Designing Nonlinear Filters Based on Daum's Theory

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The purpose of this paper is to present a method for designing nonlinear filters based on work by Frederick E. Daum. The evolution of a probability density function on an interval between measurements can be described by the Fokker-Plank equation that, under certain conditions, can be written as the product of a scalar function and an exponential function. The parameters defining the latter satisfy coupled ordinary differential equations and can be updated. However, it is very difficult to obtain the mean and covariance at this stage in the development of the theory. A major theoretical result communicated in this paper is the derivation of sufficient conditions, stated in terms of the nonlinear function defining the dynamic system, under which a probability density function exists satisfying Daum's conditions. This leads to algorithms for propagating the mean and covariance that generalize the Kalman-Bucy equations. A nonlinear filter for the exoatmospheric intercept of an intercontinental ballistic missile is given as an example.

I. Introduction

Consider a dynamic system described by the stochastic differential equation

\[ x' = f(x', t) + \omega \]  

Here \( x' \) is the system state vector and \( t \) is time. The forcing function \( \omega \) is a stochastic process. Measurements are taken at discrete time intervals and are functions of the states and another stochastic process \( \nu \).

\[ z = u(x') + \nu \]  

The purpose of this paper is to present a method for designing nonlinear filters for such systems based on the work of Daum. \(^3\) Daum's work is theoretically elegant, but it is difficult to generate computational algorithms based on the current state of the theory.

The portion of Daum's work that forms the basis for this paper will be summarized in Sec. II, along with a discussion of a new point of view providing for easier formulation of computational algorithms. The new conditions that, if satisfied, lead to computational algorithms are derived in Sec. III. The formulas for propagating the mean and covariance of the probability density function (pdf) for the estimates \( x' \) of \( x' \) are derived in Sec. IV. An example is given in Sec. V, and conclusions are summarized in Sec. VI.

II. New Perspective on Daum's Results

The specific formulation of Daum's results used here is presented in Ref. 2. Two of the assumptions that Daum makes are common to most developments. The first is

\[ E[\omega(t)\omega(r)] = Q(t)\delta(t-r) \]  

\[ E[\omega(t)\nu(r)] = 0 \]  

\[ E[\nu(t)\nu(r)] = R(t)\delta(t-r) \]  

In Eq. (3), \( E \) denotes expected value and \( \delta \) the delta function. The matrices \( Q(t) \) and \( R(t) \) are called process and measurement noise matrices, respectively.

The second common assumption is that the measurements are linear in the state variables

\[ z = H(t)x'(t) + \nu \]  

where \( H(t) \) is an \( m \times n \) time-dependent matrix. Several observations may be made before proceeding to Daum's other conditions. It is well known that the pdf \( p(x'(t), \Sigma(t)) \) of the state variables defined by Eq. (1) satisfies the Fokker-Plank equation

\[ \frac{\partial p}{\partial t} = -\frac{\partial}{\partial x'} \left( f - p \right) + \frac{1}{2} \frac{\partial^2}{\partial x'^2} \left( \sigma p \right) \]  

in the interval \( (t_{n-1}, t_n) \) between measurement updates. Here, \( \sigma p/\partial x' \) is the Jacobian of \( f \) with respect to \( x' \); \( tr \) denotes "trace," and \( \sigma p/\partial x'^2 \) is the Jacobian of the transpose of \( \sigma p/\partial x' \).

There are three elements common to recursive filters. The first is the assumption that the initial pdf \( p(x'(0), \sigma(0)) \) is defined in terms of a given set of parameters. The second is the solution to Eq. (5) on \( (t_{n-1}, t_n) \) can be described by the same set of parameters. The third is that the update at \( t_n \) based on the measurements at \( t_n \) yields a pdf that again has the given form. One can then continue to propagate and update the pdf through as many measurements as necessary.

In Ref. 2, Daum assumes an unnormalized pdf of the form

\[ p(x'(t), t_n) = \Psi(x'(t), t_n) e^{-\frac{1}{2}(x'(t) - m(t_n))' \Sigma(t_n)^{-1}(x'(t) - m(t_n))} \]  

The vector \( m(t) \) is \( n \) dimensional, \( P(t) \) is an \( n \times n \) positive-definite matrix, and \( s \) is a real number between 0 and 1. Conditions on \( \Psi(x', t_n) \) will follow shortly. Only \( s = 1/2 \) is considered here. An unnormalized pdf \( p(x'(t), t_n) \) can be made into a pdf by dividing it by the integral of the function over the whole state space. That is,

\[ p(x'(t), t_n) = \frac{p(x'(t), t_n)}{\int p(x'(t), t_n) dx'} \]  

The next observation is that since one assumes that \( p(x'(t), t_n) \) is known, the mean \( x' \) of the state vector at \( t_n \) is known. Therefore, it is possible to express \( f(x'(t)) \) as a series in the variable \( x = (x' - x) \). Assume that Eqs. (1), (2), and (3) are written in terms of \( x \), where \( x' \) is the mean of the state variables \( x' \) at the left-hand endpoint of the interval between two measurements.

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Each \( G_i \) is a symmetric matrix. Also,

\[
\frac{df}{dx} = d + S'x + x'Lx + \text{higher-order terms}
\]

for some symmetric matrix \( L \), vector \( S \), and scalar \( d \). If conditions 2 and 3 are true up to and including second-order terms, then

\[
\varphi(x) \approx e^{\kappa x^2}
\]

where \( \kappa(x) \) must have the form

\[
h(x) = x' \gamma x + g'x + \delta(x)
\]

for some symmetric matrix \( P \), vector \( g \), and scalar function \( \delta \).

The partial differential equation for \( h \) is obtained by substituting Eq. (23) into Eq. (9):

\[
\frac{df}{dx} = -\frac{dh}{dt} + \frac{1}{2} \int \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial g} \right) dt
\]

Using Eqs. (17) and (23), we get

\[
\alpha(t) = -2x' \gamma x + h(x'B + U) - d - S'x + x'Lx + \frac{1}{2} \int (Q' + 2x'Lx + x'\gamma x + h(x') dt)
\]

\[+ \text{higher-order terms} \]

Define \( J(x) \) as follows:

\[
J(x) = x'Qxx' - 2B'x - L - \gamma 0 - \cdots - \gamma 0 G_i x
\]

\[+ \text{higher-order terms} \]

where \( \gamma_0 \) depends on time alone. Equation (34) may be rewritten as

\[
\rho(x,t) = C_0 e^{\kappa/2} \int (Q - x'Lx + 2x'Lx + \gamma 0 G_i x + h(x') dt
\]

\[+ \text{higher-order terms} \]

Distribution (35) is Gaussian. The covariance \( M \) and mean \( x \) are propagated using the formulas derived next.

Daum's equation (14) can be written

\[
d \frac{d^2 x}{dt^2} = P \mathbf{B} + D \mathbf{P} + D \mathbf{Q} + Q
\]

From Eqs. (9), (10), (11), (17), and (23), we find that

\[
a = L + VQV'
\]

\[b = S' + x' \gamma x
\]

\[c = d + (x'Qxx' + h(x')) dt
\]

\[d = B - QV
\]

\[e = h - (Q-x'M) \]

It can now be shown that

\[
M = B + MB' + MQL + \cdots + \gamma 0 G_i xM + Q
\]

This equation for the propagating covariance differs from that of an EKF by the factors \( MQL + \cdots + \gamma 0 G_i xM \).

To obtain the formula for propagating the mean, first multiply both sides of Eq. (37) by \( PM^{-1} \).

\[
PM^{-1} \mathbf{x}_0 = \mathbf{m} + \frac{1}{2} \mathbf{P}_0
\]

Differentiate Eq. (45) and simplify to show that

\[
\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} - \mathbf{C} \mathbf{y} + \mathbf{d}
\]

This formula for propagating the mean differs from that of an EKF by the factors \( MQL + \cdots + \gamma 0 G_i xM \).

The equations for updating \( M \) and \( \mathbf{x} \) are obtained by substituting Eqs. (37) and (38) into Eqs. (13) and (16). The matrix \( P \) and vector \( g \) cancel out of these equations, leaving them to be run with \( \mathbf{P} \) and \( \mathbf{m} \) replaced by \( M \) and \( \mathbf{x} \) (Ref. 7).

V. State Equations for Guidance Filters

A nonlinear filter is developed to illustrate the result of the preceding section. Consider the engagement of a space-based interceptor against an intercontinental ballistic missile (ICBM). The interceptor has an angle-only seeker with a given acquisition range. The system's coordinate system, called the terminal homing coordinate system (THCS), is defined by taking the \( x \) axis to be the line of sight from the interceptor to the ICBM at acquisition, and the \( y \) and \( z \) axes to be the \( y \) and \( z \) body axes of the interceptor at acquisition. The THCS moves with the interceptor, but does not rotate in inertial space. The interceptor is equipped with rate gyroes so that attitude of the interceptor body axis relative to the THCS can be estimated. The two angle measurements taken relative to the seeker boom can therefore be transformed into two angle measurements, azimuth and elevation, in the THCS. The azimuth angle lies in the \( x,y \) plane and remains small throughout terminal homing, so no loss of performance results if the elevation angle is thought of as being in the \( x,y \) plane. Both measured angles remain small because of the limited divergent capability of the interceptor. Thus, the measurement equation is essentially linear.

The \( x,y \) axis of the target ICBM is along its linear axis of symmetry, and is centered relative to the THCS by the line of sight through the \( x \) axis from the near side to the target. The \( x,y \) plane plane filter equations will be derived here. Denote thrusts, \( m \) and \( I_0 \), \( m_0 \), \( I_0 \), the specific impulse of the ICBM, and \( \gamma_0 \), denote the direction cosine of target acceleration with respect to the \( x \) axis. The states for the \( x,y \) plane are defined as follows:

\[
y_1 = \text{azimuth angle}
\]

\[y_2 = \text{azimuth angle rate}
\]

\[y_3 = T/(\gamma_0 m)
\]

Using the fact that, for a rocket,

\[m = -T/\gamma_0 m_0
\]

one sees that

\[y_1 = y_2 + y_3 \cdot \gamma_0 \cdot T/m
\]

\[y_4 = y_2 + y_3
\]

The nonlinearities in the filter equations are due to the way in which target acceleration is described in Eqs. (49) and (50) and in the choice of a coordinate system. If the range and range rate are \( R \) and \( \dot{R} \), then the following states, which are called range axis states, appear in the filter equations:

\[R_1 = x + y + \gamma_0 \cdot T/m
\]

\[R_2 = y + y_3 \cdot \gamma_0 \cdot T/m
\]

\[R_3 = y + y_3
\]
The direction cosine of target acceleration is denoted $c_i$.

\[
x_1 = R/R
\]

\[
x_2 = 1/R
\]

\[
x_3 = c_i, c_{i+1}, c_{i+2}/(T/m)
\]

\[
x_4 = T/(g_{e,m})
\]

The seeker does not provide a range measurement, and these range axis states are not observable unless the seeker is caused to fly particular types of trajectories. These trajectories will not be considered here. An initial estimate of the range axis state is made that contains errors (quantified later), and the range axis states are propagated without update. These errors must be small, otherwise the filter diverges.

The $x$ and $y$ coordinates of the target in the STHCS are

\[
y = \sin \phi R
\]

\[
x = \cos \phi R
\]

Differentiating twice, using small angle approximations, and simplifying yield the equation for $y_3$. Because the degree of target maneuver reflected in $c_i$ is not known, the term $c_i^2(T/m)$ is represented by a stochastic process. A stochastic process $w_4$ is also added to the equation for $y_4$ because of the uncertainty in $w_{e,m}$ and lack of knowledge of $T/m$ at the start of homing. The filter-state equations are thus

\[
f_1 = y_2
\]

\[
f_2 = -x_3 y_3 y_4 - 2x_3 y_2 + x_4 y_4 - A_y y_4
\]

\[
f_3 = y_3 y_4 + w_4
\]

\[
f_4 = x_4 + w_4
\]

Furthermore, $Q_i$ and $Q_2$ are zero, $Q_1$ has 0.5 in the (3, 4) and (4, 3) positions, $Q_2$ has 1 in the (4, 4) position, and $Q_3$ has a matrix in the (3, 3) and (4, 4) positions. Otherwise, all elements of $Q_i$, $Q_3$, and $Q_4$ are zero.

Conditions $A$ are satisfied by a vector $g$ that is zero except for the fourth element that we denote by $g$ again, and a matrix $T$ that is zero except for the (4, 4) element, which we label $v$.

The matrix products in Eq. (27) all have zero entries except for the (4, 4) element. The scalar equation for this position is

\[
g = 2(q_3 y_2 - 2q_2 y_3)
\]

A block diagram illustrating the implementation of both the new nonlinear filter (NFF) and an adaptive Kalman filter (AKF) is given in Fig. 2. The elevation filter is not included in detail because it is similar to the azimuth filter. However, the filters are somewhat coupled as indicated in the diagram.

The NFF and AKF use the same maneuver detection criteria. The residuals for both filters are normalized by dividing by the appropriate angle covariance and stored along with the data necessary to refilter a portion of the measurements once a maneuver is detected. When these residuals in either filter exceed a given value $m$ in $m$ out of $n$ times, a maneuver detection switch is set. Refiltering the stored data, the $q_i$ process noise and acceleration covariance for each filter are reinitialized. These terms in both filters are reinitialized to the same value. Of course, the NFF has additional terms in the covariance matrix propagation, and the best way to use these terms appears to be reinitialization of the (3, 4) cross-covariance term. After a maneuver detection, the update of the fourth state is suppressed because a change in $T/(g_{e,m})$ due to a change in acceleration direction cannot be distinguished from that due to the rocket equation, which is really what it is supposed to reflect.

Since maneuver detection is being done, the value of process noise is being used to decrease exponential function. The initial value and time constant are chosen for best AKF performance after maneuver detection and a steady-state value is chosen for best AKF steady-state performance. The same process noise is used in both filters and the same initial covariance matrix is used for both filters. Therefore, the improvement the NFF shows over the AKF comes only from designing with the additional terms suggested by Eqs. (44) and (46).

The intercept and target are closing at 10,000 m/s and the engagement lasts 10s. The $v$ and $x_1$ errors in initial position, velocity, and acceleration were 700 m, 75 m/s, and 7.5 m/s², respectively. Perfect range axis information was used to generate the data of Figs. 3 and 4 to illustrate the
difference between filters. Range axis errors equal to the
y and z axes errors were used in Fig. 5. The target acceleration
at impact was 120 m/s² and the zero effort miss 2386 m in each
axis. The seeker tracked a point in the ICBM target plume,
and an aimpoint bias was used to move the intended impact from
this point in the plume to the target center. The seeker accura-
cy was 100 μrad at 100 km coming down to a floor of
50 μrad. For miss distance computation and probability of hit
calculations, the ICBM body was 3 m in diameter and 13 m
long. The interceptor had a 100-m/s² acceleration limit.

The acceleration occurs along the y axis to illustrate filter
performance over the widest range of accelerations, i.e., large
accelerations in the azimuth filter and no acceleration in the
elevation filter. The most significant difference in performance
occurs in the third- and fourth-state estimates, as illus-
trated in Figs. 3 and 4. This is to be expected, since this is
where the nonlinearities occur. The NNF performs better by
converging faster and having smaller 1σ steady-state errors.
When the target acceleration is partitioned between the y
and z axes, the fourth-state convergence is not as good, but it is still
better than the AKF and convergence in the acceleration states
remained better than for the AKF.

The following guidance law was used:

\[ U_{gy} = (1 + at^2_j) \Delta \left[ V_2y_2 - bx_yx_y + (0.5)z_2 \right] \]  \hspace{1cm} (61)

where \( U_{gy} \) is acceleration in the xy plane. A similar formula
holds for the xz plane. Here \( \Delta, a, \) and \( b \) are constants, \( V_r \)
the closing velocity, and the aimpoint bias term is \( bx_yx_y \).
Time to go is \( t_g \). There are a number of important issues concern-
ing the implementation of the guidance loop that cannot
be covered here. The NNF shows improvement (in terms of
probability of hit) over the AKF only when the target maneu-
vers and an aggressive guidance law are necessary to take
advantage of this.

Probability of hit vs maneuver angle for discrete times start-
ing at 8 s (2 s before impact) is presented in Fig. 5. Here a
discrete turn is one where the target starts with a rotation rate
of zero, the nozzle is gimbaled for 1 s during which the target
rotates through a given angle, and then the rotation rate is
zeroed out via a control law. The target starts out accelerating
along the y axis and then yaws, so all of the acceleration
remains perpendicular to the line of sight. This case best illus-
trates the difference between NNF and AKF performance.

VI. Conclusion

This paper presents a new mathematical foundation for de-
signing nonlinear filters. Conditions stated in terms of the state
equations defining the dynamic system are derived, under
which the Kalman-Bucy equations for propagating the state
estimates and covariance may be generalized. The interpreta-
tion of the additional terms that appear in the state and
covariance propagation equations and the way in which one can
use these terms to improve filter performance are far from
clear. However, we have presented an example that dem-
strates significant performance improvement can be obtained
using these nonlinear terms and therefore justifies calling the
attention of the filtering and tracking community to these
results. It is also significant, at least theoretically, that in some
cases the pdf remains Gaussian even if the filter equations are
nonlinear.

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