Comment on "Low-Noise Linear Combination of Triple-Frequency Carrier Phase Measurements"

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Abstract

Elucidating the nature of the familiar closed-form solution arising in the optimization of [1].

1 The underlying optimization problem

Based on prior experience with quadratic forms as cost functions arising in optimization problems whose solutions are useful in navigation applications [2]-[8][1], we recognized the underlying problem posed in [1] to be a familiar minimization of a convex paraboloidal function $y = f(x)$ going from Euclidean n-space x to a scalar y (i.e., $f : E^n \rightarrow \mathcal{R}$, where the linear weighting coefficients of [1, Eq. 1]: $[\alpha, \beta, \gamma] \in E^3$), with symmetric positive definite $n \times n$ inner product matrix P (with units being the square of those of x), in seeking to minimize:

$$y = x^T P^{-1} x \text{ (corresponding to [1, Eq. 16], notice that } y \text{ is sans units as an amplification factor)}, \quad (1)$$

subject to the single constraint of lying on an intersecting plane of specified orientation in 3-space:

$$g^T x = d \text{ (corresponding to [1, Eq. 14], notice that all units can divide out)}, \quad (2)$$

where, in the above, g is an n-vector corresponding to the direction numbers perpendicular to the plane, d is a scalar with the same units as x (representing the y-intercept for $x^T = (0, 0, 0)$), and superscript T denotes the transpose. More insight is availed here from Fig. 1 into the true

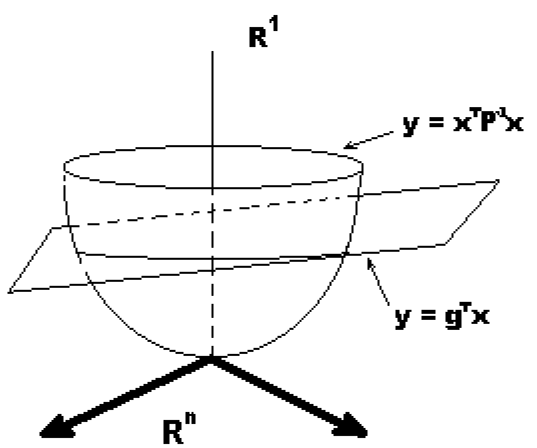

Figure 1: A 3-D perspective view of the underlying optimization posed in [1], with n=2 or n=3.

nature of the underlying optimization than offered by the planar views of [1, Figs. 1, 2], which merely rehash orthogonal properties in [1] exhibited by **all** Lagrange multiplier applications with

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[1]We also offered two counterexammples [9] to the methodology used in establishing a recent gravity model.

equality constraints [10]. This constrained optimization problem can be posed using a single scalar Lagrange multiplier λ (with reciprocal units) within an associated scalar Lagrangian cost function [10, pp. 242-247], [11]:

$$L(x,\lambda) = x^T P^{-1} x + \lambda[-g^T x + d] \tag{3}$$

to be extremized below to obtain the solution to the original minimization problem of Eqs. 1, 2.

2 Its closed-form solution

Taking the requisite partial derivatives of the above Lagrangian cost function in order to establish the associated stationary *saddle point* of Eq. 3, the two intermediate results are:

$$0 = \frac{\partial L}{\partial x} = \quad 2P^{-1}x + \lambda[-g] \quad \Longrightarrow x = \frac{1}{2}\lambda P g, \tag{4}$$

$$0 = \frac{\partial L}{\partial \lambda} = \quad -g^T x + d \quad \Longrightarrow \text{returns planar constraint back.} \tag{5}$$

By substituting the result of Eq. 4 into Eq. 5, yields a closed-form solution for the scalar multiplier:

$$-\frac{\lambda}{2} g^T P g + d = 0 \Longrightarrow \lambda = \frac{2d}{g^T P g} \text{ (in reciprocal units of } x\text{)}, \tag{6}$$

which is substituted back into Eq. 4 yielding the final closed-form solution that minimizes Eq. 1 and satisfies Eq. 2:

$$x_{\min} = \frac{d}{g^T P g} P g \text{ (in units of } x\text{) and the associated minimum is } y_{\min} = \frac{d^2}{g^T P g} \text{ (also being without units)}, \tag{7}$$

where, for the application of [1, Eq. 14], $d \equiv 1$.

3 Correcting a few minor oversights in [1]

A rather obvious typo in [1, Eq. 1] is that the subscript for the third component of the three vector on the right should be corrected to be φ_{L5} rather than φ_{L3}. We compliment the excellent discussion of units, amplification factors, and physical motivation, justification, and insights provided in [1], and, in particular, the correct delineation of when such an optimization is appropriate: "for short baselines where the ionospheric and tropospheric transmission errors are negligible" and "thermal noise and multipath are the primary sources of error that can be lessened" through choice selection of the "linear combinations of the" available "carrier phase measurements" by the optimization addressed herein. While only two vectors and three vectors are addressed in [1, Table 1] as completed optimizations for $x \in E^2$ and $x \in E^3$, respectively, we hasten to remind that a possible situation for further improvement that was overlooked in [1] was for $x \in E^4$ (corresponding to optimizing selection of weightings: $x^T = [\alpha, \beta, \gamma, \delta]$) that would arise in considering the available GPS frequencies L1, L2, L3, and L5, which are all integer multiples of a single onboard oscillator (at least L1, L2, and L3 are [12, p. 594]).

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