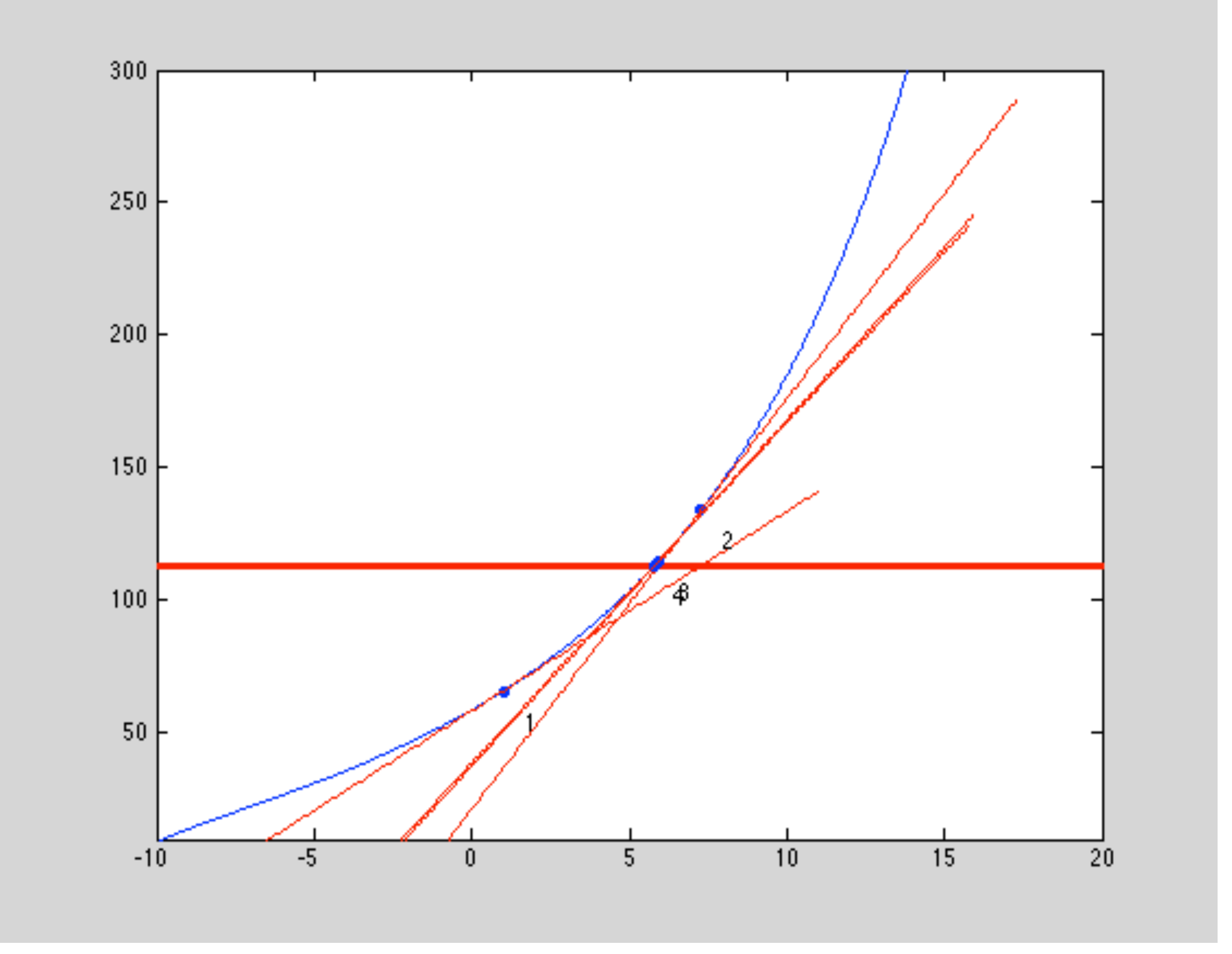
Note: Newton Raphson method



Note: Newton Raphson method



Note: Newton Raphson method



Note: Newton Raphson method

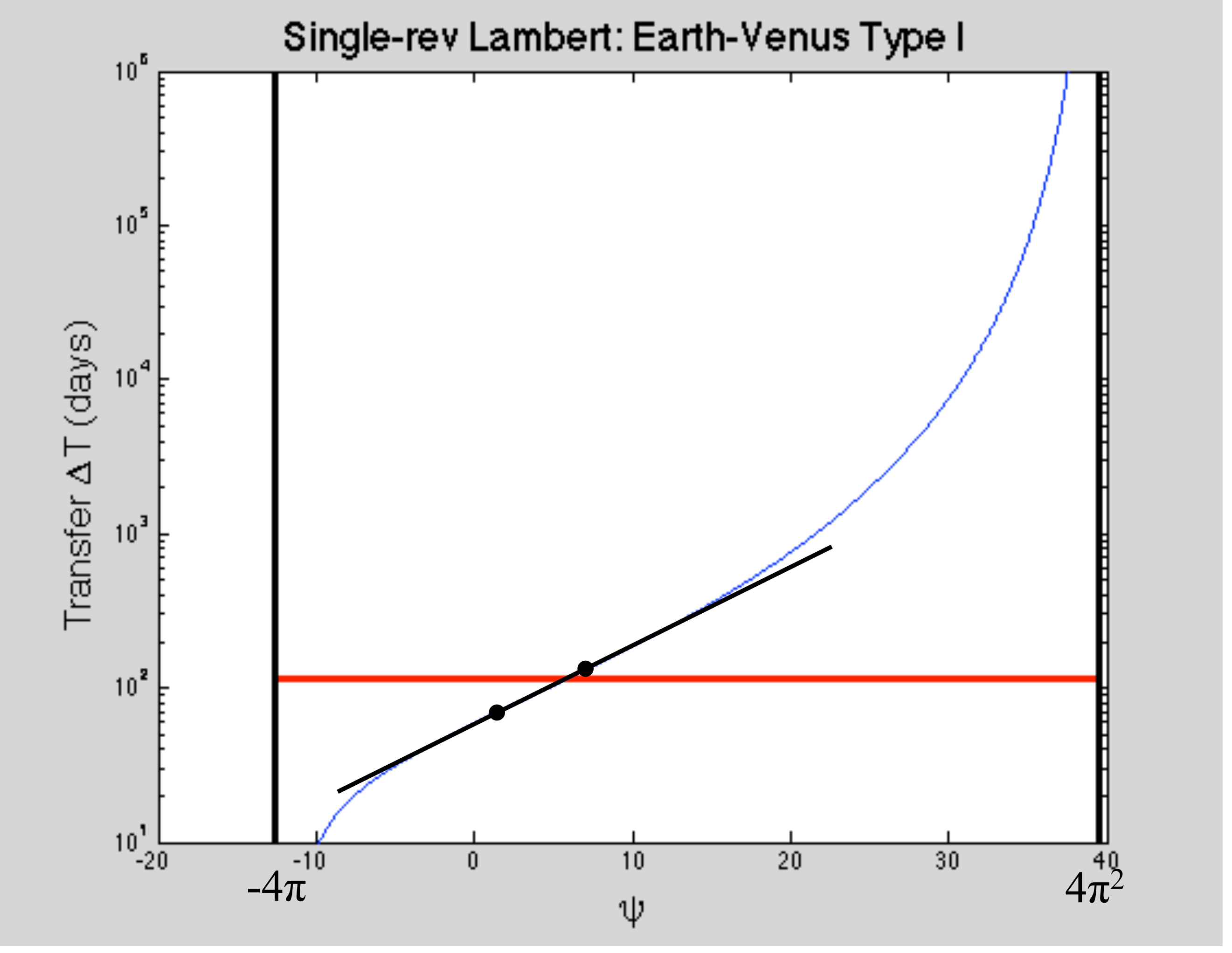
- Time history of Newton Raphson method:

```
Step 01:   Tol:  4071827.045073 sec
Step 02:   Tol:  1814604.134515 sec
Step 03:   Tol:   147501.379528 sec
Step 04:   Tol:     1176.622979 sec
Step 05:   Tol:        0.076250 sec
Step 06:   Tol:        0.000000 sec
```

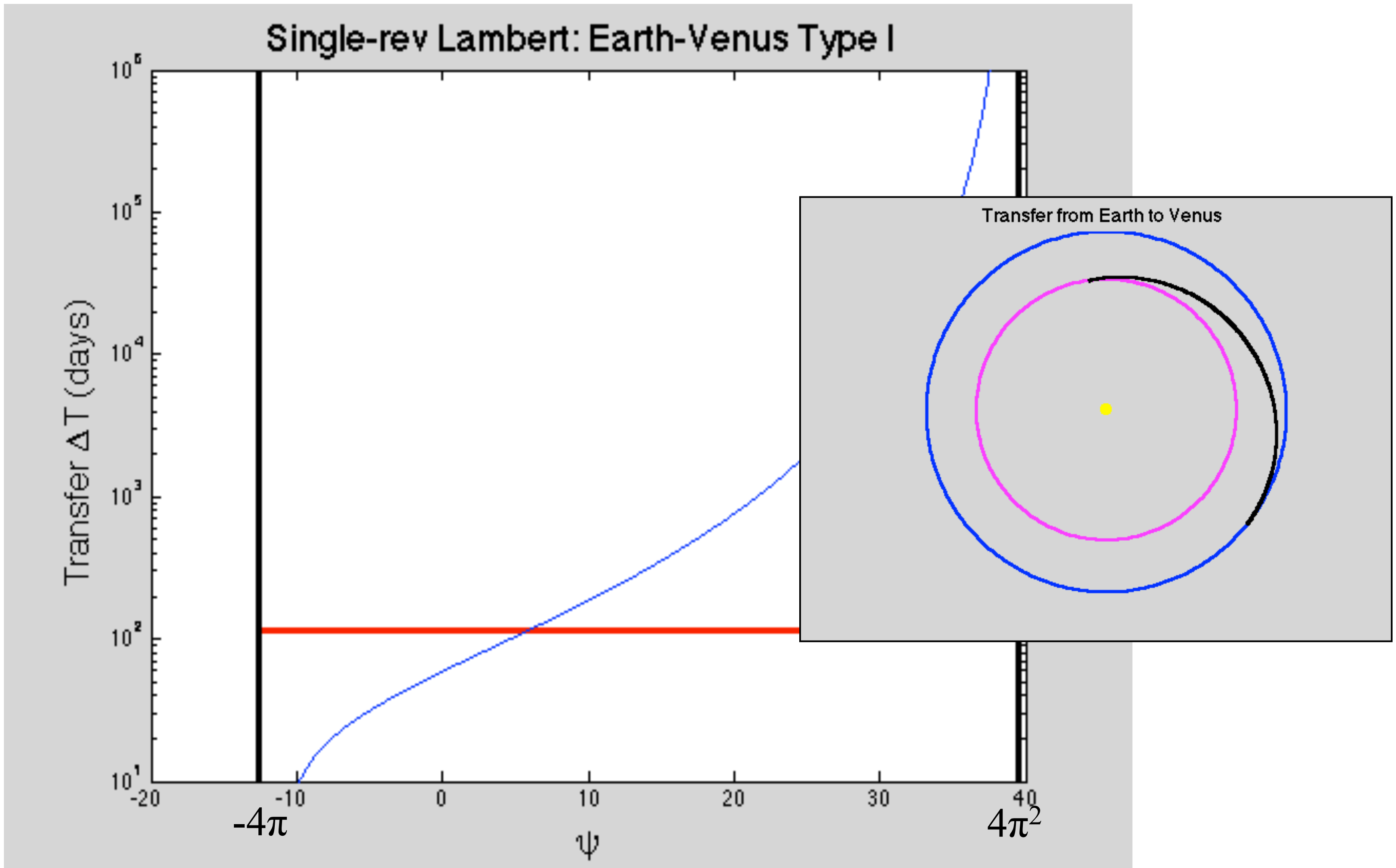
- Requires 6 steps to hit a tolerance of 10^{-5} seconds!
- Note: This CAN break in certain circumstances.
- With current computers, this isn't a HUGE speed-up, so robustness may be preferable.

Note: Newton Raphson Log Method

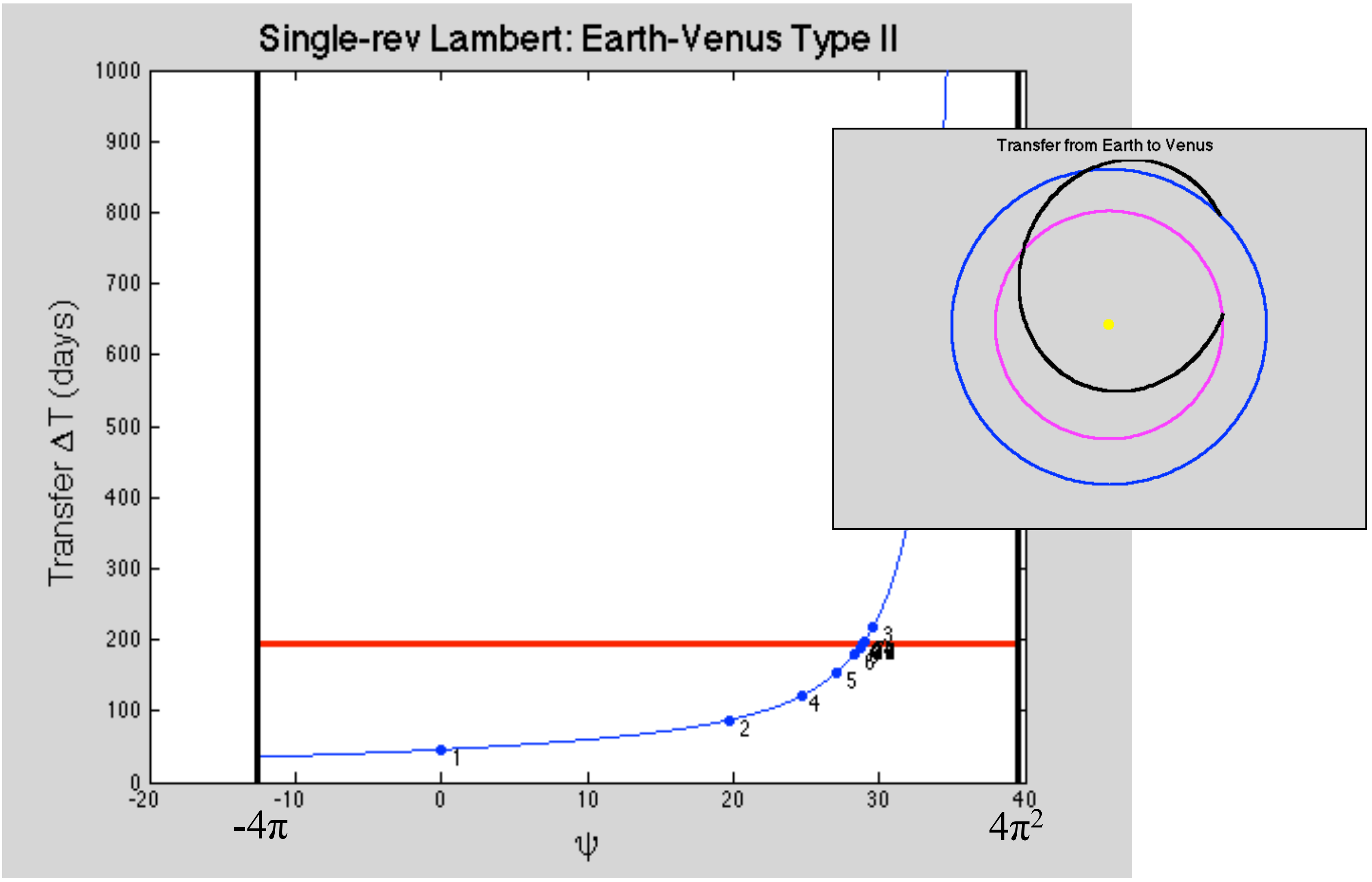
Note log scale



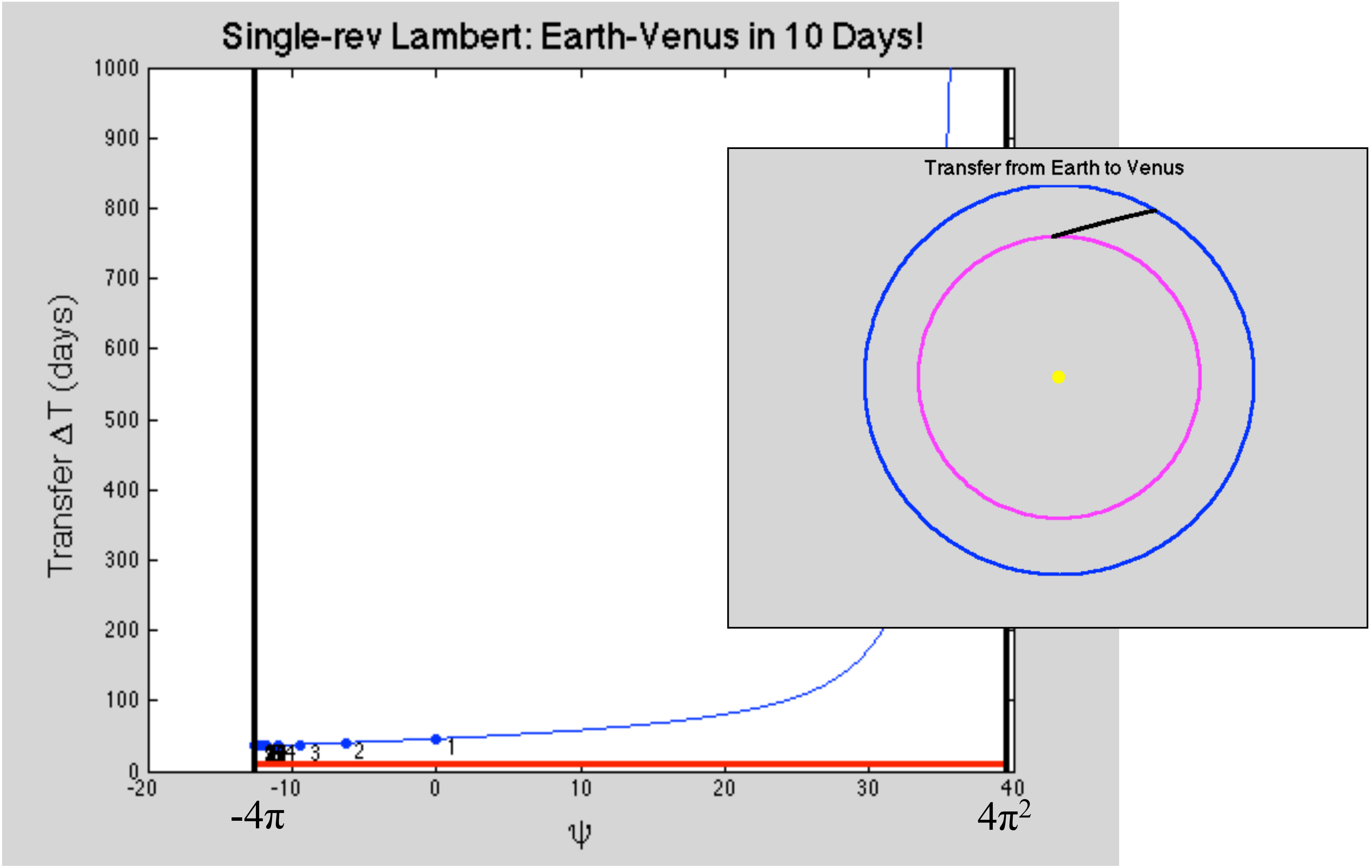
Single-Rev Earth-Venus Type I



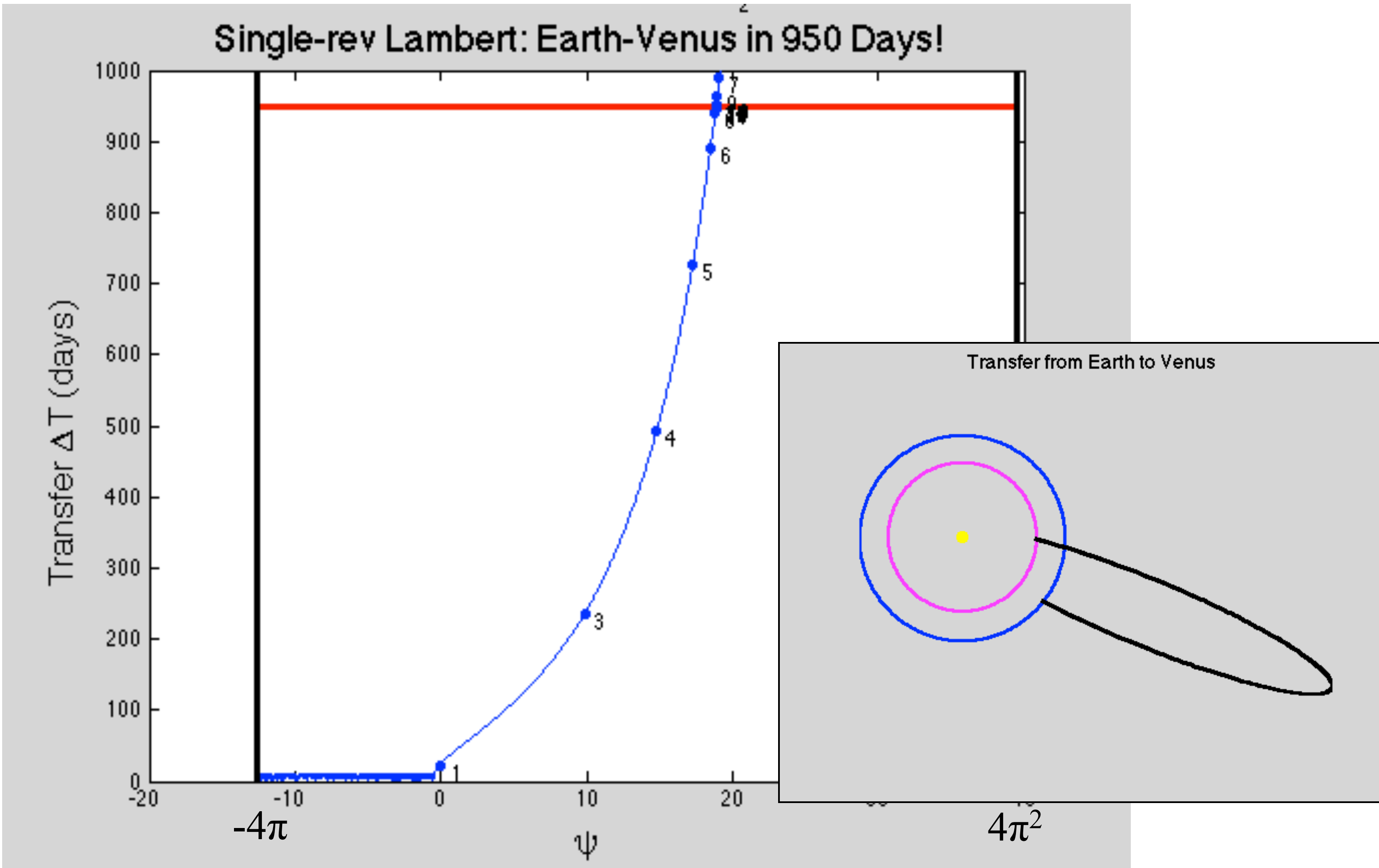
Single-Rev Earth-Venus Type II



Interesting: 10-day transfer



Interesting: 950-day transfer



Multi-Rev

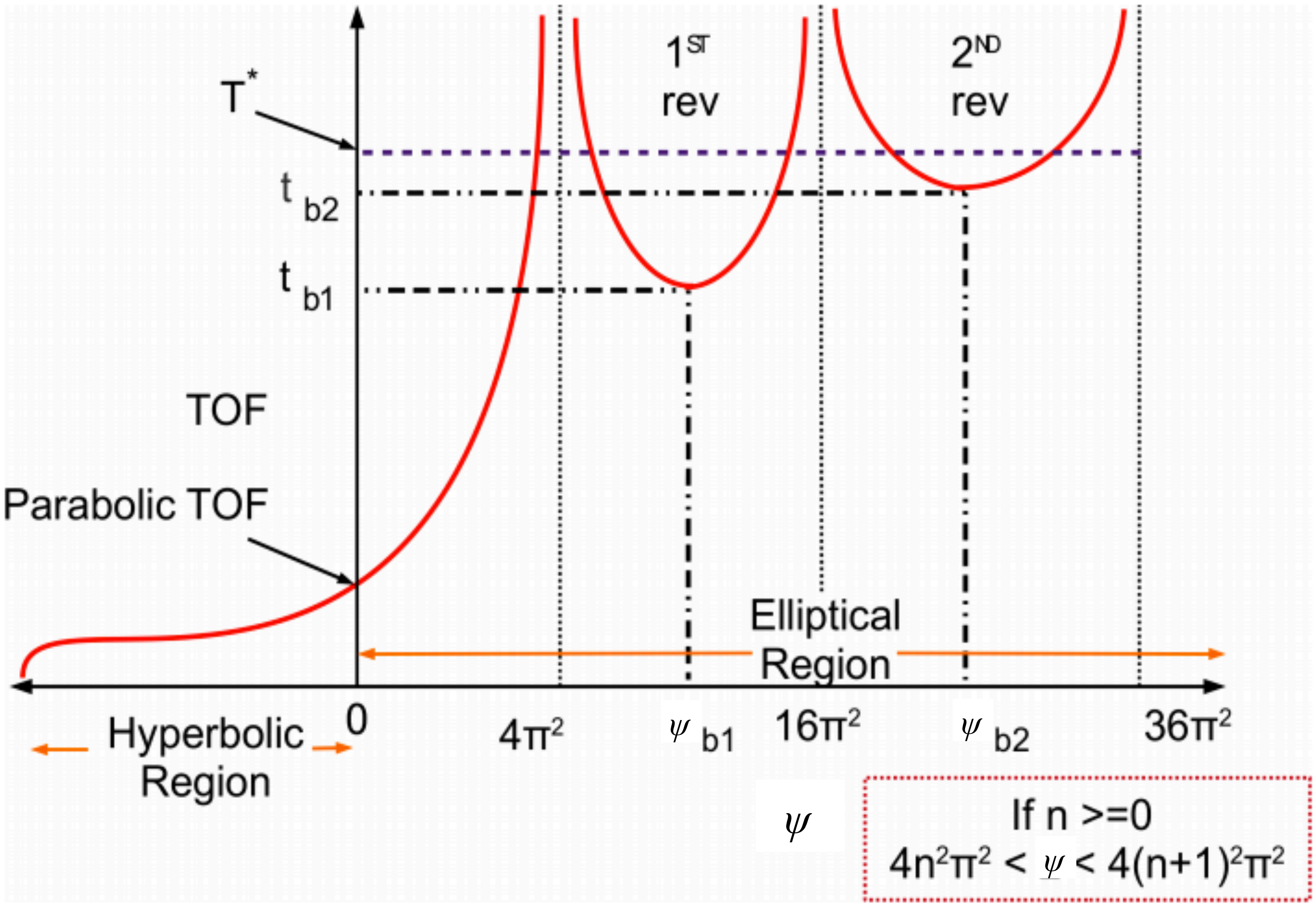
- Seems like it would be better to perform a multi-rev solution over 950 days than a Type II transfer!

A few details

- The universal variables construct ψ represents the following transfer types:

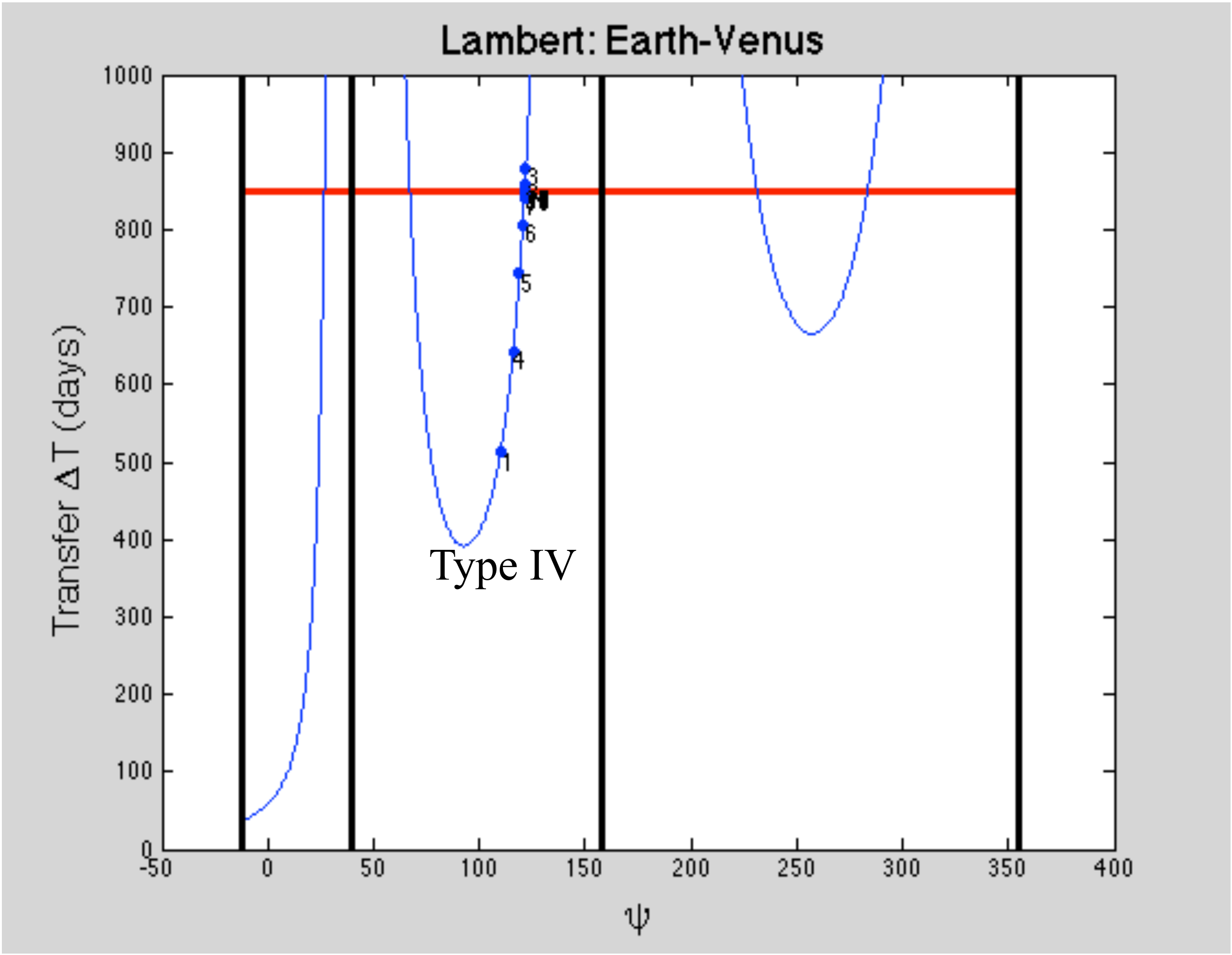
$$\text{Type of Transfer}\begin{cases} \psi < 0, & \text{Hyperbolic} \\ \psi = 0, & \text{Parabolic} \\ 0 < \psi < 4\pi^2, & 0 \text{ revolutions elliptical} \\ 4n^2\pi^2 < \psi < 4(n+1)^2\pi^2, & n \text{ revolutions elliptical} \end{cases}$$

Multi-Rev



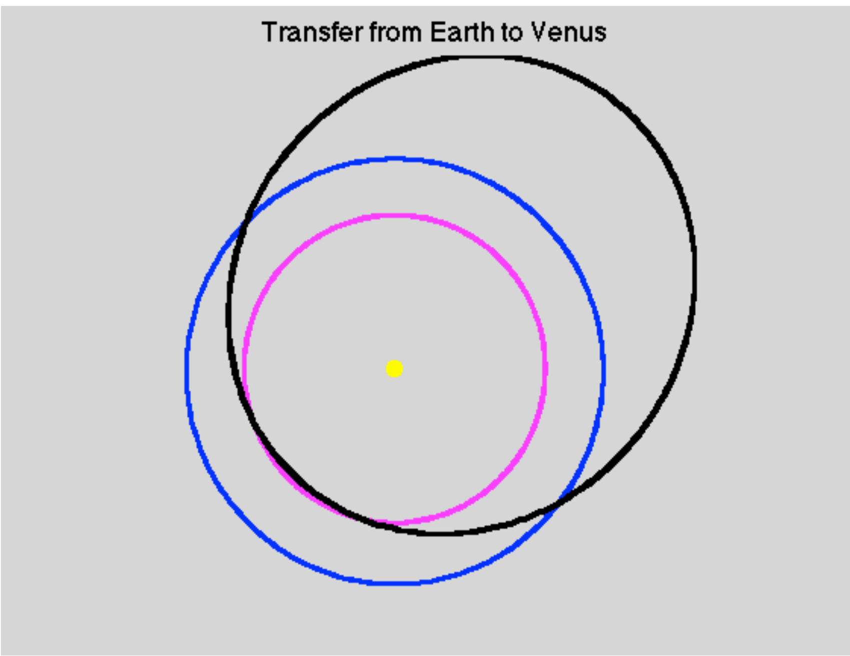
T*
t b2
t b1
TOF
Parabolic TOF
1ST rev
2ND rev
Elliptical Region
Hyperbolic Region
0
4π2
ψ b1
16π2
ψ b2
36π2
ψ
If n >=0
4n2π2 < ψ < 4(n+1)2π2

Earth-Venus in 850 days

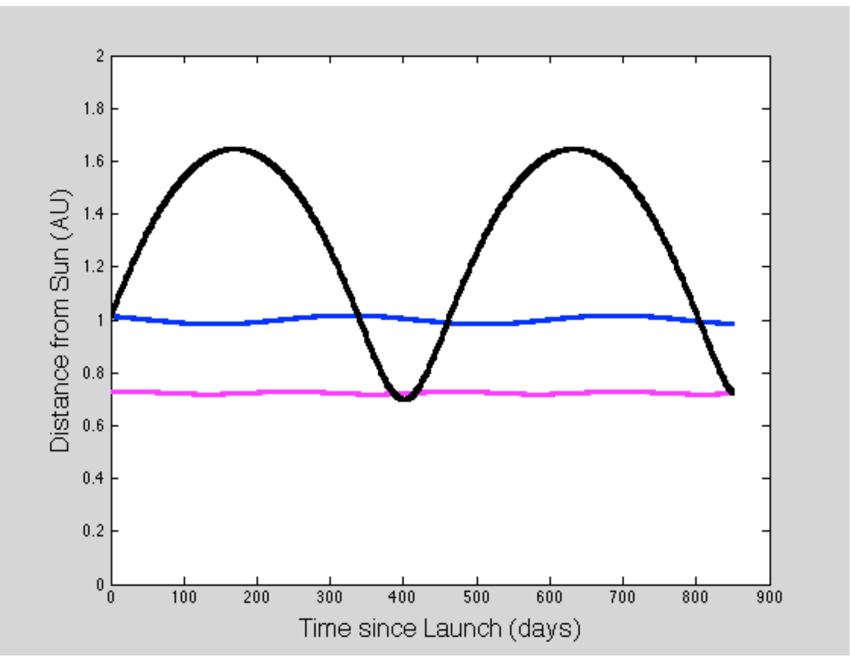


Earth-Venus in 850 days

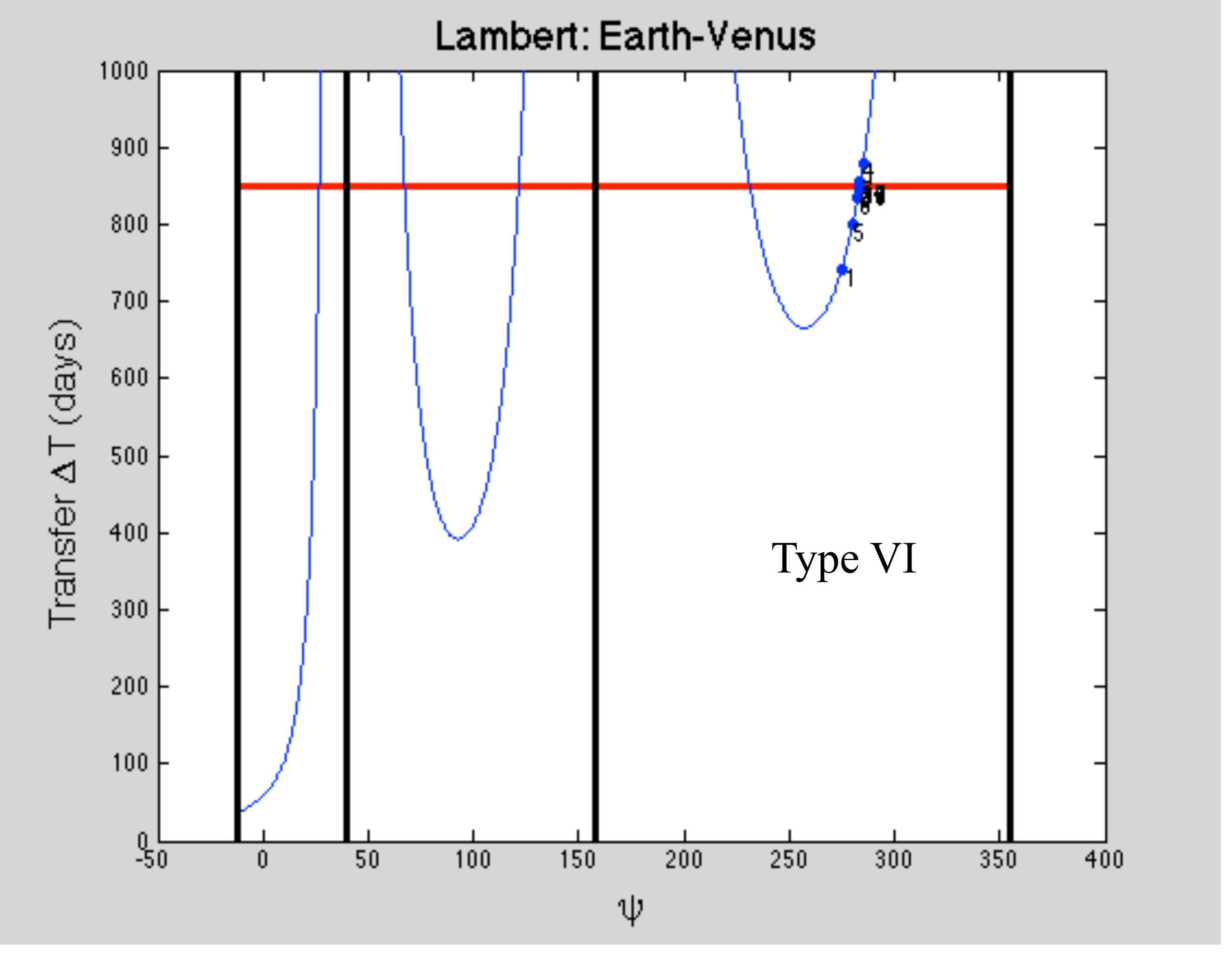
Heliocentric View



Distance to Sun

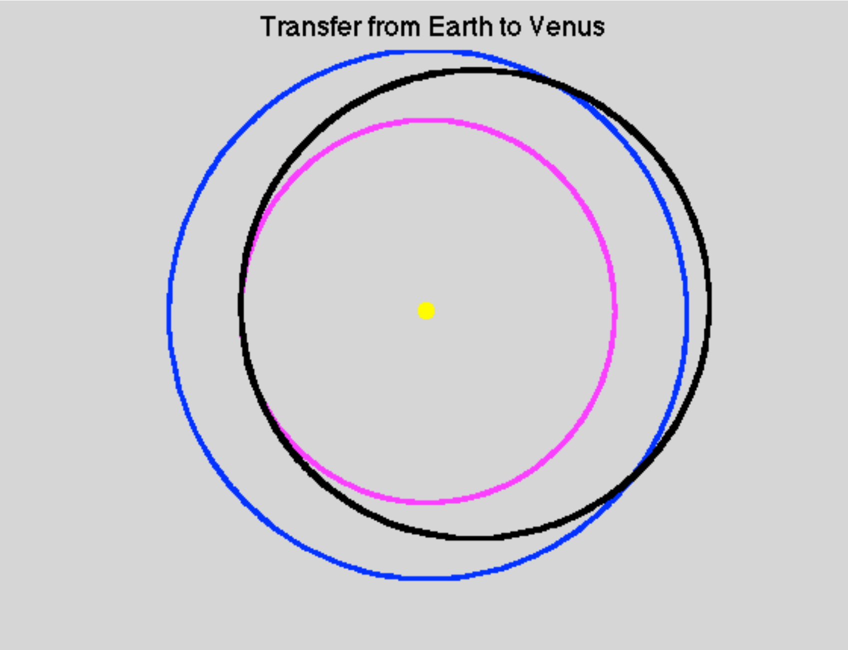


Earth-Venus in 850 days

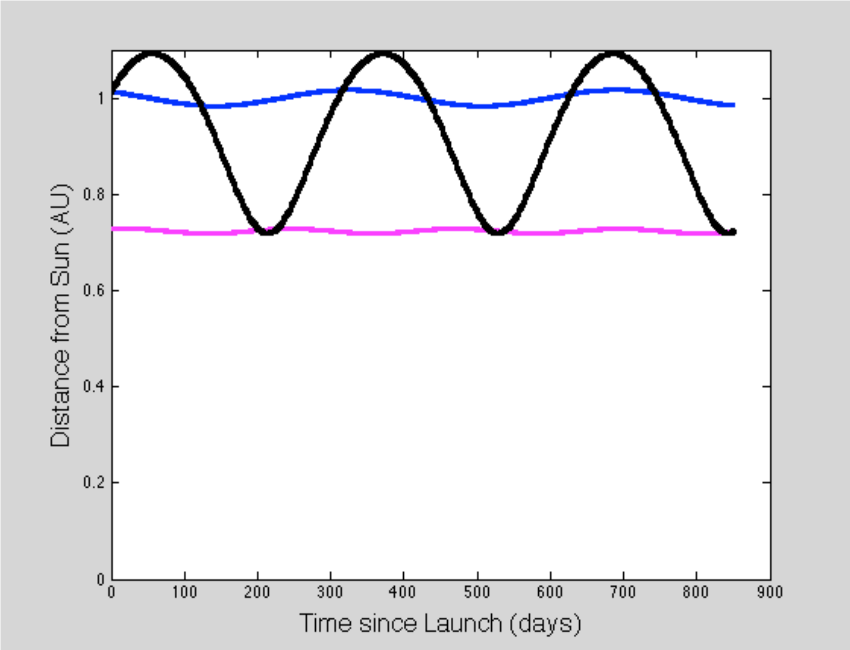


Earth-Venus in 850 days

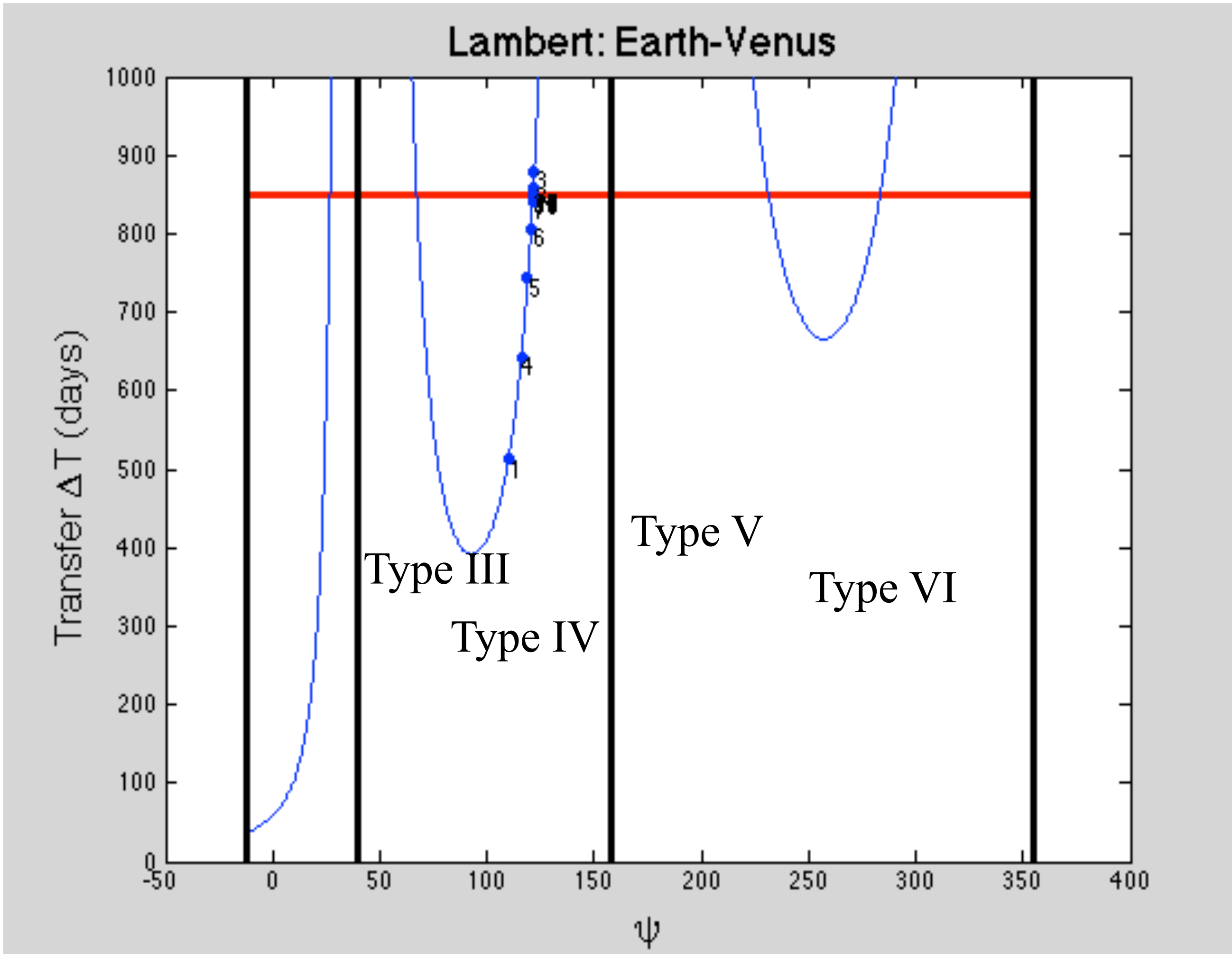
Heliocentric View



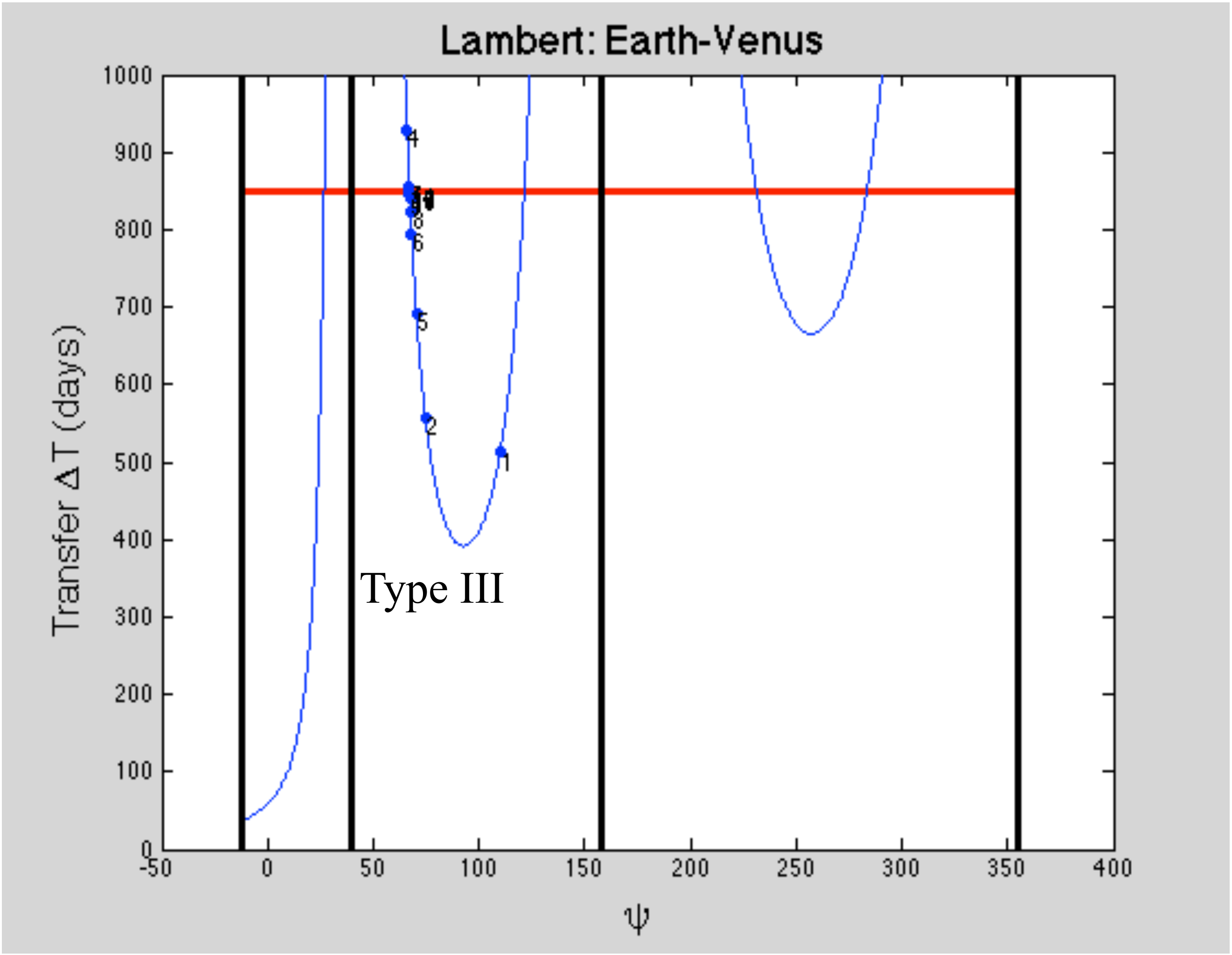
Distance to Sun



What about Type III and V?

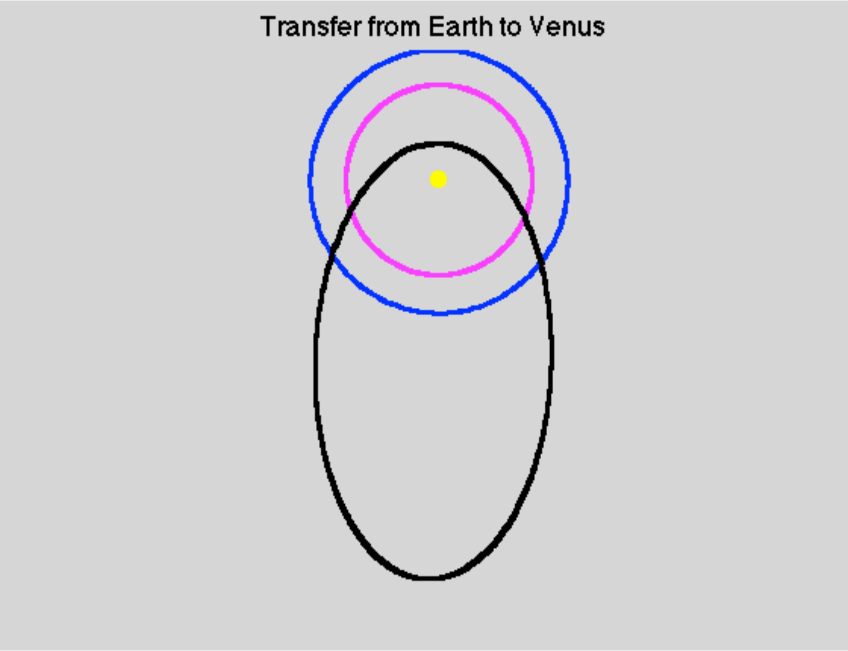


Earth-Venus in 850 days

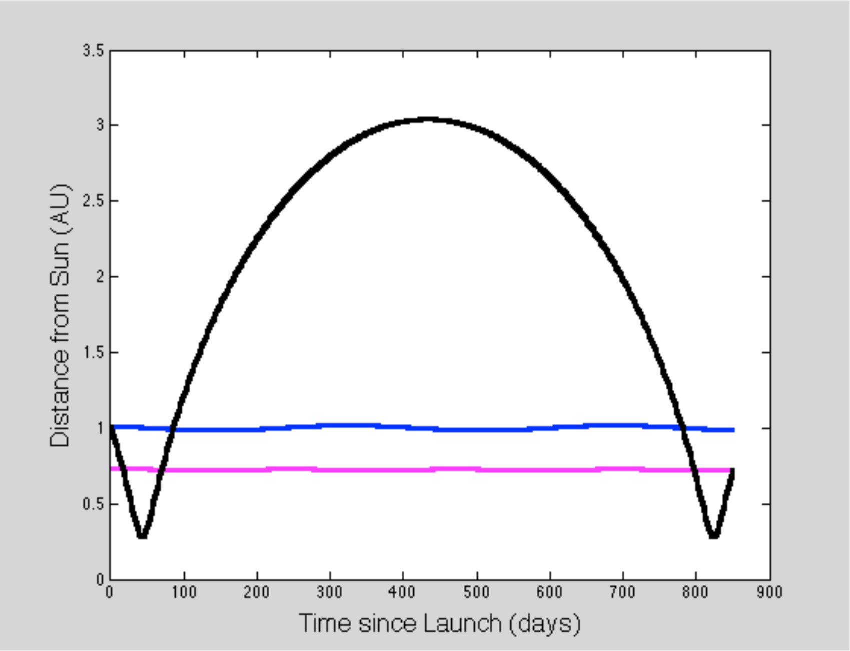


Earth-Venus in 850 days

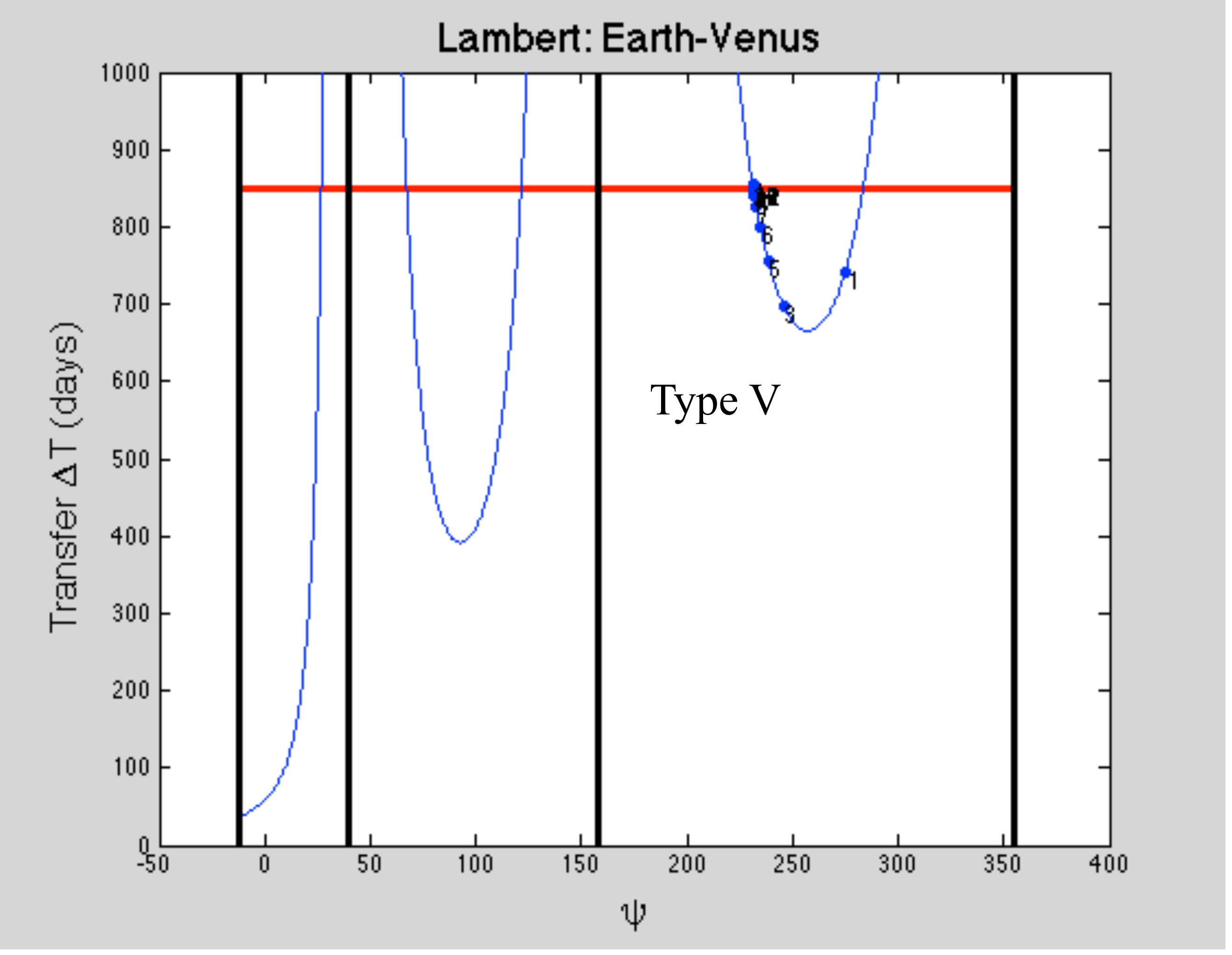
Heliocentric View



Distance to Sun

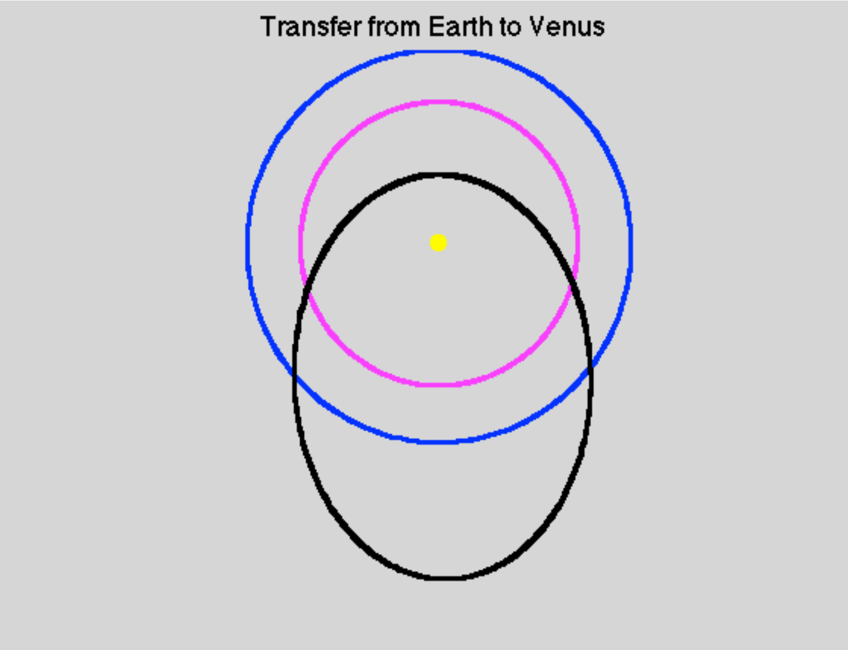


Earth-Venus in 850 days

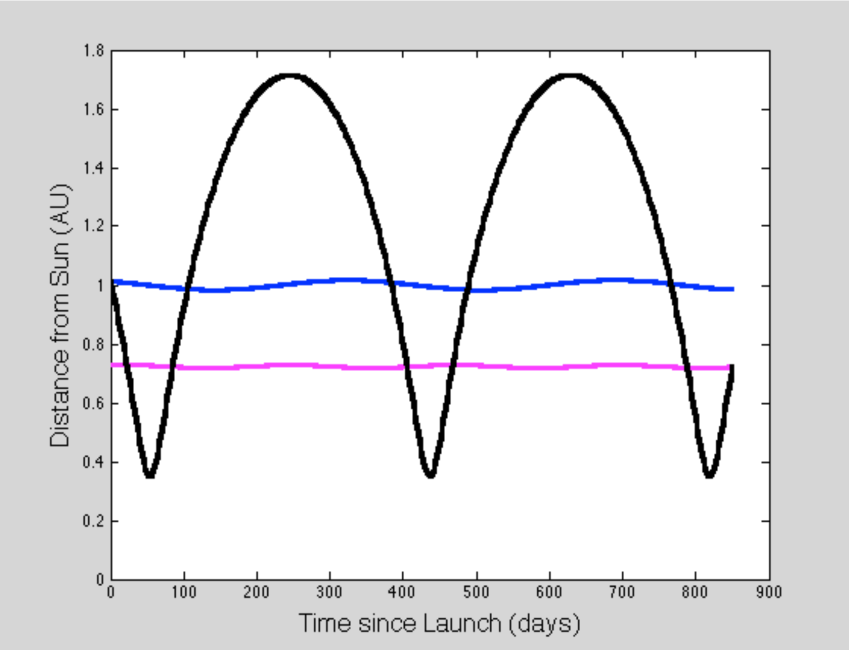


Earth-Venus in 850 days

Heliocentric View



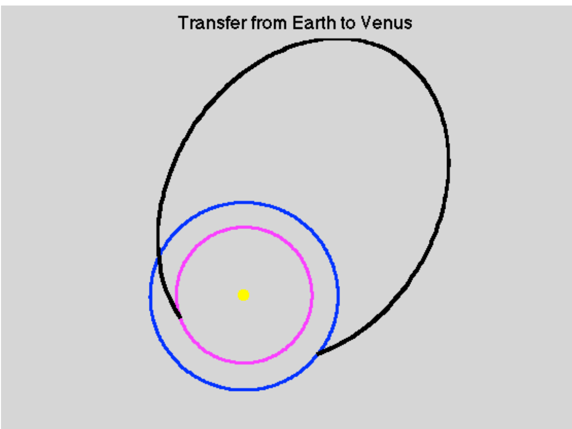
Distance to Sun

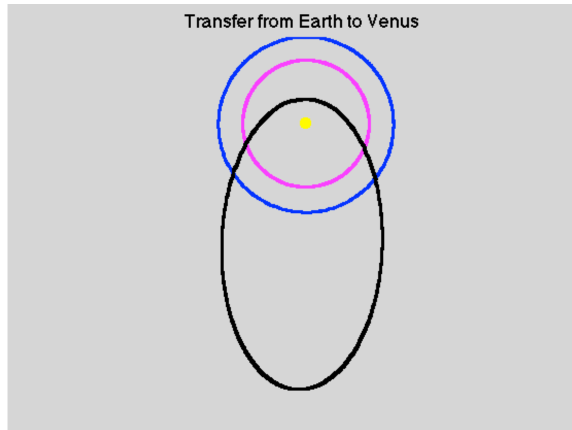


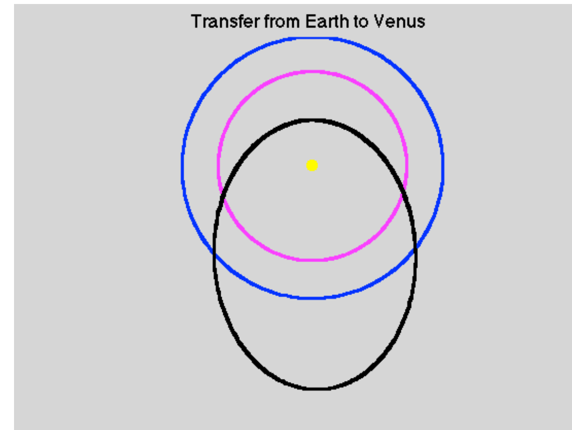
Summary

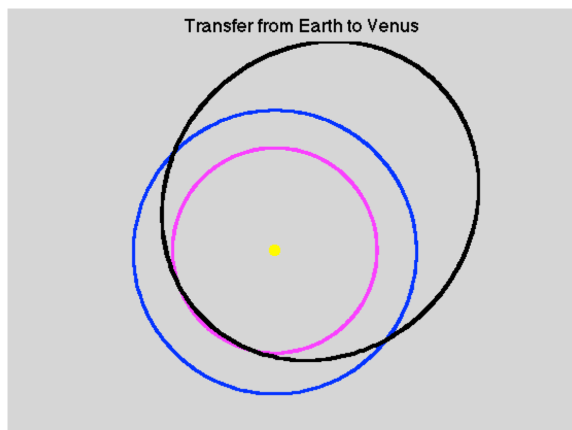
- The bisection method requires modifications for multi-rev.
- Also requires modifications for odd- and even-type transfers.
- Newton Raphson is very fast, but not as robust.
- If you're interested in surveying numerous revolution combinations then it may be just as well to use the bisection method to improve robustness

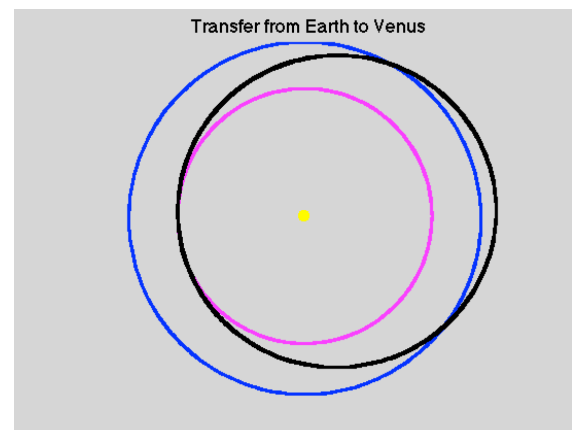
Types II - VI


Transfer from Earth to Venus


Transfer from Earth to Venus


Transfer from Earth to Venus


Transfer from Earth to Venus


Transfer from Earth to Venus

Announcements

- Homework #5 due Thursday
- Quiz #10 tomorrow

- Mid-term to be handed out on Thursday, Oct 17th. It will be due on Tuesday, Oct 22nd. CAETE due date is Tuesday, Oct 29th. Each person will have the same amount of time for the test – CAETE students just have more flexibility to schedule when they work on the test. Open book open note, no working with others.

- Reading: Chapter 7

- Space News:
  - Juno's Earth flyby is tomorrow!

Credit: NASA/JPL-Caltech