Comment on "Low-Noise Linear Combination of Triple-Frequency Carrier Phase Measurements"

THOMAS H. KERR III
TeK Associates, Lexington, Massachusetts 02420-5336

Received October 2009; Revised March 2010

ABSTRACT: *Elucidating the nature of a familiar closed-form solution arising in the optimization of [1].*

THE UNDERLYING OPTIMIZATION PROBLEM

Based on prior experience with quadratic forms as cost functions arising in optimization problems whose solutions are useful in navigation applications ([2–5] (endorsed in [6]), [7–9]) and practice in having also provided two timely, critical counterexamples [10] to the methodology used in establishing a recent gravity model, we recognized the underlying problem posed in [1] to be a familiar minimization of a convex paraboloidal function, $y = f(x)$, going from Euclidean n-space x to a scalar y (i.e., $f: E^n \rightarrow R$, where the linear weighting coefficients of [1, Eq. 1]: $[\alpha,\beta,\gamma] \in E^3$), with symmetric positive definite $n \times n$ inner product matrix, P (with units being the square of those of x), in seeking to minimize:

$$y = x^T P^{-1} x \text{ (corresponding to [1, Eq. 16], notice that y is sans units as an amplification factor)}, \quad (1)$$

subject to the single constraint of lying on an intersecting plane of specified orientation in 3D-space:

$$g^T x = d \text{ (corresponding to [1, Eq. 14], notice that all units can divide out)}, \quad (2)$$

where, in the above, g is an n-vector corresponding to the direction numbers perpendicular to the plane, d is a scalar with the same units as x (representing the y-intercept for $x^T = (0,0,0)$), and superscript T denotes the transpose. More insight is availed here
F1 from Figure 1 into the true nature of the underlying optimization than is offered by the planar views of [1, Figures 1, 2], which merely rehash orthogonal properties in [1] exhibited by **all** Lagrange multiplier applications with **equality constraints** [11].

This constrained optimization problem can be posed using a single scalar Lagrange multiplier, λ (with reciprocal units), within an associated scalar Lagrangian cost function [11, pp. 242–247, 12]:

$$L(x,\lambda) = x^T P^{-1} x + \lambda[-g^T x + d] \quad (3)$$

to be extremized below to obtain the solution to the original minimization problem of Eqs. (1) and (2).

CLOSED-FORM SOLUTION

Taking the requisite partial derivatives of the above Lagrangian cost function in order to establish the associated stationary *saddle point* of Eq. (3), the two intermediate results are:

$$0 = \frac{\partial L}{\partial x} = 2P^{-1}x + \lambda[-g] \Longrightarrow x = \frac{1}{2}\lambda P g, \quad (4)$$

$$0 = \frac{\partial L}{\partial \lambda} = -g^T x + d \Longrightarrow \text{returns planar constraint back.} \quad (5)$$

By substituting the result of Eq. (4) into Eq. (5), yields a closed-form solution for the scalar multiplier:

$$-\frac{\lambda}{2} g^T P g + d = 0 \Longrightarrow \lambda = \frac{2d}{g^T P g} \text{ (in reciprocal units of } x), \quad (6)$$

which is substituted back into Eq. (4) yielding the final closed-form solution that minimizes Eq. (1) and satisfies Eq. (2):

$$x_{\min} = \frac{d}{g^T P g} P g \text{ (in units of } x) \text{ and the associated minimum is } y_{\min} = \frac{d^2}{g^T P g} \text{ (also without units)}, \quad (7)$$

where, for the application of [1, Eq. (14)], $d \equiv 1$.

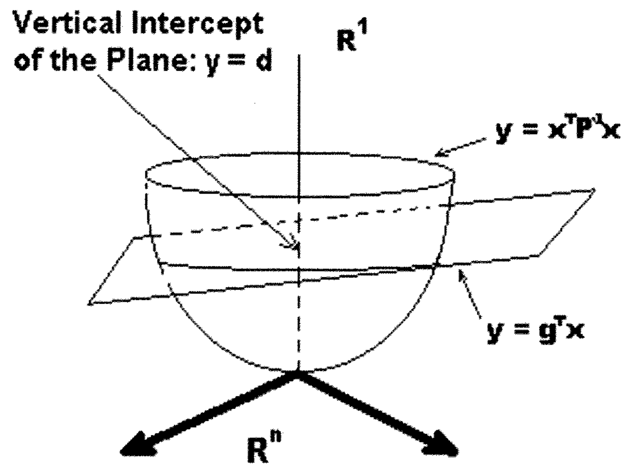

Fig. 1–A 3-D perspective view of the underlying optimization posed in [1], with $n = 2$ or $n = 3$.

CORRECTING A FEW MINOR OVERSIGHTS IN [1]

A rather obvious typo in [1, Eq. (1)] is that the subscript for the third component of the three vector on the right should be corrected to be φ_{L5} rather than φ_{L3}. We compliment the excellent discussion of units, amplification factors, and physical motivation, justification, and insights provided in [1], and, in particular, the correct delineation of when such an optimization is appropriate: "for short baselines where the ionospheric and tropospheric transmission errors are negligible" and "thermal noise and multipath are the primary sources of error that can be lessened"" through choice selection of the "linear combinations of the" available "carrier phase measurements" by the optimization addressed herein. While only two vectors and three vectors are addressed in [1, Table 1] as completed optimizations for $x \in E^2$ and $x \in E^3$, respectively, we hasten to remind that a possible situation for further improvement that was overlooked in [1] was for $x \in E^4$ (corresponding to optimizing selection of weightings: $x^T = [\alpha, \beta, \gamma, \delta]$) that would arise in considering the available GPS frequencies L1, L2, L3, and L5, which are all integer multiples of a single common onboard oscillator (at least L1, L2, and L3 are [13, p. 594]). The only computation needed to solve the optimization of Eqs. (1) and (2), as discussed herein and provided as $x_{\min}$ in Eq. (7), is obtained by merely a symmetric positive definite matrix inversion (n^3 operations) and no more than the two indicated matrix-vector multiplications and a scalar division and scalar-vector multiplication ($3n^3 + n + 1$), as ($3n^3 + n + 1$) additions and ($3n^3 + n + 1$) multiplications for an upper bound of $2(3n^3 + n + 1)$ total flop count per computed output time step.

REFERENCES

1. Richert, T., El-Sheimy, N., "Low-Noise Linear Combination of Triple-Frequency Carrier Phase Measurements," *NAVIGATION*, Vol. 53, No. 1, Spring 2006, pp. 61–67.
2. Kerr, T. H., "Real-Time Failure Detection: A Static Nonlinear Optimization Problem that Yields a Two Ellipsoid Overlap Test," *Journal of Optimization Theory and Applications*, Vol. 22, No. 4, August 1977, pp. 509–535.
3. Kerr, T. H., "Statistical Analysis of a Two Ellipsoid Overlap Test for Real-Time Failure Detection," *IEEE Transactions on Automatic Control*, Vol. 25, No. 4, August 1980, pp. 762–773.
4. Kerr, T. H., "False Alarm and Correct Detection Probabilities Over a Time Interval for Restricted Classes of Failure Detection Algorithms," *IEEE Transactions on Information Theory*, Vol. 28, No. 4, July 1982, pp. 619–631.
5. Kerr, T. H., "Integral Evaluation Enabling Performance Trade-offs for Two Confidence Region-Based Failure Detection," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 3, May-June 2006, pp. 757–762.
6. Brumbeck, B. D., and Srinath, M. D., "A Chi-Square Test for Fault-Detection in Kalman Filters," *IEEE Transactions on Automatic Control*, Vol. 32, No. 6, June 1987, pp. 532–554.
7. Kerr, T. H., "Comments on Determining if Two Solid Ellipsoids Intersect," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 1, January-February 2005, pp. 189–190 (see two line derivation on p. 190 following Eq. (6), and as summarized two paragraphs before Eq. (1) of [5], for a more straightforward structural interpretation and simpler implementation).
8. Kerr, T. H., "Modeling and Evaluating an Empirical INS Difference Monitoring Procedure Used to Sequence SSBN Navaid Fixes," *NAVIGATION*, Vol. 28, No. 4, Winter 1981–82, pp. 263–285.
9. Kerr, III, T. H., "Sensor Scheduling in Kalman Filters: Evaluating a Procedure for Varying Submarine Navaids," *Proceedings of the 57th Annual Meeting of the Institute of Navigation*, Albuquerque, NM, June 2001, pp. 310–324.
10. Kerr, T. H., "Comment on Precision Free-Inertial Navigation with Gravity Compensation by an Onboard Gradiometer," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 30, No. 4, July-August 2007, pp. 1214–1215.
11. Luenberger, D. G., *Optimization by Vector Space Methods*, John Wiley & Sons, Inc., NY, 1969.
12. Luenberger, D. G., *Linear and Nonlinear Programming*, 2nd Ed., Addison-Wesley, Inc., Reading, MA, 1984.
13. Kerr, III, T. H., "Further Critical Perspectives on Certain Aspects of GPS Development and Use," *Proceedings of the 57th Annual Meeting of The Institute of Navigation*, Albuquerque, NM, June 2001, pp. 592–608.