Basic Inertial Navigation

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FOREWORD

This report is intended as a tutorial on inertial navigation. The basic principles are presented with a minimum of mathematical detail. This work was funded by the AIM-9X Program Office.

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ABSTRACT *(Maximum 200 words)*

(U)This document describes the basic prinicples of alignment and navigation of an inertial system.

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INTRODUCTION

Navigation is the art of knowing where you are, how fast you are moving and in which direction; and of positioning yourself in relation to your environment in such a way as to maximize your chances for survival. Navigation may consist of examining the bark on a tree, reading the display of a GPS receiver, or something in between. In this document we discuss a particular method of navigation known as inertial navigation.

Inertial navigation is accomplished by integrating the output of a set of sensors to compute position, velocity, and attitude. The sensors used are gyros and accelerometers. Gyros measure angular rate with respect to inertial space, and accelerometers measure linear acceleration, again with respect to an inertial frame. Integration is a simple process, complexities arise due to the various coordinate frames encountered, sensor errors, and noise in the system.

TWO-DIMENSIONAL MOTION

To begin our discussion, we reduce the problem to its simplest terms by assuming a flat earth and confining our motion to the surface. To indicate position we must establish a grid or coordinate system. We label one direction on the surface as N and another, perpendicular to it, as E. If the third direction is perpendicular to the surface in the down direction, a north-east-down (NED), right-handed coordinate frame is established.

For the navigating system (platform), we label the longitudinal axis as x and the cross axis as y (Figure 1). The z axis is down in order to complete a right-handed coordinate system. This coordinate frame (xyz) moves with the vehicle and is fixed in the body. In general, this frame is rotated about the vertical with respect to the NED frame by some azimuth angle, ψ.

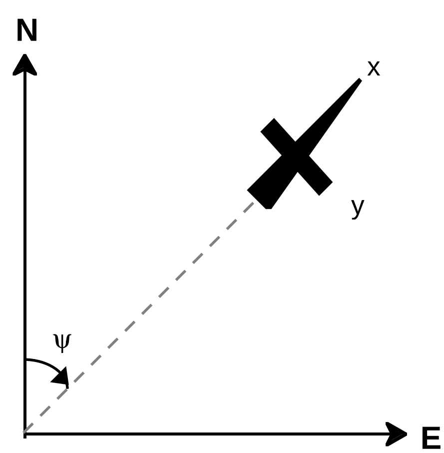

FIGURE 1. Coordinate Frames.

Assume that, initially, the x axis is oriented north and the y axis is oriented east. (We will consider alignment later.) To track our motion with respect to the grid, we mount an accelerometer on the x axis and another on the y axis. (This is a strapdown configuration in which the inertial sensors are fixed with respect to the body of the navigating system.) If we assume that the NED frame is an inertial frame, the accelerometers will measure acceleration with respect to that frame, but the output will be in the xyz frame. Therefore, we will need to know the azimuth angle to navigate in the NED frame. We can sense the change in the azimuth angle with a gyro mounted on the z axis. The output of this gyro is $\omega_z = d\psi/dt$. Since ψ is initially zero, the integral of ω_z gives us ψ as a function of time. For now we assume perfect sensors. Once we have ψ, we can rotate the accelerometer outputs to the NED frame.

$$\begin{pmatrix} a_N \\ a_E \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} \tag{1}$$

Then we can easily integrate a_N and a_E, once to obtain the velocity in the NED frame and again to obtain position.

In a practical system, we must have some means of determining our initial position, velocity, and attitude; integrations require initial conditions. If we are at rest, we know our initial velocity (zero) and we can measure our initial position. The initial azimuth angle also must be measured by some external means as no system error is produced as a result of an initial misalignment. By system error we mean position, velocity, and attitude errors. Thus, for our flat earth model, a stationary alignment must be accomplished by external measurements. If our platform is moving with a constant velocity, we also must have an external measurement of velocity. If there is no accelerated motion, there is no observability of our system errors. Observability means that some condition exists that excites one or more of the system error states and causes a measurable response of the system. An example is given in the following.

Suppose we have a co-moving platform (i.e., our platform and a reference platform are attached to the same structure) that has been accurately aligned. We use information from this reference system to align our platform. This process is transfer alignment. When we refer to alignment from this point on, we will mean velocity-matching alignment unless otherwise specified. In a velocity-matching alignment, we compare our system velocity with the velocity of some external reference. If an alignment error or sensor error propagates into a velocity error, then the error is observable for this type of alignment. Note that observability may depend on the conditions under which the alignment is being performed, as well as the type of alignment. We will touch on other types of alignment in the appendix on Kalman filtering (Appendix A).

If we accelerate the system now, any azimuth error becomes observable. Let

$$\hat{\psi} = \psi + \delta\psi \tag{2}$$

be our estimate of azimuth, where ψ is the true azimuth angle and $\delta\psi$ is the error in our estimate. We measure a change in velocity due to acceleration of

$$\begin{aligned} \Delta\vec{V}_m &= \int_{t_1}^{t_2} \begin{pmatrix} \cos\hat{\psi} & -\sin\hat{\psi} \\ \sin\hat{\psi} & \cos\hat{\psi} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} dt \\ &\cong \begin{pmatrix} 1 & -\delta\psi \\ \delta\psi & 1 \end{pmatrix} \int_{t_1}^{t_2} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} dt \\ &= \begin{pmatrix} 1 & -\delta\psi \\ \delta\psi & 1 \end{pmatrix} \Delta\vec{V} \end{aligned} \tag{3}$$

where

$\Delta \overset{\rightharpoonup}{V}$ = true change in velocity in the NED frame
$\Delta \overset{\rightharpoonup}{V}_m$ = measured change in velocity in NED frame.

We have used Equation (1) and the approximations

$$\cos(\psi+\delta\psi) \cong \cos\psi - \delta\psi\sin\psi$$

and

$$\sin(\psi+\delta\psi) \cong \sin\psi + \delta\psi\cos\psi.$$

Since we know $\Delta\mathbf{V}$ from our reference system, we have a direct measure of our azimuth error. If no noise is present in the system, we could determine ψ as accurately as we pleased with one measurement by taking higher order expansions of the trigonometric functions in Equation (3) (assuming we could solve the resulting nth-order polynomials in $\delta\psi$). If noise is present in the system, we must use filtering techniques to increase the accuracy of our alignment. The most common method of filtering in navigation systems is the Kalman filter, which is discussed in Appendix A. Note that Equation (3) is overdetermined; we have two equations in one unknown. If we were to estimate $\delta\psi$ from a noisy measurement, we would use least-squares techniques.

So far we have assumed perfect sensors. Some common gyro and accelerometer error models are discussed in Appendix B. In general, different error sources generate navigation errors that are integral or half-integral powers of time. An accelerometer bias, for example, generates a velocity error that is linear in time and a position error that is quadratic in time. A bias on the vertical gyro would generate an azimuth error that is linear in time. The velocity and position errors due to this heading error would be a function of the accelerations present (Equation (3)). A common misconception about how inertial navigation systems work is that the heading error translates directly to position error. We can see from Equation (3) that this is not so. If we know the north and east components of velocity, we will navigate accurately, no matter how erroneous our heading estimate is, as long as we do not accelerate. A human operator might use azimuth to steer by, but that would mean he was heading to the wrong place. The navigation system would know its own position. Of course, as soon as the system accelerated due to a course change or for some other reason, the heading error would generate velocity and position errors.

THREE-DIMENSIONAL MOTION

Our two-dimensional model exhibits, to a limited extent, all the characteristics of an inertial navigation system. The next level of complexity we consider is to remove the constraint of two-dimensional motion. Once we remove this constraint, we also turn on a gravitational field that is normal to the north-east plane. This field produces an acceleration of approximately 32.2 ft/s^2 in the down direction.

A short digression is necessary at this point to explain how an accelerometer measures gravitational acceleration. No matter how an accelerometer is constructed, we may think of it as shown in Figure 2.

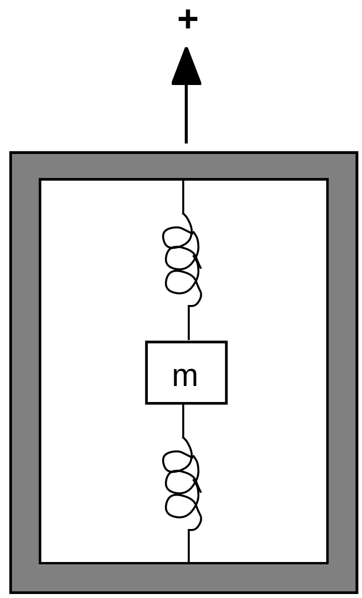

FIGURE 2. Accelerometer.

The accelerometer consists of a proof mass, m, suspended from a case by a pair of springs. The arrow indicates the input axis. An acceleration along this axis will cause the proof mass to be displaced from its equilibrium position. This displacement will be proportional to the acceleration. The amount of displacement from the equilibrium position is sensed by a pick-off and scaled to provide an indication of acceleration along this axis. The equilibrium position of the proof mass is calibrated for zero acceleration. An acceleration in the plus direction will cause the proof mass to move downward with respect to the case. This downward movement indicates positive acceleration. Now imagine that the accelerometer is sitting on a bench in a gravitational field. We see that the proof mass is again displaced downward with respect to the case, which indicates positive acceleration. However, the gravitational acceleration is downward. Therefore, the output of an accelerometer due to a gravitational field is the negative of the field acceleration. The output of an accelerometer is called the specific force and is given by

$$\mathbf{f} = \mathbf{a} - \mathbf{g} \tag{4}$$

where

$\mathbf{f}$ = specific force
$\mathbf{a}$ = acceleration with respect to the inertial frame
$\mathbf{g}$ = gravitational acceleration.

This relation is the cause of much confusion. The easy way to remember this relation is to think of one of two cases. If the accelerometer is sitting on a bench, it is at rest so **a** is zero. The force on the accelerometer is the normal force of reaction of the bench on the case or negative **g**. Or imagine-dropping the accelerometer in a vacuum. In this case **f** reads zero and the actual acceleration is **a** = **g**. To navigate with respect to the inertial frame, we need **a**, which is why, in the navigation equations, we convert the output of the accelerometers from **f** to **a** by adding **g**.

Now, we return to our consideration of three-dimensional motion over a flat earth. The platform attitude can no longer be specified by just the azimuth angle. We will specify the orientation of the platform by azimuth, pitch, and roll, as shown in Figure 3. In this document, platform refers to the body axes of the system being aligned. These axes are the same as the xyz coordinate frame previously defined.

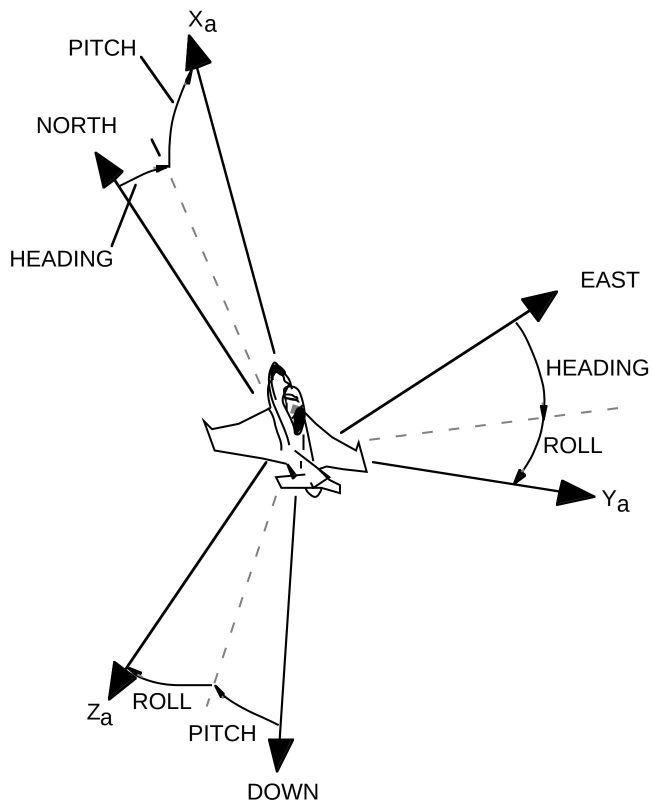

FIGURE 3. Platform Orientation.

The transformation matrix from the NED frame to the xyz frame (body axes) is given by

$$C_N^B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

where

ψ= azimuth

θ = pitch

ϕ = roll.

This matrix is to be considered as a coordinate transformation rather than a rotation of a vector. That is, when the matrix multiplies the components of a vector expressed in the NED frame, it produces the components of the same vector, expressed in the body frame. The inverse of this matrix, or the transformation from the body to the NED frame, is the transpose of this matrix. Transformations and coordinate frames are further discussed in Appendix C.

Since we now have three degrees of freedom, we will need an accelerometer on the z body axis and gyros on the body x and y axes. We now measure both acceleration and angular rate about each body axis.

Let us return to the case where we are at rest at some known point in the north-east plane. To align the platform, we must determine three angles; pitch, roll, and heading. To be clear on this point, alignment does not mean we physically rotate the platform, we merely determine its attitude with respect to some reference frame. Fortunately, we now have the means of determining pitch and roll. Let us assume our platform is rotated by some angle $\delta\theta$ about the y or pitch axis. The situation appears as shown in Figure 4. The y axis is out of the plane of the paper.

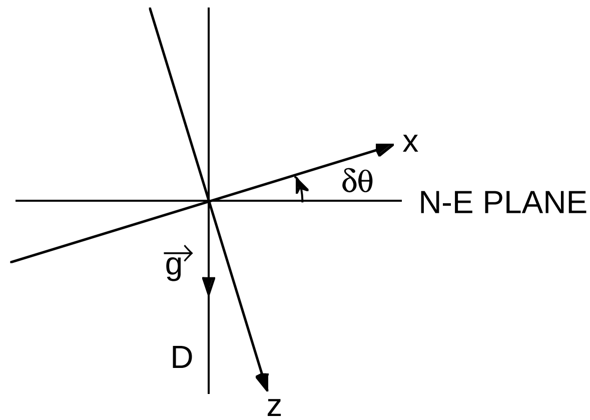

FIGURE 4. Tilt Errors.

This tilt is observable. The reason is that we are now sitting in a gravitational field. The accelerometer will measure a component of **g** in the z direction of

$$f_z = -g\cos\delta\theta$$

and in the x-direction of

$$f_x = g\sin\delta\theta$$

where g is the magnitude of the gravitational field. For small angles, the measured values are

$$f_z = -g$$
$$f_x = g\delta\theta.$$

The output of the x accelerometer gives us a direct measurement of the tilt about the y axis. Similarly, the output of the y accelerometer provides a measure of the tilt about the x axis. We still have no observability on heading.

This discussion provides a simple example of correlation of errors. If the x accelerometer has a bias error of δbx, the output of the accelerometer is

$$f_x = \delta bx + g\delta\theta$$

If we are trying to level the platform (i.e., determine its attitude, we do not actually rotate anything) with the accelerometer output, we cannot tell the difference between accelerometer bias and tilt. A filter will zero out this signal by apportioning the corrections to bias and tilt according to its a priori estimate of their initial errors.

Note that even though we have residual tilt and bias errors, these errors, once correlated, will produce no velocity errors. This will be true as long as we remain at rest or proceed without turning. As soon as we turn, the errors start to become decorrelated and no longer cancel, which is why it is desirable to rotate the platform before the alignment ends. A rotation of 180 degrees would allow us to compute both the accelerometer bias and the platform tilt. If we align without turning, then turn 180 degrees after the alignment ends, the bias and tilt errors will now add rather than cancel. This is a worst case scenario.

To summarize, we can level our platform for a stationary alignment. This alignment can be done very accurately as we know the reference velocity accurately (it is zero) and there is little noise on the velocity measurement. For our flat nonrotating earth, we still cannot estimate heading in a stationary alignment. As we shall see, this situation will change once we allow the earth to rotate.

If we now consider a moving base alignment, we find that little new is added to the equation. Since we can now initialize the system at any arbitrary attitude, the reference must supply us with an estimate of our attitude as well as position and velocity. Alignment in this case means reducing the errors in these quantities as much as possible. Gravity still excites the level axis errors and allows us to estimate pitch and roll errors by comparing our velocity to a reference velocity. We still have the same problem of the correlation of the tilt and accelerometer bias errors. We still require maneuvering in order to estimate heading error.[1] The main difference is in the accuracy of our measurements. Our reference velocity is usually provided by another navigation system. The accuracy of this reference system will limit the accuracy of our alignment. Another contributor to measurement accuracy is measurement noise. The system under alignment may be connected to the reference system by a nonrigid structure. The velocity measurement will thus be corrupted by vibration and flexure. These effects, which are not measurable or predictable, appear as noise on the measurement.

A navigation solution for our three-dimensional model is shown in block diagram form in Figure 5. The inputs, $\mathbf{f^B}$ and $\omega^{\mathbf{B}}$, are the accelerometer and gyro outputs where

$\mathbf{f^B}$ = specific force in body coordinates
$\omega^{\mathbf{B}}$ = angular rate in body coordinates.

These inputs will, in general, contain error terms. The angular rate is used to compute the NED to body transformation, C_N^B. Pitch, roll, and heading can be found from the elements of this matrix.

The inverse of C_N^B is used to transform the specific force to the NED frame. The acceleration due to gravity is added to this quantity to obtain acceleration in the NED frame (Equation (4)). This acceleration is integrated twice to produce velocity, **V**, and position, **R**. Both **V** and **R** will be with respect to the NED frame. The navigation outputs may then be compared to some reference quantity to compute measurements to be processed by a Kalman filter (or equivalent). The filter may estimate corrections to both the navigation solution and some of the sensor errors, depending on the filter model.

[1] Our treatment of heading error as different from pitch and roll error is somewhat arbitrary for the three-dimensional case. Resolution of the gravity vector will depend on the azimuth angle error for large pitch angles, thus making it observable. The reason these errors are treated differently is that, in practice, we almost always align at a nominally level attitude.

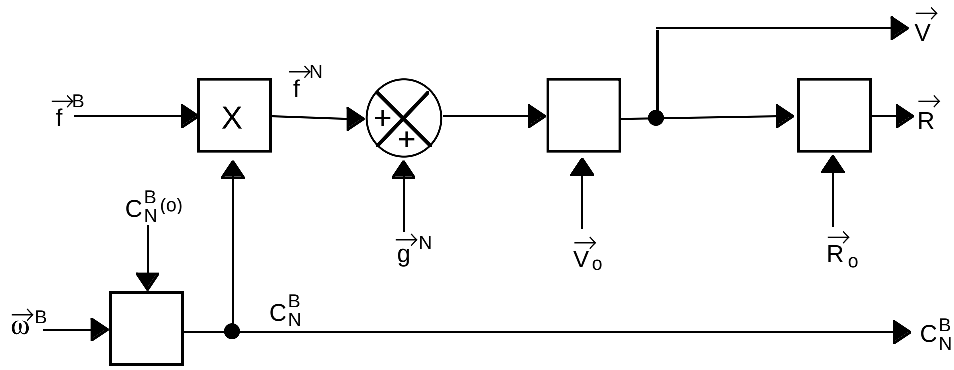

FIGURE 5. Flat Earth Navigator.

Numerically, the only part of this process that presents any degree of difficulty is the integration of ω^B to obtain $C_N{}^B$. This is because to body may rotate at high rates. Since integration errors tend to accumulate, this algorithm must be chosen carefully to ensure numerical accuracy.

We mention in passing two effects that affect sensor accuracy, coning in gyros and sculling in accelerometers. These are due to rotational effects in strapdown systems. We will not go into the details here. Coning and sculling are usually compensated at high frequencies in the sensor electronics.

NONROTATING SPHERICAL EARTH

We now come to our spherical earth model. We will assume, for the moment, that the earth is not rotating. We can see that our NED coordinate system is no longer appropriate for indicating position with respect to the earth's surface. We now describe our platform's position in terms of latitude, longitude, and altitude (Φ,λ,h) above the earth's surface (Figure 6). We still use the NED

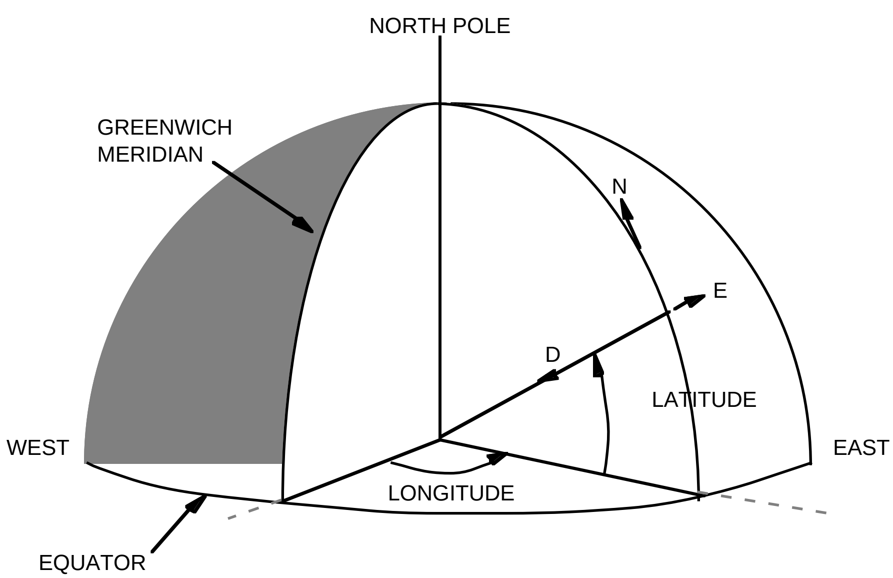

FIGURE 6. Spherical Earth.

frame, but now it represents a frame that is tangent to the surface of the earth at the platform's present position. This frame is referred to as a locally level frame. Note that the NED frame now moves about the surface of the earth along with the platform. We also define a coordinate frame, which is fixed with its origin at the center of the earth, as the z axis through the North Pole, the x axis through the Greenwich Meridian, and the y axis to complete a right-handed coordinate frame. This is the earth-centered, earth-fixed (ECEF) frame. The transformation from the ECEF frame to the NED frame is defined by the latitude and longitude. The attitude of the platform is described with respect to the NED frame exactly as before.

Gravitational acceleration can no longer be considered as a constant. Gravitational acceleration decreases inversely as the distance from the center of the earth, hence **g** is a function of altitude. This leads to an instability in the navigation solution in the vertical direction. A positive altitude error causes us to compute a weaker value for the gravitational acceleration than is actually present. Hence we will not correctly compensate for the gravitational field in our navigation equations. Since this residual component of gravity appears as an upward acceleration (Equation (4)), it will cause a velocity error in the upward direction that will, in turn, increase the altitude error. This increase is positive feedback, the vertical solution will diverge at an exponential rate. Near-earth navigators require some external means of stabilizing the vertical channel if they are navigating for more than a few minutes. This stabilization can be any external measure of altitude such as barometric pressure or the output of an altimeter. We will not include stabilization in the rest of our analysis.

Figure 7 shows a block diagram for a navigation solution for a nonrotating spherical earth model. This model is only slightly more complicated than the flat earth model. Our inertial frame is no longer the NED frame, it is the ECEF frame. The gyros and accelerometers now measure angular rate and acceleration of the body with respect to the ECEF frame. First, we integrate the angular rates to obtain the ECEF to body transformation, C_E^B. The inverse of this transformation is then used to rotate the specific force to the ECEF frame. As before, we add gravity to this quantity to obtain the acceleration of the body in the ECEF frame. Note the feedback path for the gravity computation. The acceleration is integrated twice to obtain velocity and position in the ECEF frame. These quantities are useful when the inertial navigator is being used in conjunction with a GPS set, as GPS uses ECEF coordinates. However, to provide a useful interface for a human

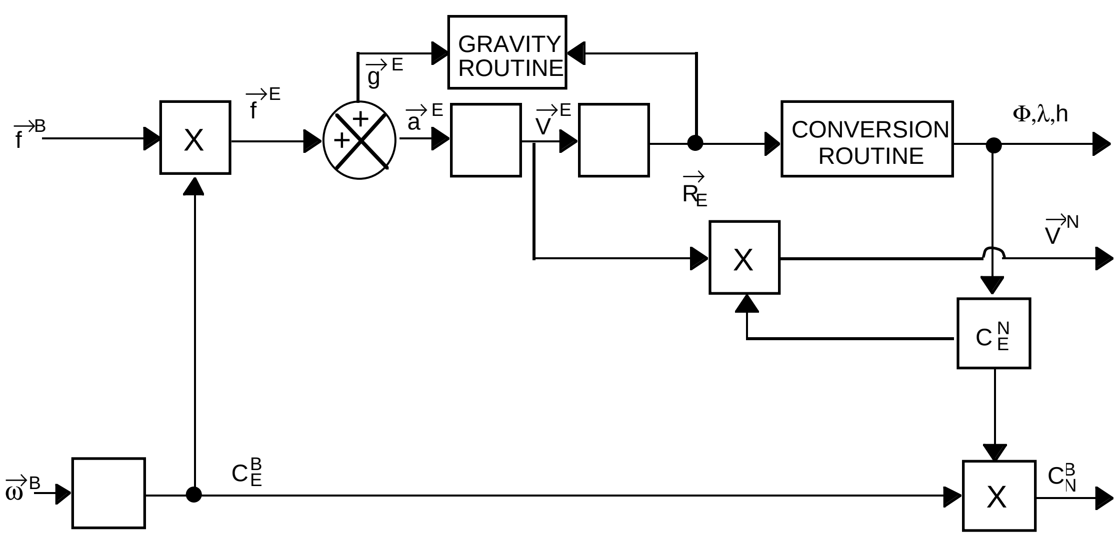

FIGURE 7. Nonrotating Spherical Earth Navigator.

operator, we convert position to latitude, longitude, and altitude (Φ,λ,h), and rotate the velocity to the NED frame. The ECEF-to-NED transformation matrix, C_E^N , is defined by the position. We use this matrix along with C_E^B to compute the attitude matrix, C_N^B. We have omitted showing initialization for the integrators in Figure 7 to keep it from becoming too busy.

The only new feature we have encountered when going from a flat earth to a nonrotating spherical earth, other than dealing with spherical coordinates, is the positive feedback in the gravity computation. We still align the platform to the NED frame as before. Previous remarks on alignment still apply. Minor complications will occur because the NED frame is no longer an inertial frame, but, in principle, everything works as before.

One difference between a flat earth model and a spherical earth model should be noted, the phenomenon of Schuler oscillation. If we initialize our system with some initial system error, the position, velocity, and attitude errors will be bounded (assuming perfect sensors). These errors will oscillate with a period of approximately 84.4 minutes. In a flat earth model, these errors would be unbounded. Any sensor errors will behave the same in both cases, they will produce system errors that are powers of time. A derivation of Schuler oscillation is given in Appendix D.

ROTATING SPHERICAL EARTH

The next step in our analysis is to allow our spherical earth to rotate. We define a new coordinate frame called the inertial frame, which is fixed at the center of the earth. Ignoring the earth's orbital motion, we regard the orientation of this frame as fixed with respect to the distant stars. The inertial frame is defined to be coincident with the ECEF frame at zero time. Figure 8 shows the relationship between the inertial and ECEF frames. The ECEF frame rotates with respect to the inertial frame with an angular velocity (Ω) of approximately 15.04 degrees per hour.

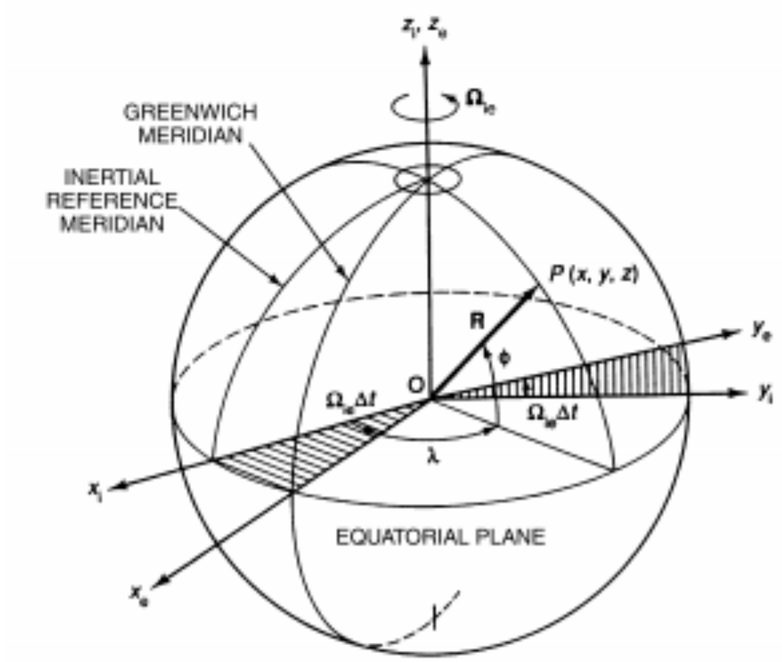

FIGURE 8. Rotating Spherical Earth.

The resulting navigation system is shown in Figure 9. The gyros and accelerometers now measure angular rate and linear acceleration of the body with respect to the inertial frame. We integrate the angular rate to obtain the inertial-to-body transformation, C_I^B. The inverse of this transformation is used to rotate the specific force to inertial coordinates. We again add gravity to obtain the acceleration of the body with respect to inertial space. The integral of this term is the time rate of change of the position vector with respect to inertial space. (This is not a navigation velocity. We define our navigation velocities with respect to a coordinate frame fixed in the earth for near-earth navigators). We correct this term for rotational effects to obtain ECEF velocity, as shown in Figure 9. The second integral of the inertial acceleration gives us the position vector in inertial coordinates. We rotate this position vector with the inertial-to-ECEF transformation, C_I^E, which we obtained from integrating the earth rate. The rest of the navigation solution goes much as before, the ECEF position vector is converted to latitude, longitude, and altitude. These are used to compute the direction cosine matrix (DCM) from the ECEF frame to the NED frame, C_E^N. Then C_I^B, C_I^E, and C_E^N are combined to form the NED-to-body transformation, C_N^B.

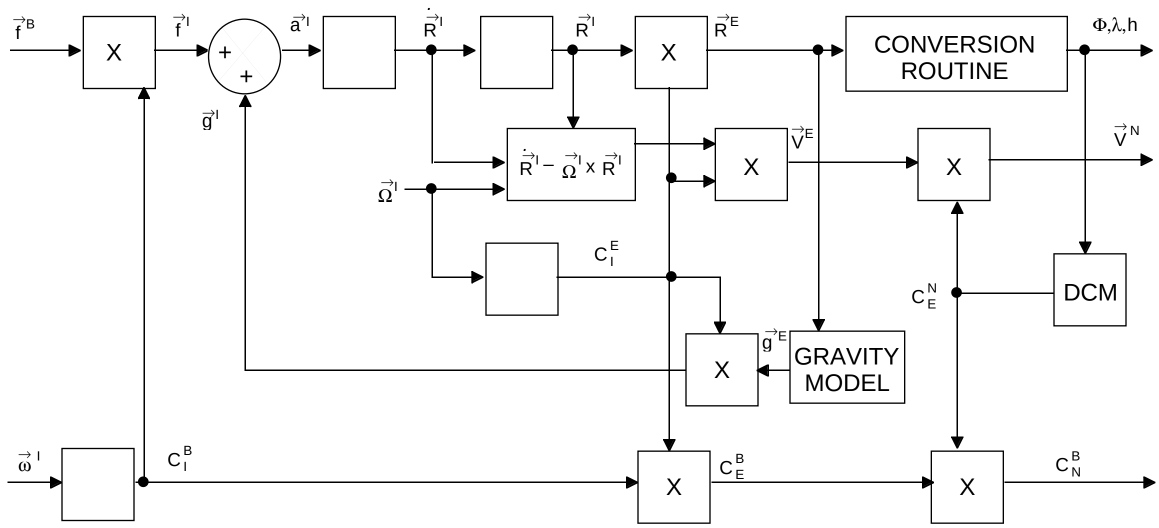

FIGURE 9. Rotating Spherical Earth Navigator.

One new consequence of the earth's rotation other than increased mathematical complexity exists. If we now perform a stationary alignment, we can estimate heading as well as tilts because the earth's rotation rate sensed by the gyros, $\omega^{\mathbf{B}}$, will be resolved in the wrong axes due to heading error. Imagine a gimbaled platform that is leveled but has some heading error. Since, in a gimbaled platform mechanization, the platform is torqued to keep it locally level, the platform must be torqued to compensate for the earth's rotation. This torque is applied about the north and vertical axes. The north component is $\Omega\cos\Phi$, where Ω is the angular rotation rate of the earth and Φ is latitude. A small heading error of $\delta\psi$ will cause a tilt about the north-axis of $(\delta\psi\Omega\cos\Phi)$ multiplied by time. This tilt will result in an east velocity error, which can be used to estimate heading error. Since the error signal depends on $\cos\Phi$, we see that the nearer we get to the poles, the less effective this technique is. This technique is known as gyrocompassing, which can produce very accurate alignments at latitudes below 70 degrees.

Neither of the navigation mechanizations shown in Figures 7 and 9 perform any of their navigation functions in spherical coordinates. ECEF and inertial coordinates are well defined everywhere. Quantities, such as longitude, north, and east, are meaningless at the poles, but these are only outputs. Both systems will continue to navigate across the poles. Some of the outputs will become undefined, but once the pole is crossed, the outputs will recover. Transfer alignments that depend on locally level velocities for their measurements also become increasingly inaccurate as the poles are approached. For these reasons, and the fact that the ECEF frame is the natural coordinate frame for GPS, most navigation systems today are mechanized in the ECEF frame. Quantities, such as latitude, longitude, and north and east velocities, are provided only as a human interface.

WGS84

The earth is actually approximated by an oblate spheroid rather than a sphere. We will not go into the details here, as going to the more accurate model merely increases the mathematical complexity of our solution, it does not provide any new insights. The current standard model for the reference ellipsoid is the WGS-84 system defined by the Defense Mapping Agency (see bibliography).

This completes our introduction to inertial navigation. Some special topics are addressed in more detail in the appendixes. In the bibliography we list additional references that cover some of this material in more depth. We have only touched on the alignment process. Alignment and calibration are really part of the measurement process, and will be examined in more detail in the appendix on Kalman filtering (Appendix A).

CONCLUSION

We hope the uninitiated reader has learned the basic simplicity of the navigation process without being overwhelmed with the mathematical details, and has developed an appreciation of the basic principles of alignment and navigation of an inertial system.

Appendix A

KALMAN FILTER

The Kalman filter is a recursive algorithm designed to compute corrections to a system based on external measurements. The corrections are weighted according to the filter's current estimate of the system error statistics. The derivation of the filter equations requires some knowledge of linear algebra and stochastic processes. The filter equations can be cumbersome from an algebraic point of view. Fortunately, the operation of the filter can be understood in fairly simple terms. All that is required is an understanding of various common statistical measures.

We begin with the concept of a state vector, which is central to the formulation of the filter algorithms. A state vector is a set of quantities we have chosen to describe the "state" of a system. For a navigation system, the state is naturally described by position, velocity, attitude, sensor errors, and related quantities. Since we are trying to correct the errors in a navigation system, working in terms of error states is convenient. Any quantity can be described in terms of its true value plus some unknown error. For some arbitrary state x, our estimate of will be

$$\hat{x} = x + \delta x$$

where x is the true value and δx is the error in our estimate. The purpose of the filter is to estimate the error states (δx) and use them to correct our state estimates, $\hat{x}$.

We need to assume that all system equations and measurement equations can be expanded in a Taylor series in the error states. Since these error states are assumed to be small, we keep only first order terms. Therefore, our equations are linear. This formulation, where we are filtering the error states instead of the states themselves, is referred to as an extended Kalman filter.

To illustrate the basic principles of the filter, we will begin by making the simplest assumptions we can. We will assume that the mean of all error states is zero. (If they were not, we could make them so by redefining the states.) We also will assume a one-state system, our error state will be δx. We will have some *a priori* estimate of the initial standard deviation (σ) of this error state. To keep the notation consistent with the literature, we will call the variance (σ^2) of the error state P.

The dynamics of our error state will, in general, be described by some differential equation. To begin, we assume that this equation takes the simple form

$$\delta\dot{x}(t) = F\delta x(t) \qquad \text{(A1)}$$

where F is some constant. This equation has the formal solution

$$\delta x(t+\Delta t) = (\exp F\Delta t)\delta x(t) = \Phi(\Delta t)\,\delta x(t), \qquad \text{(A2)}$$

where we have defined the state transition matrix (STM) as $\Phi(\Delta t) = (\exp F\Delta t)$. The STM describes the evolution of the system error states in time, when no measurements are being processed.

By definition, the variance of our error state is $P = E[\delta x^2]$,where the operation E[ ] denotes an ensemble average. By assumption, $E[\delta x] = 0$. Applying this operator to Equation (A2), we obtain

$$P(t+\Delta t) = \Phi(\Delta t)\, P(t)\, \Phi(\Delta t). \tag{A3}$$

We have written the equation in this form to correspond to the form it will take when we consider state vectors of more than dimension.

We now consider the effect of a measurement on the system. Assume that we have an external measurement of x that is corrupted by noise.

$$\tilde{x} = x + \eta, \tag{A4}$$

where $\tilde{x}$ is the measured value of x and η is a zero-mean Gaussian white sequence of variance R. We define a measurement residual as

$$z = \hat{x} - \tilde{x} = \delta x + \eta. \tag{A5}$$

If it were not for the noise term, we would correct our state as

$$\hat{x} = \hat{x} - z. \tag{A6}$$

Since this measurement is corrupted by noise, we need to know how to weight the measurement in correcting our state variable. This problem of computing optimal gains is the heart of the Kalman filter. We wish to compute the Kalman gains, K, in the equation

$$\hat{x} = \hat{x} - Kz \tag{A7}$$

where Kz is our best estimate of the error state.

Since R is the variance of the measurement error and P is the variance of our current estimate of δx, we logically expect the gains to be a function of these two variables. We state the result for one dimension,

$$K = P/(P + R). \tag{A8}$$

This result makes sense. If our measurement is more accurate than our system error, then $R \ll P$ and $K \cong 1$. In this case, Equation (A7) is approximately the same as (A6), as it should be. If we know our system error more accurately than our measurement, we should make very little correction due to the measurement. In this case $R \gg P$ and $K \cong P/R$ so the correction is very small. We can see that the Kalman gains make a great deal of sense. Of course, this is not a derivation of the gain equations, but we are not concerned with mathematical rigor here.

Once we make a measurement, we must change the variance of P to reflect this new information. The result for one dimension is

$$P = PR/(R + P). \tag{A9}$$

In the case where $R \ll P$, this reduces to $P \cong R(1\text{-}R/P)$. The error state variance thus becomes approximately equal to the measurement error variance, as we would expect. In the case where $R \gg P$, Equation (A9) reduces to $P \cong P(1\text{-}P/R)$. The error state variance changes very little in the case, again as we would expect.

Since x is in general an n-dimensional vector, P will be an nxn matrix. The above equations will become correspondingly more complicated, but the principle will remain the same. Before we consider the complications introduced by higher dimensional spaces, we summarize the process described. The sequence of events is as follows:

1. The variance of δx is initialized as P_0.
2. The variance is propagated forward in time to the first measurement as

$$P_1 = \Phi(\Delta t)\ P_0\ \Phi(\Delta t).$$

3. The gain is computed based on this value of P and the measurement error variance, $K_1 = P_1/(P_1 + R)$.
4. The measurement residual is computed as $z = \hat{x} - \tilde{x}$.
5. The state vector is corrected according to $\hat{x} = \hat{x} - K_1 z$.
6. The variance of δx is updated to reflect the measurement as

$$P'_1 = P_1 R/(R + P_1).$$

7. This variance is propagated forward to the next measurement as

$$P_2 = \Phi(\Delta t)\ P'_1 \Phi(\Delta t)$$

and the process begins all over.

Note that we have used P'_k to denote the value of P immediately after the kth measurement and P_k to denote its value immediately before.

In this description, we have neglected the effect of process noise (or plant noise). This effect takes the form of a random forcing function in Equation (A1),

$$\delta \dot{x}(t) = F\delta x(t) + \nu \tag{A1$'$}$$

where ν is a zero-mean Gaussian white noise of power spectral density N. This will add a term to Equation (A2) of

$$\begin{aligned} \delta x(t+\Delta t) &= (\exp F\Delta t)\delta x(t) + \int_t^{t+\Delta t} \left[\exp F(\tau - t)\right] \nu d\tau \\ &= \Phi(\Delta t)\ \delta x(t) + \int_t^{t+\Delta t} \left[\Phi(\tau - t)\right] \nu d\tau \end{aligned} \tag{A2$'$}$$

and in Equation (A3) of

$$P(t+\Delta t) = \Phi(\Delta t)\ P(t)\ \Phi(\Delta t) + Q \tag{A3$'$}$$

where

$$Q = N\int_t^{t+\Delta t} \left[\exp 2F(\tau - t)\right] d\tau \cong N\Delta t \quad \text{for} \quad 2F\Delta t << 1.$$

All other equations remain the same.

As we mentioned, δx is, in general, a vector defined as

$$\delta \mathbf{x} = \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{pmatrix} \quad \text{(A10)}$$

P now becomes an nxn matrix defined by

$$P = E[\delta \mathbf{x} \delta \mathbf{x}^{\mathbf{T}}] \quad \text{(A11)}$$

where $\delta \mathbf{x}^{\mathbf{T}}$ is the transpose of Equation (A10). This is referred to as the covariance matrix. The diagonal elements will be the variances of the individual error states. The off-diagonal elements will be a measure of the correlations between the corresponding diagonal elements. Correlations are important as they permit the indirect estimate of a state by measuring a correlated state. Correlations may arise through the STM or through measurements.

The measurement Equation (A5) now appears as

$$\mathbf{z} = H\, \delta \mathbf{x} + \eta. \quad \text{(A5)}'$$

H is the measurement (or observation) matrix. The vector form of the measurement assumes that more that one measurement is taken. This equation also assumes that the measurements consist of quantities that can be expanded in terms of the components of the error state vector. We made this linearization assumption at the beginning. Equation (A3) generalizes to

$$P(t+\Delta t) = \Phi(\Delta t)\, P(t)\, \Phi(\Delta t)^T + Q \quad \text{(A3)}''$$

where Q is a matrix generalization of the process noise term. The STM will not, in general, be a simple exponential (the F matrix in Equation (A1) will not be a constant). Computation of this matrix will usually require some form of numerical integration.

We will merely state the form Equations (A8) and (A9) take. For optimal gains, these equations become

$$K = PH^T(HPH^T + R)^{-1} \quad \text{(A8)}'$$

and

$$P' = [I - KH]P. \quad \text{(A9)}'$$

In the above, R is now a matrix of the noise associated with each element of the measurement vector. For more detail, the reader is referred to one of the many texts on Kalman filtering.

Kalman filters are useful when processing measurements in the presence of measurement noise and system noise.Kalman filters provide a method of incorporating the measurements in an optimal fashion. Measurements are required in inertial navigation systems to align the system and bound the system errors over time. These measurements may take many forms: position measurements, Doppler velocity measurements, GPS measurements, or any external system that gives us a useful measure of one or more of the navigation parameters.

Alignment may be self-alignment such as the gyrocompassing technique discussed in the text or alignment to some external source. When the external source is another inertial system, the process is usually referred to as transfer alignment. The measurements used in the transfer alignment process can be based on the accelerometer output or the gyro output. Examples of accelerometer-based alignments are velocity matching and velocity integral matching. Examples of gyro-based alignments are angular rate matching and attitude matching (a common attitude-matching application is a SINS alignment where the platform is aligned to a ship's inertial navigation system). GPS position and velocity measurements are used to align a system and to bound the system errors.

Alignment often includes calibration of some of the sensor errors. This calibration is especially useful for sensor errors that can be modeled as bias terms. All that is required is to include these sensor error terms as part of the error state vector. For a more detailed treatment of the measurement process, the reader is referred to the bibliography.

Appendix B

NOISE AND SENSOR ERRORS

Most of the error sources that corrupt our navigation solution are sensor errors or random disturbances. In this appendix, we describe some of the more common gyro and accelerometer errors. The error model for any sensor will, to some extent, depend on its construction. A ring laser gyro will behave somewhat differently than a tuned rotor gyro. The errors discussed are common to all types of gyros and accelerometers. We also discuss some of the basic types of noise models.

Bias Errors—The simplest type of error. A bias is, by definition, a constant signal on the output of a sensor, independent of the input. A bias will not change during a run, but may vary from turn-on to turn-on. A bias is modeled as a random constant. For navigation grade sensors, bias is usually specified in deg/hr or milligees.

Scale Factor Errors—A linear error that is proportional to the input signal. Scale factor is usually specified in parts per million.

Misalignment—Refers to mechanical misalignment. Ideally, the gyros and accelerometers define an orthogonal triad. This idealized coordinate frame is the platform frame. Since achieving perfect mechanical alignment in a practical system is impossible, we describe the alignment error of each sensor to the platform axes as a random constant. This requires six numbers for the gyros and six for the accelerometers. For example, one number describes the misalignment of the x gyro from the x platform axis in the y direction and another the misalignment in the z direction. This error is sometimes referred to as nonorthogonality error and is given in terms of microradians.

These errors are idealizations. For example, bias errors are only constant for short terms, they typically exhibit drift that might be modeled as a Markov process (see below) superimposed on a constant bias. Scale factor errors usually exhibit some degree of non-linearity. Also some error terms are dependent on the stresses on the sensor. These error terms are due to mechanical deformations of the sensor.

Another important error source is due to temperature. Navigation grade sensors model the effects of temperature and compensate for this in their internal electronics. This compensation is never perfect. The most important residual errors over temperature are a temperature dependence of bias and scale factor errors, making calibration of the sensor difficult.

Another error source is quantization error. The output of a gyro or accelerometer is given in terms of some smallest unit, we do not have infinite precision. This produces a white noise on the output proportional to the magnitude of the quantization.

Many error sources are random and can only be described in terms of stochastic processes. A stochastic process is a random time process. Many of these processes may be described by differential equations with white noise forcing functions. A white noise signal is a mathematical idealization that contains infinite energy and bandwidth. This signal is described in terms of its power spectral density (PSD). Mathematically, if η is a zero-mean white process with PSD N, then

$$E[\eta] = 0$$

and

$$E[\eta(t)\eta(t+\tau)] = N\delta(\tau)$$

where $\delta(\tau)$ is a delta function.

The most important noise associated with a gyro is random walk. Random walk results from the integration of white noise. Random walk can be described by the differential equation

$$\dot{x} = \eta$$

where η is a white noise term with PSD N. The variance of x will be

$$E[x^2] = Nt.$$

This time-dependent error restricts the accuracy of an alignment and cannot be estimated or compensated for. Accelerometers also exhibit random walk, but the effect on a navigation system is usually minor.

Some noise sources (system disturbances) are correlated in time. Their current value depends to some extent on one or more past values. The most common processes of this type are Markov processes. First and second order Markov processes are described by the differential equations

$$\dot{x} + \beta x = \eta \qquad \text{(first order)}$$

and

$$\ddot{x} + 2\alpha\beta\dot{x} + \beta^2 x = \eta. \qquad \text{(second order)}$$

Wing flexure in a aircraft, for example, is commonly modeled as a second order Markov process. This effect is important effect in transfer alignment. Disturbances of this type may be modeled in the filter by state vector augmentation. For a more detailed discussion of stochastic processes, the reader is referred to the bibliography.

The degree of accuracy with which we must model all these error sources is dependent on the application as well as the quality of the sensor.

Appendix C

COORDINATE TRANSFORMATIONS

In the most general sense, a coordinate transformation over a real three-dimensional space is a set of nine real numbers that transforms the components of a vector in one frame to another. Suppose we have two coordinate frames that we call A and B. We write an arbitrary vector, $\vec{X}$, in the A frame as

$$\vec{X}^{A} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

where the subscripts 1, 2, and 3 refer to the coordinate axes (e.g., xyz, NED). We denote the same vector with coordinates in the B frame as

$$\vec{X}^{B} = \begin{pmatrix} X'_1 \\ X'_2 \\ X'_3 \end{pmatrix}$$

where we have used the primes to distinguish the components of the vector in the B frame from the components in the A frame.

A coordinate transformation relates the components in one frame to the components in another. Thus, if C_A^B is the transformation from A to B,

$$\vec{X}^{B} = C_A^B \vec{X}^{A} \tag{C1}$$

similarly

$$\vec{X}^{A} = C_B^A \vec{X}^{B} \tag{C2}$$

where C_B^A is the inverse of C_A^B .

Equation (C1) can be written in vector matrix notation as

$$\begin{pmatrix} X'_1 \\ X'_2 \\ X'_3 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{C3}$$

with a similar expression for Equation (C2). This equation can be written in a compact form as

$$X'_k = C_{kj} X_j \cdot (j = 1, 2, 3) \tag{C4}$$

In the matrix elements, C_{kj}, k is the row index and j is the column index. According to the summation convention, repeated indices are summed from 1 to 3 unless otherwise indicated.

Orthogonality conditions on the transformation reduce the number of independent elements in the transformation matrix from nine to three. This means that we may write the matrix elements as functions of as few as three parameters. There are many ways to do this.

One of the most common ways of parameterizing the transformation matrix is by the use of Euler angles. There are many conventions for specifying Euler angles, but for aircraft navigation systems, it is almost always done as follows:

$$C_N^B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{C5}$$

This is the transformation from NED to body axes, a product of three rotations in the following sequence:

1. A positive rotation about the down (z) axis by an angle ψ
2. A positive rotation about the new y axis by an angle θ
3. A positive rotation about the resulting x axis by an angle ϕ

These angles define leading (ψ), pitch (θ), and roll (ϕ). Note that order is important, rotations about different axes do not commute. For example,

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Since rotations are orthogonal, the inverse of Equation (C5) is just its transpose or

$$C_B^N = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{C6}$$

This is easily verified by direct multiplication of Equations (C5) and (C6) (in either order). In this discussion, we have illustrated Euler angles with a particular example, but any rotation can be paramaterized by three rotations in any sequence.

The rotation from one frame to another can be accomplished by a sequence of three rotations in an infinite number of ways. One particularly useful way of describing a transformation by a single rotation is given by

$$C_A^B = (I) + \left(\vec{\theta}x\right)\frac{\sin\theta}{\theta} + \left(\vec{\theta}x\right)^2 \frac{(1-\cos\theta)}{\theta^2} \tag{C7}$$

In this equation (I) is the 3 x 3 unit matrix

$$(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and, by definition

$$\left(\vec{\theta}x\right) = \begin{pmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{pmatrix}$$

The angle θ is the magnitude

$$\theta = \left[\theta_x^2 + \theta_y^2 + \theta_z^2\right]^{1/2}$$

Note that this form of the transformation still requires three parameters; θ_x, θ_y, and θ_z. This represents a rotation by the angle θ about an axis defined by the unit vector

$$\vec{n} = \frac{1}{\theta}\begin{pmatrix} \theta_x \\ \theta_y \\ \theta_z \end{pmatrix}$$

The components of $\vec{\theta}$ are the same in both frames.

Transformations are not always parameterized with three parameters. A common method of describing rotations is by the use of quaternions. A quaternion has four components so there is some redundancy. A quaternion is a set of four quantities, q_0, q_1, q_2, and q_3. They may be thought of as a combination of a vector and a scalar that obey the following multiplication rules: For two quaternions ($q_0 + \vec{q}$) and ($p_0 + \vec{p}$), their product is

$$\left(p_0 + \vec{p}\right)\left(q_0 + \vec{q}\right) = p_0 q_0 - \vec{p}\cdot\vec{q} + p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q} \tag{C8}$$

The details of quaternion algebra are beyond the scope of this document. We mention them because they are commonly encountered in navigation algorithms. The quaternion may be used to construct a 3 x 3 rotation matrix as

$$(R) = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2\left(q_1q_2 + q_0q_3\right) & 2\left(q_1q_3 - q_0q_2\right) \\ 2\left(q_1q_2 - q_0q_3\right) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2\left(q_2q_3 + q_0q_1\right) \\ 2\left(q_1q_3 + q_0q_2\right) & 2\left(q_2q_3 - q_0q_1\right) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \tag{C9}$$

The reason quaternions are widely used is because of their integration properties. The derivative of a quaternion is a linear differential equation. For example:

let

ω_i = components of the angular velocity of the body frame with respect to inertial space, in body coordinates

Ω_i = components of the angular velocity of the NED frame with respect to inertial space, in NED coordinates

The quaternion that represents the rotation from NED to body obeys the differential equation

$$\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = 1/2 \begin{pmatrix} 0 & -(\omega_1 - \Omega_1) & -(\omega_2 - \Omega_2) & -(\omega_3 - \Omega_3) \\ (\omega_1 - \Omega_1) & 0 & (\omega_3 + \Omega_3) & -(\omega_2 + \Omega_2) \\ (\omega_2 - \Omega_2) & -(\omega_3 + \Omega_3) & 0 & (\omega_1 + \Omega_1) \\ (\omega_3 - \Omega_3) & (\omega_2 + \Omega_2) & -(\omega_1 + \Omega_1) & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{C10}$$

This equation is linear in the q's. The corresponding equation for Euler angles, for example, is highly nonlinear.

Another common way of keeping track of a rotation matrix is to use the elements of the direction cosine directly. Since the rows and columns of this matrix are orthogonal, any two rows (or columns) determine the other row (or column). The differential equations for these matrix elements (the C_{kj}) are again linear. The usual Euler angles can be recovered from the matrix elements by use of inverse trigonometric relations.

We give a very useful relation involving the derivative of a transformation matrix,

$$\dot{C}_A^B = C_A^B \Omega_{AB}^B = \Omega_{AB}^B C_A^B \tag{C11}$$

In this equation, C_A^B is a 3 x 3 orthogonal tranformation and

$$\Omega_{AB}^A \equiv \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{C12}$$

In this antisymmetric form, the ω_k are the components of the angular velocity of the B frame with respect to the A frame, with coordinates in the A frame. The matrix Ω_{AB}^B is the same, except that the coordinates of the angular velocity are in the B frame. This relation is, in fact, the basis for the direction cosine method of integrating the transformation matrix discussed in the previous paragraph.

As an example of how Equation (C11) is used, let us take the derivative of Equation (C1) with respect to time. The result is

$$\vec{x}^B = \dot{C}_A^B \vec{x}^A + C_A^B \vec{x}^A \tag{C13}$$

In this equation, $\dot{\vec{x}}^{B}$ is the time derivative of the vector with respect to the B frame, while $\dot{\vec{x}}^{A}$ is the derivative with respect to the A frame. Using Equation (C11), this equation becomes

$$\dot{\vec{x}}^{B} = C_{A}^{B}\left(\dot{\vec{x}}^{A} + \Omega_{AB}^{A}\vec{x}^{A}\right)$$

By matrix multiplication, we verify that

$$\Omega_{AB}^{A}\vec{x}^{A} = -\vec{\omega}_{AB}^{A} \times \vec{x}^{A}$$

Substituting into the previous equation, we have

$$\dot{\vec{x}}^{B} = C_{A}^{B}\left(\dot{\vec{x}}^{A} - \vec{\omega}_{AB}^{A} \times \vec{x}^{A}\right), \tag{C14}$$

which we recognize as the Coriolis law that relates derivatives with respect to different coordinate frames.

Finally, we write down the relations between the derivatives of the Euler angles and the angular velocity of one frame with respect to another. To be definite, we refer to Equation (C5). Let $\vec{\omega}$ be the angular rate of the body with respect to the NED frame, expressed in the NED frame.

Using Equation (C11) we find

$$\begin{aligned}
\omega_N &= \dot{\phi}\cos\theta\cos\psi - \dot{\theta}\sin\psi \\
\omega_E &= \dot{\theta}\cos\psi + \dot{\phi}\cos\theta\sin\psi \\
\omega_D &= \dot{\psi} - \dot{\phi}\sin\theta
\end{aligned} \tag{C15}$$

Similarly, we can relate the components of $\vec{\omega}$ in the body frame to the Euler angles as

$$\begin{aligned}
\omega_x &= \dot{\phi} - \dot{\psi}\sin\theta \\
\omega_y &= \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta \\
\omega_z &= \dot{\psi}\cos\phi\cos\theta - \dot{\theta}\sin\phi
\end{aligned} \tag{C16}$$

As we can see from the material presented in this appendix, the subject of coordinate transformation is far from trivial. We have only scratched the surface here. However, we have given the most important results that apply to inertial navigation.

Appendix D

SCHULER OSCILLATION

In this appendix we present a simple derivation of Schuler oscillation. For simplicity, we assume a platform located at the equator of a spherical nonrotating earth. Also assume no initial position error and the x axis of the platform is pointing north. A tilt about the north (x) axis results in an acceleration error in the east (y) direction of

$$\delta a_y = -g\theta_x$$

where g is the gravitational acceleration. The velocity error due to this tilt is then given by

$$\delta V_y = \int_0^t \delta a_y dt = -g\int_0^t \theta_x d\tau .$$

This results in an east position error of

$$\delta y = \int_0^t \delta v_y d\tau .$$

The navigation algorithm will rotate its notion of local level according to its sensed velocity. This corresponds to a tilt error rate of

$$\ddot{\theta}_x = \delta v_y / R = -g / R\int_0^t \theta_x d\tau$$

where R is the radius of the earth. Differentiating this equation with respect to time, we have

$$\ddot{\theta}_x + (g/R)\,\theta_x = 0.$$

We recognize this as the differential equation for a simple harmonic oscillator. The frequency of oscillation is

$$\omega = \sqrt{g / R} .$$

The period of oscillation is

$$T = 2\pi/\omega .$$

For g = 32.17 ft/s and R=20925646 ft, we have

$$T = 84.46 \text{ min},$$

which is the period of Schuler oscillation. The equation for the frequency of oscillation is formally identical to that for a simple pendulum of length R, which is why we sometimes see the statement that an inertial navigation system acts like a simple pendulum with a length equal to the earth's radius. By taking the appropriate derivatives, it is easy to see that both position and velocity in the y direction obey the same differential equation and, therefore, oscillate with the same frequency. Since, in a practical system, we have sensor errors, we normally see the Schuler oscillation superimposed on an error curve due to sensor errors. In general, we observe Schuler oscillations about both level axes.

Appendix E

NED ROTATION RATES

The orientation of the NED frame is constantly changing with respect to the inertial frame because of the rotation of the earth and the motion of the navigator over the earth's surface. In this appendix we give the expressions for the angular velocity of the earth fixed frame with respect to inertial space and the angular velocity of the NED frame with respect to the earth fixed frame.

Differentiating the ECEF to NED transformation

$$C_E^N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E1}$$

we obtain

$$\begin{aligned} \dot{C}_E^N &= C_E^N \begin{pmatrix} 0 & \dot{\lambda} & \dot{\phi}\cos\lambda \\ -\dot{\lambda} & 0 & \dot{\phi}\sin\lambda \\ -\dot{\phi}\cos\lambda & -\dot{\phi}\sin\lambda & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\dot{\lambda}\sin\phi & \dot{\phi} \\ \lambda\sin\phi & 0 & \dot{\lambda}\cos\phi \\ -\dot{\phi} & -\dot{\lambda}\cos\phi & 0 \end{pmatrix} C_E^N \end{aligned} \tag{E2}$$

where

ϕ = latitude

λ = longitude

From equation (C11) we find the angular rate of the NED frame with respect to the ECEF frame in both NED and ECEF coordinates,

$$\vec{\omega}_{EN}^{E} = \begin{pmatrix} \dot{\phi}\sin\lambda \\ -\dot{\phi}\cos\lambda \\ \dot{\lambda} \end{pmatrix} \tag{E3}$$

and

$$\vec{\omega}_{EN}^{N} = \begin{pmatrix} \dot{\lambda}\cos\phi \\ -\dot{\phi} \\ -\dot{\lambda}\sin\phi \end{pmatrix}$$

These rates are often referred to as transport rates.

The longitude and latitude rates are

$$\dot{\lambda} = \frac{V_E}{\left(R_p + h\right)\cos\phi} \tag{E4}$$

and

$$\dot{\phi} = \frac{V_N}{R_m + h}$$

where

V_E = east velocity
V_N = north velocity
h = altitude above reference ellipsoid

We have defined the radii of curvature of the reference ellipsoid as

$$R_p = \frac{a}{\left(1 - \varepsilon^2 \sin^2\phi\right)^{1/2}} \tag{E5}$$

and

$$R_m = \frac{a\left(1 - \varepsilon^2\right)}{\left(1 - \varepsilon^2 \sin^2\phi\right)^{3/2}}$$

where

a = earth’s semimajor axis = 20,925,646 ft
ε^2 = square of the earth’s eccentricity = 0.00669438

Combining Equations (E4) and (E3), we have the alternate form of the transport rates in NED as

$$\vec{\omega}_{EN}^{N} = \begin{pmatrix} V_E / R_p + h \\ -V_N / R_m + h \\ -V_E \dfrac{\tan\phi}{R_p + h} \end{pmatrix} \tag{E6}$$

The earth rate term is found from

$$\vec{\omega}_{IE}^{N} = C_E^N \begin{pmatrix} 0 \\ 0 \\ \Omega_E \end{pmatrix} = \begin{pmatrix} \Omega_E \cos\phi \\ 0 \\ -\Omega_E \sin\phi \end{pmatrix} \tag{E7}$$

where

Ω_E = angular rotation rate of the earth with respect to inertial space
= 15.041067 deg/hr
= 7.292115×10^{-5} rad/s

These expressions have been included for reference only. The reader is referred to the bibliography for a detailed description of the earth's reference ellipsoid.

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