

Department of Electrical Engineering
The University of Iowa

Ph.D. Thesis Proposal

**Analysis of the Control of Nonlinear Systems
subjected to Stochastic Disturbances**

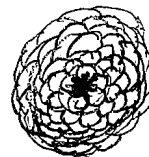
by

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I. Introduction

Much of the phenomena of the real world is nonlinear and should be analyzed **using** nonlinear differential equations **without recourse** to linearized versions of these differential **equations** which, when solved, yield answers strikingly different from the observed phenomena and, hence, having little **value** as a mathematical model. **Many** of the control systems encountered today have essential nonlinearities resulting from the physical limitations of the devices employed. In order to obtain meaningful results from **the** mathematical model, satisfactory methods of analysis must be found for **both the** case of deterministic inputs and the case of stochastic inputs. Suitable analysis techniques are needed as a first step **in** the development of synthesis techniques.

Below are presented some **of** the current techniques employed in the analysis of nonlinear systems with stochastic inputs. Also mentioned are **the** limitations associated with each of these methods.

II. The **Analysis of Nonlinear** Control Systems with Random Inputs

According to R. C. Boston (1953), who first introduced the technique , the method of statistical linearization , is an inexact method of allowing systems with a certain class of nonlinearities, subject to random inputs, to be analyzed. The **class** of nonlinearities allowing the method of statistical **linearizations** to be used **are** zero-memory **nonlinearities** in both open loop **and** feedback configurations.

Zero-memory nonlinearities are those **nonlinearities** with responses determined completely by the instantaneous amplitude of **the** input to the element. (The method does **not** apply to **nonlinearities with** memory, e.g., hysteresis).

The purpose of the analysis is to **allow** the computation of the probability **density** function (pdf.) of the output when the pdf of the input is **known**. The configuration **and nonlinearities of** the system must be known **before** the method of statistical **linearization can** be applied.

The **method** involves replacing the **nonlinear element of** the system by a **linear** element with a **parameter**. The parameter, the equivalent gain, K_{eg} , **is evaluated by statistical considerations**. After the gain is evaluated the **analysis** of the whole system proceeds using the methods of **linear** systems.

The relationship between the output, y , and the input, x , for zero-memory **nonlinearities can** be expressed as $y = f(x)$. The method of statistical linearization involves assuming the form of $y = K_{eg} x + x_H$ as **the** input-output relationship and neglecting x_H , the "distortion function". K_{eg} is **chosen** by satisfying the criterion of minimizing **the mean** square error, $\overline{[y - K_{eg} x]^2}$.

The mean square error is **determined** by applying **the** fundamental theorem of expectation, as shown below:

$$M = \overline{[y - K_{eg} X]^2} = E\{[y - K_{eg} X]^2\} = E\{|f(x) - K_{eg} x|^2\} = \\ \int_{-\infty}^{\infty} [f(x) - K_{eg} x]^2 p_x(x) dx = \int_{-\infty}^{\infty} f^2(x) p_x(x) dx - 2K_{eg} \int_{-\infty}^{\infty} x f(x) p_x(x) dx \\ + K_{eg} \int_{-\infty}^{\infty} x^2 p_x(x) dx .$$

By differentiating M with respect to $K_{e\bar{g}}$ and setting the result equal to zero an equation results which can be solved for the value of $K_{e\bar{g}}$ that minimizes M .

$$\frac{\partial M}{\partial K_{e\bar{g}}} = -2 \int_{-\infty}^{\infty} x f(x) p_x(x) dx + 2 K_{e\bar{g}} \int_{-\infty}^{\infty} x^2 p_x(x) dx = 0$$

$$\therefore K_{e\bar{g}} = \frac{\int_{-\infty}^{\infty} x f(x) p_x(x) dx}{\int_{-\infty}^{\infty} x^2 p_x(x) dx}$$

The method of **statistical linearization** is especially well adopted to gaussian inputs since, after the conversion of the non-linear system to an approximate linear system, the theory of linear systems assures that every variable in the system is gaussianly distributed and so, therefore, is the output.

Under the assumption of a gaussian input the expression for the computation of the $K_{e\bar{g}}$ is greatly simplified. The pdf, of the input to the nonlinearity approximation is $p_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2} \frac{(x-\bar{x})^2}{\sigma_x^2}}$.

$$\int_{-\infty}^{\infty} x^2 p_x(x) dx = E(x^2) = \sigma_x^2 + (E[x])^2$$

$$\text{So } K_{e\bar{g}} = \frac{\int_{-\infty}^{\infty} x f(x) \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2} \frac{(x-\bar{x})^2}{\sigma_x^2}} dx}{\sigma_x^2 + \bar{x}^2}$$

The nonlinearity is now replaced by the linear systems with $K_{e\bar{g}}$. The resulting linear system is analyzed using the usual linear system analysis techniques.

According to Borton (1953), the above analysis is also applicable to servomechanisms and feedback control systems with unity feedback. The method of statistical linearization is not so straight forward for these configurations and leads to an analysis equivalent to the analysis of a linear system with a parameter. The result of the analysis indicates that multimodal behavior of

the response is indicated.

This type of response is physically verified, but the method of statistical linearization does not predict when the response will be of a particular mode.

In a more recent work, Pervozvanskii (1965) has generalized the method of statistical linearization and extended it to nonlinear time-varying systems.

An example of statistical linearization is given in the appendix.

I . Approximate Analysis of Nonstationary Nonlinear Systems by Seminvariants

The purpose of the analysis is to find the probability density function (pdf) of the output of a system which can be represented as a nonlinear, time-varying, differential equation of the form $\frac{dx(t)}{dt} = a(x(t), t) + f(t) w(t)$, where $w(t)$ is white gaussian noise.

As shown by M. L. Dashevskii (1966, 1967), the method involves a generalization of a technique that is very familiar from statistics. Recall the familiar technique of using the characteristic function, $M = E[e^{jzX}]$, to calculate the moments of the variables, $\overline{X^k} = (-j)^k \frac{\partial^k M}{\partial z^k} \Big|_{z=0}$. The function $\Psi = \ln M$, called the "second characteristic function", was used to compute the semi-invariants, $\lambda_k = (-j)^k \frac{\partial^k \Psi}{\partial z^k} \Big|_{z=0}$. (This method can be found in the exercises of Wozencraft and Jacobs' PRINCIPLES OF COMMUNICATIONS). The generalization involves extending the techni-

que to the case when all the above quantities are functions of time,

$$t. \quad X(t)^k = (-j)^k \left. \frac{\partial^k M(t, z)}{\partial z^k} \right|_{z=0} \quad \text{and} \quad \lambda_k(t) = (-j)^k \left. \frac{\partial^k \Psi(t, z)}{\partial z^k} \right|_{z=0}$$

are the two generalizations upon which the method is based.

From the Fokker-Planck or Kolmogorov's equation the partial differential equation for the pdf of the output $\frac{\partial p(x, t)}{\partial t}$

$= -\frac{\partial}{\partial x} [p(x, t) a(x, t)] + \frac{1}{2} b^2(x, t) \frac{\partial^2 p(x, t)}{\partial x^2}$, is manipulated into an integro-partial differential equation in $M(z, t)$, $a(x, t)$, $p(x, t)$, and $b(t)$:

$$\frac{\partial M(z, t)}{\partial t} = jz \int_{-\infty}^{\infty} e^{jz x(t)} a(x(t), t) p(x(t), t) dx(t) - \frac{b^2(t)}{2} z^2 M(z, t).$$

From the equation $\Psi(z, t) = \ln M(z, t)$ the

integro-partial differential equation involving $\Psi(z, t)$, $b(t)$, $a(x, t)$,

and $p(x, t)$ are obtained: $\frac{\partial \Psi(z, t)}{\partial t} = e^{-\Psi(z, t)} jz \int_{-\infty}^{\infty} e^{jz x(t)} a(x(t), t) p(x(t), t) dx(t) - b^2(t) z^2 / 2$

Now, differentiating the above with respect to z , multiplying

by $(-j)$, and setting $z = 0$.

$$\text{Yields: } \left. \frac{\partial^2 \Psi(z, t)}{\partial z \partial t} \right|_{z=0} = j e^{-\Psi(z, t)} \int_{-\infty}^{\infty} [1 + z(jx - \Psi'(z, t))] \cdot a(x, t) e^{jzx} p(x, t) dx \Big|_{z=0} - b^2(t) z \Big|_{z=0}$$

Yielding:

$$\frac{d\lambda_1(t)}{dt} = (-j) \frac{\partial^2 \Psi(z, t)}{\partial z \partial t} = \int_{-\infty}^{\infty} a(x(t), t) p(x(t), t) dx(t).$$

By taking $(-j) \frac{\partial}{\partial z}$ to operate on the above

yields:

$$d\lambda_2(t) = \frac{\partial^3 \Psi(z, t)}{\partial z^2 \partial t} = \int_{-\infty}^{\infty} (x - \lambda_1) a(x(t), t) p(x(t), t) dx(t) + b^2(t).$$

Similarly the above operation is continued until $\frac{d\lambda_1(t)}{dt}$, $\frac{d\lambda_2(t)}{dt}$,

$\frac{d\lambda_3(t)}{dt}$, $\frac{d\lambda_4(t)}{dt}$, and $\frac{d\lambda_5(t)}{dt}$ are obtained.

The $\lambda_k(t)$ are the semi-invariants and are of **fundamental** importance in **the** method. The still unknown $p(x(t), t)$ is expanded in an Edgeworth series **which** involves the unknown lower parameters $\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t),$ and $\lambda_5(t)$. The above integro-differential equations **are** solved simultaneously for **the** $\lambda_k(t), k = 1, 2, \dots, 5,$ and used in the Edgeworth series as a good representation of the pdf, $p(x(t), t),$ of the output.

Only a finite number of the $\frac{d\lambda_k(t)}{dt}$ **are** solved for **simul-** taneously to **keep** the work load as **low** as possible. **It** is assumed that **all** $\lambda_k(t) = 0$ for $k \geq 6$.

The power of **this** method lies in the **fact that** it easily handles the time-varying nonlinear **differential** equations that have general **nonlinearities** involving **time**.

Owe of the main drawbacks of the method is that the **number** of integro-differential equations **that** must be **solved simultaneously** in **order** to obtain the required semi-invariants greatly increases **as** the order **of** the differential equations describing the system increases.

IV. Volterra Functional Analysis of Nonlinear Time-Varying Systems with Random Inputs

According to Y. H. Ku (1965, 1967), the recent **Volterra** functional method is **extremely** powerful in that it can be used to **analyze** **systems that** can be represented as differential equations that are nonlinear, time-varying, and that have **dererministic** or stochastic inputs.

Since the method of analysis for the case of stochastic inputs is very similar but slightly more complicated than the analysis for the case of **deterministic** inputs, the deterministic case will be discussed here first. The method is applicable when the systems can be represented by **differential** equations of the following form:

$$i) \quad \mathbb{Z}(D)x(t) + F(x, \dot{x}, \dots, x^{(n-1)}) = r(t), \quad D = \frac{d}{dt}$$

$$ii) \quad \mathbb{Z}(D)x(t) + g(t) F(x, \dot{x}, \dots, x^{(n-1)}) = r(t)$$

$$iii) \quad L(t; D)x(t) + F(x, \dot{x}, \dots, x^{(n-1)}) = r(t)$$

$$iv) \quad L(t; D)x(t) + g(t) F(x, \dot{x}, \dots, x^{(n-1)}) = r(t)$$

where $L(t; D)$ denotes a linear operator in both t and D ; F is an **analytic** nonlinear function of the response, $x(t)$, and its time derivatives, $g(t)$ denotes a function of time, and $r(t)$ denotes the input, either deterministic or a **sample function** $\{r(t), -\infty < t < \infty\}$ from a strict sense stationary source with bounded moments of all orders.

For $\mathbb{Z}(D)x(t) + F(x, \dot{x}, \dots, x^{(n-1)}) = r(t)$ and a deterministic $r(t)$, the method yields a solution of the form $x(t) = \sum_{n=1}^{\infty} x_n(t)$, $x_n(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(t_1, t_2, \dots, t_n) r(t-t_1) \dots r(t-t_n) dt_1 \dots dt_n$, and where the kernel for the n th term is an n -dimensional kernel $h_n(t_1, t_2, \dots, t_n)$. Thus, the method is a generalization of the **convolution** integral used in linear system analysis. Indeed; the first term of the series, $x_1(t)$, is simply a **convolution** of the first input, $r(t)$, and the impulse $h_1(t)$ of the linear portion of the overall system. Recurrence relations exist for computing

the **Volterra** kernels in terms of previous kernels **and** previously computed terms of the **series**. These recurrence **relations** make the evaluation of the several **convolutions** less tedious.

Only a finite number of **terms of the series** are required to closely approximate the **nonlinear system**, a situation analogous to **the** use of a finite number of **terms** of a Fourier series to represent a function.

For $L(t;D) x(t) = F(x, \dot{x}, \dots, x^{(n-1)}) = r(t)$ and a **deterministic** $r(t)$, the method is very similar to the method for the case given above **except** that instead of **using** an impulse response $h_1(t)$ for calculating $x_1(t)$, there is a time-varying system function $k_1(t, T)$ **such that** $x_1(t) = \int_{-\infty}^{\infty} k_1(t, T) r(t-T) dT$. The $k_1(t, T)$ can be found by first finding $K_1(t, s)$ by Zadeh's method and inverse **Laplace** transforming.

The other **two** cases of a nonlinearity of the **form** of $g(t) F(x, \dot{x}, \dots, x^{(n-1)})$ are treated in the **same way** as the above two cases.

The analysis of the four different forms of differential **equations** for the case of a stochastic input **deviates only** slightly from **the** analysis for the case of a **deterministic** input. For the differential equation of **form 1**).

$$Z(D) x(t) = F(x, \dot{x}, \dots, x^{(n-1)}) = r(t) \text{ where } \{ r(t), -\infty < t < \infty \}$$

is a sample function from a strict **sense stationary** source **with** moments of all orders **bounded**, the linear systems is given by

$$Z(D) x_1(t) = r(t). \text{ The solution for the linear part is } x_1(t) = \int_{-\infty}^t h_1(t-T) r(T) dT(1), \text{ where } h_1(t) \text{ represents the impulse response}$$

of the linear portion system. Denoting the ensemble average of $\mathbf{r}(t)$ and $\mathbf{x}(t)$ by $\langle \mathbf{r}(t) \rangle_r$ and $\langle \mathbf{x}(t) \rangle_x$ and applying these averages to both members of (1) gives $\langle X_1(t) \rangle_x = \int_0^t h_1(t-\tau) \langle r(\tau) \rangle_r d\tau$
 $= \langle r(t) \rangle_r \int_0^t h_1(t-\tau) u(\tau) d\tau = \langle r(t) \rangle_r X_{1u}(t)$
 where the interchange of expectation and integration has taken place and $X_{1u}(t)$ is the response of the system to a deterministic unit step function $U(t)$. For the higher order term $\langle X_i(t) \rangle_x = \langle r^i(t) \rangle_r X_{iu}(t)$ where the $\langle r^i(t) \rangle_r$ is the ith moment of the input and X_{iu} is a result identical to the deterministic case with $r(t) = U(t)$.

The other forms ii, iii, and iv, of the nonlinear differential equations are treated in exactly the same way except that for $L(t;D)$ the system function is used. All follow the form $\langle X_i(t) \rangle_x = \langle r^i(t) \rangle_r X_{iu}(t)$ with $\langle X(t) \rangle_x = \sum_{i=1}^{\infty} \langle x_i(t) \rangle_x$ being the final solution,

As Ku mentions, the method described above is less tedious when digital computer programs are used in performing the convolutions and summations.

An example using the Volterra functional analysis is given in the appendix.

V. Sequential Estimation of States and Parameters in Noisy Nonlinear Dynamical Systems

Put forward by D. M. Detchmندی and R. Sridhar (1966) this method is one of the most promising in that it can yield an estimate of X , \hat{X} , when $X = g(x,t) + k(x,t)u$ and $y(t) = h(x,t) +$ (observation error) where U represents an unknown input.

This method uses a variational approach **and** then uses invariant imbedding equations to solve the resulting two point boundary value problem.

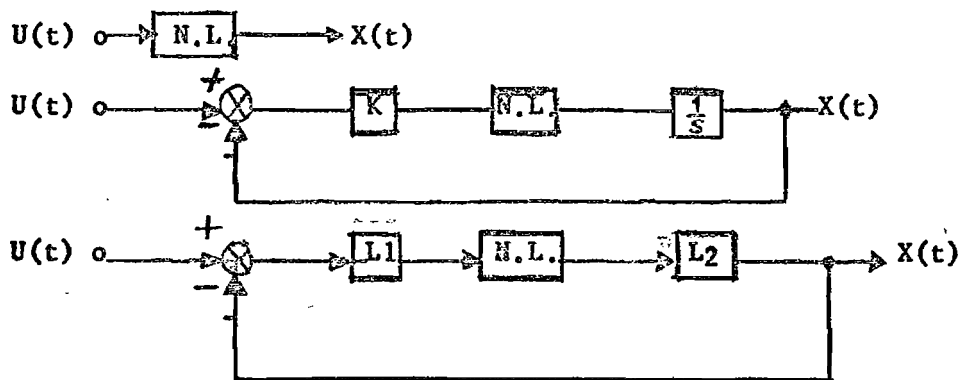
An **important** advantage is that **the** estimator obtained by this scheme can be **implimented** in real time.

There is also the possibility that this method **can** be used in conjunction with statistical linearization in helping to **determine** the gain **for** the **feedback** configuration, when, **as** mentioned **above**, the analysis proceeds **as with** a linear system with a **parameter**.

VI. Proposed Area of Research

In **summary**, we **have** the following **four** methods or tools to apply to nonlinear systems with stochastic inputs:

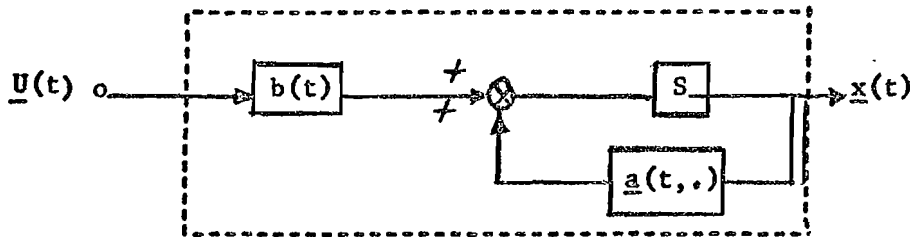
- (1.) The method of statistical linearization **can** be applied to control systems of **the** following configurations:



where $U(t)$ is white gaussian noise and the nonlinearity, **N.L.** is of an acceptable type (zero-memory but **time-varying** or time-invariant).

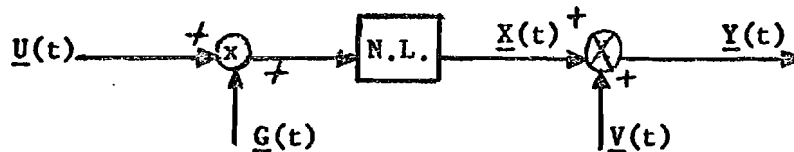
L_1 and L_2 in the above are linear systems.

- 2.) The method of semi-invariant^o is applicable to the following configuration:



or any **configuration** that allows a nonlinear differential equation to be **explicitly** written. Again, $\underline{U}(t)$ is **gaussian white** noise. The precision of the results using **the** method of semi-invariants is **much** better than the method of statistical linearization and it can be used in the analysis of a larger class of **nonlinear** systems (i.e., time-varying nonlinearities).

- (3.) The **Volterra** functional **analysis** method is applicable if the system can be **formulated** as a differential equation with a linear part and a nonlinear part of the form $g(t) F(x, \dot{x}, \dots, x^{(n-1)})$. The only **approximation** involved in this method is that of using only a finite number of the terms in the series. The random inputs must be strict-sense stationary and have all movements bounded.
- (4.) The sequential estimation **method** is **good** in that it can be used when the input is a composite of control function and noise. The allowed system configuration is:



where $\underline{U}(t)$ and $\underline{V}(t)$ are unknown noises and $\underline{G}(t)$ is a deterministic control force.

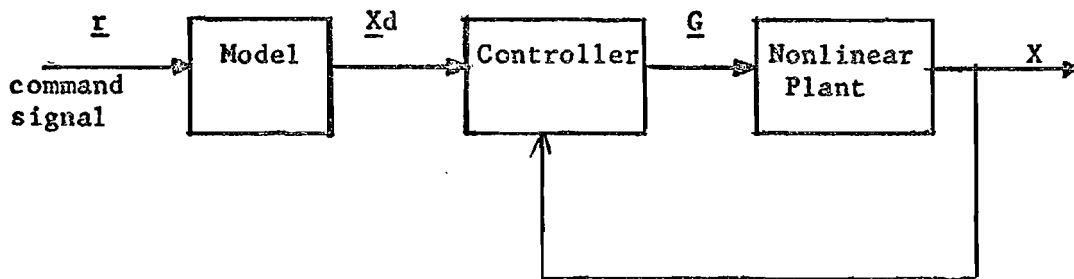
The response of a linear **system** subjected to stochastic inputs and a **deterministic** control force **can be obtained** by adding the response of the system due to **the** stochastic inputs **alone** to the response of the **system** due to the **deterministic control force** alone. That the sum of the responses to the individual inputs is the responses of the system to the combined inputs follows from the validity of the superposition principle for linear systems.

A **common** characteristic of three of the four approaches to the analysis of nonlinear systems subjected to **random inputs given** above **was the** absence of a **deterministic** control force. **No such** convenient superposition principle exists **for** nonlinear systems so that, in general, the response of a nonlinear system to a stochastic input and a deterministic input is not the **sum** of the response due to the stochastic part **alone and** the response due to the **deterministic** part alone. Because superposition is **not** valid for nonlinear systems, the response to stochastic inputs alone is of no value **in trying** to **determine** the response to stochastic inputs and a deterministic control.

In the proposed research, it is **my** desire to attempt to solve the problem of analysis of nonlinear systems subjected to stochastic inputs **and** deterministic control. I would also like to consider the **problem** of **filtering** under the above conditions **when** the observation is contaminated by measurement noise. **My** approach will be to **learn how** to derive the **Stratonovich-Kushner-**

Bucy **fittering** equation for **this** general nonlinear system having stochastic and deterministic **inputs** and to learn the **Ito** and **Stratonavich's** approach to manipulating this equation. I would then try to **generalize** the four approaches mentioned above to this problem.

Another approach to **the** problem that I plan to take in the proposed research will be in following the lead of a recent paper by H. F. **VanLandingham** and W. A. **Blackwell** (1967) on the **design** of a technique that generates a control **signal** which forces the **state** of a nonlinear plant to be close to the state of a **reference** model by **ingeniously** applying **Liapunov's** second method. The system configuration is **shown** below.



The **model** used is a **linear autonomous system** and represents "ideal" **system** behavior. The state variable representation of this model is $\dot{\underline{X}}_d = A_0 \underline{X}_d + B_0 \underline{r}$, where \underline{X}_d is the state of the linear **constant** system model, \underline{r} is the input vector, and A_0 and B_0 are constant matrices.

The actual nonlinear system is characterized by the **non-linear** differential equation $\dot{\underline{X}} = f(\underline{x}, \underline{G}, t)$, where \underline{X} is the actual state, \underline{G} is the deterministic control vector, and t is time,

The error, \underline{e} , is defined as $\underline{X}_d - \underline{X}$ and can be conveniently manipulated into the differential equation $\dot{\underline{e}} = A_0 \underline{e} + A_0 \underline{X} - f(\underline{x}, \underline{G}, t) + B_0 \underline{r}$.

Liapunov's second method is applied to this differential equation by assuming a form for the Liapunov function of $V = \underline{e}^T P \underline{e}$, where P is a symmetric positive definite matrix. The derivative of V with respect to time is

$$\dot{V} = \underline{e}^T [A_0^T P + P A_0] \underline{e} + \left\{ 2 \underline{e}^T [A_0 \underline{X} - f(\underline{x}, \underline{G}, t) + B_0 \underline{r}] \right\}$$

A deterministic control \underline{G} is synthesized that makes V negative definite. This condition ensures that the error, \underline{e} , is asymptotically stable and, hence, goes to zero.

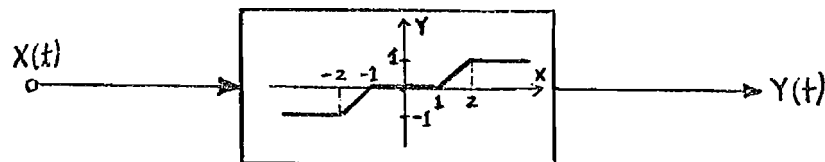
My proposed research on this aspect of the problem will be to investigate the possibility of replacing the Liapunov function of the deterministic case with a "stochastic Liapunov function" in synthesizing a \underline{G} to ensure \underline{e} to be asymptotically stable and thus generalize the above method to the control of nonlinear systems with random inputs.

The use of stochastic Liapunov functions has received a good deal of attention in the last three years. Kushner (1967) has done quite a bit in the area of stochastic Liapunov functions.

In working with stochastic Liapunov functions, I realize that this part of the research will be less practical-oriented since, as Kushner (1965) mentioned, a shortcoming common to both deterministic and stochastic Liapunov functions is the difficulty of finding them.

VII. AppendixStatistical Linearization Example

The figure below shows a nonlinear element representing saturation with a dead zone.



In equation form, $Y(t) = f[X(t)]$, where $X(t)$ is a zero mean gaussian random process with unit variance and having a pdf of $P_x(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} x^2)$.

The assumption made is that $X_H(t)$ is negligible in the representation $Y(t) = \text{Keg } X(t) + X_H(t)$. Therefore $X_H(t)$ is dropped, yielding $\hat{Y}(t) = \text{Keg } X(t)$.

The mean squared error which is to be minimized as the criterion satisfied by this technique is $[Y(t) - \hat{Y}(t)]^2 = [Y(t) - \text{Keg } X(t)]^2$
 $= [f(X(t)) - \text{Keg } X(t)]^2$.

So Keg is to be chosen to minimize

$$M = \int_{-\infty}^{\infty} [Y(t) - \hat{Y}(t)]^2 P_x(x) dx$$

The general result for Keg, shown earlier, is

$$\text{Keg} = \frac{\int_{-\infty}^{\infty} x f(x) p_x(x) dx}{\int_{-\infty}^{\infty} x^2 p_x(x) dx}$$

Since $p_x(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$,

$$\int_{-\infty}^{\infty} x^2 p_x(x) dx = \int_{-\infty}^{\infty} x^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \sigma^2 - (\mu_x)^2 = 1$$

In the above $\mu_x = 0$ since $\mu_x = \int_{-\infty}^{\infty} x \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = - \int_{-\infty}^{\infty} x' \frac{e^{-(x')^2/2}}{\sqrt{2\pi}} dx' = -\mu_x$

under $x' = -x$

and the only quantity equal to its negative is zero.

Evaluating the expression for Keg in terms of the appropriate value of $f(x)$ yields:

$$\begin{aligned} \text{Keg} = & \left[\int_{-\infty}^{-2} x(-1) p_x(x) dx + \int_{-2}^{-1} x(1+x) p_x(x) dx + \int_{-1}^{+1} x \cdot 0 \cdot p_x(x) dx \right. \\ & \left. + \int_{+1}^2 x(x-1) p_x(x) dx + \int_2^{\infty} x(1) p_x(x) dx \right] / 1. \end{aligned}$$

Substituting in the pdf yields:

$$\begin{aligned} \text{Keg} = & \left[\int_{-\infty}^{-2} x(-1) e^{-x^2/2} dx + \int_{-2}^{-1} (x+x^2) e^{-x^2/2} dx + \int_{-1}^{+1} 0 dx \right. \\ & \left. + \int_{+1}^2 (x^2-x) e^{-x^2/2} dx + \int_2^{\infty} x e^{-x^2/2} dx \right] / \sqrt{2\pi}. \end{aligned}$$

Using symmetry properties of even functions and shuffling the integrals around yields:

$$\text{Keg} = 2 \left[\int_2^{\infty} (x^2-x) e^{-x^2/2} dx + \int_1^2 x e^{-x^2/2} dx \right] / \sqrt{2\pi}.$$

To further simplify the above expression for Reg three integrations by parts are carried out below. Use is made of the error function.

$$i.) \int_2^{\infty} x e^{-x^2/2} dx = \frac{1}{2} \int_4^{\infty} e^{-u/2} du = -e^{-u/2} \Big|_4^{\infty} = -e^{-\infty} + e^{-2} = e^{-2}.$$

$$\begin{aligned} ii.) \int_1^2 x^2 e^{-x^2/2} dx &= \int_1^2 -x (-e^{-x^2/2} x dx) = -x e^{-x^2/2} \Big|_1^2 + \int_1^2 e^{-x^2/2} dx \\ &= -2e^{-2} + e^{-1/2} + \sqrt{\frac{\pi}{2}} \cdot \frac{2}{\sqrt{\pi}} \int_{\frac{1}{\sqrt{2}}}^{\sqrt{2}} e^{-u^2} du = -2e^{-2} + e^{-1/2} + \sqrt{\frac{\pi}{2}} [\text{erf}(\sqrt{2}) - \text{erf}(\frac{1}{\sqrt{2}})] \end{aligned}$$

$$iii.) \int_1^2 -x e^{-x^2/2} dx = e^{-x^2/2} \Big|_1^2 = e^{-2} - e^{-1/2}.$$